Weak Interactions of Equatorial Waves in a One-Layer Model. 
Part I: General Properties

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ABSTRACT

Dispersive equatorial waves are labeled by the zonal slowness \( s \), the meridional quantum number \( n \) and the vertical separation constant \( c \). The slowness (reciprocal of phase speed) is a variable more useful than the wavenumber to relate the interactions among equatorial waves. For instance, frequency is a simpler function of slowness than it is of wavenumber, and the four classes of equatorial waves are separated in \( s \)-space; viz., Rossby (R); \( s < -2n - 1 \), mixed Rossby-gravity (MR); \( s < 1 \), gravity (G); \( -1 < s < 1 \), and Kelvin (K); \( s = 1 \). Moreover, total energy and pseudo-momentum conservation require for the component with intermediate slowness of each triad to gain (loose) energy from (to) the other two. (If the triad is resonant, the wave with intermediate \( s \) must also have maximum absolute frequency.)

Nonlinear effects are parameterized by a single variable, the interaction coefficient \( \gamma \) for each resonant triad (RT). The interaction and resonance conditions are reduced to finding the zeros of a polynomial of, at most, sixth degree in \( s \), allowing for a classification of all possible resonant triads: There are three types of RT for \( n > 0 \): RRR, GGR, and GGG; resonant triads with \( M (n = 0) \) and/or \( K (n = -1) \) components have the properties of one of these three classes, depending on the frequency of the wave(s) with \( n < 1 \) (namely, the \( M \) and \( K \) may be taken as an \( R \) for \( s < \beta c/2 \) or as a \( G \) otherwise).

Non-local resonant triads in frequency space include: the packets of Rossby or inertia–gravity waves interacting with a long Rossby mode; short Rossby or inertia–gravity waves with different meridional quantum numbers interacting with a long Rossby or Kelvin mode (geostrophic flow); and the scattering of a short westward propagating inertia–gravity wave into a short eastward propagating inertia–gravity, mixed Rossby–gravity or Kelvin wave, by a short Rossby (or mixed Rossby–gravity) wave with twice the wavenumber.

Unlike the problems of quasi-geostrophic flow at midlatitude and internal gravity waves in a vertical plane, there are resonant triads of equatorial waves with the same speed, which have a finite interaction coefficient.

1. Introduction

Even if the weak-interactions limit (infinitesimal amplitudes) may not be appropriate to fully predict the evolution of realistic geophysical fluid systems at all scales, resonant interactions presumably play a relevant role in the description of the dynamics of those systems, at least for moderate energies. In a recent publication (Ripa, 1982; hereafter R82) the problem of nonlinear wave–wave interactions for a one-layer model in the equatorial \( \beta \)-plane (Matsuno, 1966) was posed, using a normal modes expansion (see also Tribbia, 1979). The present paper is devoted to the study of the weak interactions of that system, generalized to include the case of an isentropic atmosphere. Works by Domaracki and Loesch (1977), Loesch and Deininger (1979), and Boyd (1980a,b, 1983a,b) on related subjects are discussed in the appropriate section of this and the companion paper.

One interesting aspect of equatorial \( \beta \)-plane modes is that they approximate, for high enough meridional quantum numbers, the sphere structure functions relevant to midlatitudes. They are, in fact, a better approximation to Hough functions than the usual midlatitude \( \beta \)-plane sinusoids (Longuet–Higgins, 1968). Therefore, it is important to compare the results of this paper with the studies of resonant triads for Laplace tidal equations in the midlatitude \( \beta \)-plane. Particularly, Longuet–Higgins and Gill (1967) and Ripa (1981a, hereafter R81) studied the properties of resonant triads of planetary waves; Duffy (1974) analyzed the resonant interaction of one Rossby and two inertia–gravity waves; there are no resonant trios of inertia–gravity waves in the \( f \)-plane.

It would be presumptuous to try to explain or predict experimental facts solely from the study of the resonant triads of a model with such a poor density structure: a realistic dynamical system should also include off-resonant triads, interaction among different vertical modes, boundary effects, forcing and dissipation. Rather, this paper focuses on a particular aspect of equatorial dynamics. Simple questions that can be answered with the results of Part I and II of this paper are: how different or similar are resonant interactions in the tropics and at higher latitudes; can equatorial Rossby waves generate resonantly inertia-
gravity modes; are there resonant trios of equatorial gravity waves? (or must these modes resonate at a higher order, as they do in the f-plane?); etc.

Several comments can be made on the notation of this paper and its companion:

First, we use the slowness (Hayes, 1974) instead of the wavenumber k while dealing with the dispersive components because the relation \( \omega^2(s) \) is a rational function, whereas \( \omega(k) \) is transcendental; as a result, the expressions we encounter are much simpler in \( s \) than in \( k \). For instance, the solution of the resonance condition is reduced to finding the zeroes of a polynomial in \( s \); we do not get a polynomial in \( \omega \) or \( k \) because the relations \( s(\omega) \) and \( s(k) \) are transcendental. One can appreciate the advantage of dealing with polynomials. Why use slowness and not phase-speed as independent variable? Pseudo-momentum conservation requires a balance of energy cascade/decascade in \( s \)-space, and Rossby, gravity, and Kelvin components are ordered in \( s \). An ordering of components in slowness does not translate into an ordering in phase-speed (unless \( s \) has the same sign for all modes, which is not the case for the problem of this paper).

Second, we denote the separation constant, which characterizes the vertical structure, by \( c \) because it is the symbol widely used by oceanographers in equatorial studies. Thus, \( c \) should not be confused with the phase-speed (it coincides with the phase-speed only in the absence of rotation). Two variables often used instead of the separation constant are the equivalent-depth \( c^2g^{-1} \) and the stretched vertical wavelength \( cT_0 \) (with \( T_0 \) a typical buoyancy period); since these two variables do not add any additional physical insight, its use should be discouraged.

Third, the equatorial deformation radius used here is the \( \epsilon \)-folding half-width of a Kelvin wave: \( R = (2c\beta^{-1})^{1/2} \); other definitions found in the literature are \( R = (c\beta^{-1})^{1/2} \) and \( R = (2^{-1}\beta^{-1})^{1/2} \).

The rest of this paper is organized as follows: The model equations, conservation laws and phase-space expansion are presented in Section 2. The general properties of resonant interactions are discussed in Section 3; the problem of the stability of an infinitesimal wave is also discussed, as a first illustration of the method. Section 4 is devoted to the general solution of the interaction and resonance conditions (and the subsequent classification of all possible resonant triads). A summary is presented in Section 5. Finally, the triangular function \( \Delta(x) \) and notation are defined in the Appendix. Part II of this paper (Ripa, 1983c) is dedicated to applications of the results of this one, such as the study of resonant double-triads, interactions among resonant harmonics and among wave-packets.

It is worth pointing out that the results of Section 2 are restricted neither to weak interactions (they are valid for motion of any amplitude) nor to the equatorial \( \beta \)-plane (\( f_0 \) may take any value and zonal walls are allowed). Similarly, Section 3 is applicable to waves from any system with, at least, one homogeneous coordinate.

2. Model equations

The equatorial model with the simplest structure that, nevertheless, exhibits some interesting nonlinear phenomena, is probably one in which the dynamical fields are independent of the vertical coordinate \( z \): one active layer of homogeneous fluid in hydrostatic balance and in the \( \beta \)-plane. Let \( (x, \ y, \ t) \) be the independent variables for eastward and northward positions and time, respectively. The dependent variables are the zonal velocity \( u \), the meridional velocity \( v \), and a variable \( \eta \) related to both the pressure and mass fields (through the hydrostatic assumption). In particular, let \( h \) be proportional to the mass per unit horizontal area, and normalize it and \( \eta \) so that \( \eta \approx 1 + \eta \), with \( \beta \) and \( h \) and \( \eta \) constant; in the equatorial \( \beta \)-plane it is \( \beta = 0 \). We may leave the value of \( f_0 \) undetermined for this section. The equations of motion, in the absence of forcing and dissipation, take the form

\[
\begin{align*}
\partial_t u + u \partial_x u + v \partial_y u - f v + c^2 \partial_x \eta &= 0, \quad (2.1a) \\
\partial_t v + u \partial_x v + v \partial_y v + f u + c^2 \partial_y \eta &= 0, \quad (2.1b) \\
\partial_t h + \partial_x(hu) + \partial_y(hv) &= 0. \quad (2.1c)
\end{align*}
\]

If linearized, the set (2.1) with \( f_0 = 0 \) is Matsuno's (1966) model, which, indeed, may be used to study the temporal/horizontal evolution of the fields in an arbitrary vertical mode, standing or propagating, parameterized by the value of the separation constant (see, e.g., Moore and Philander, 1977).

Taken as a physically plausible nonlinear model, though, the set (2.1) cannot represent motion with any vertical structure: the vertical derivative of the nonlinear terms is not proportional, in general, to that of the linear ones which results in a nonlinear coupling of the vertical modes. The two-dimensional model (2.1) only applies to the case of one active layer with vanishing velocity vertical shear and buoyancy.

For an incompressible ocean, the dynamic pressure field is \( c^2 \eta - gz \), and the depth of the active layer is equal to its main value, say \( D \), multiplied by \( h \), with \( h = 1 + \eta \). Let the active layer be layered between two passive ones. The separation constant is then given by

\[
c^2 = \frac{gD(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)}{\alpha_2(\alpha_1 - \alpha_3)},
\]

where \( \alpha_1 \), \( \alpha_2 \), and \( \alpha_3 \) are the specific volumes of the top, middle, bottom layers. (This includes the possibilities of vacuum instead of the top layer, \( \alpha_1 = \infty \), or a flat bottom, \( \alpha_3 = 0 \).)
For an atmosphere with constant potential temperature \( \theta \), the absolute temperature field is \( T = \theta + (c^2 \eta - g)C_p^{-1} \), and the separation constant is given by \( c^2 = R \theta \), where \( R \) and \( C_p \) are the gas constant and specific heat at constant pressure for dry air (see, e.g., Holton, 1979). The pressure and specific volume fields are found by using \( \log(T/\theta) = \lambda \log(p/p_0) \) and \( \alpha = RT/p \), where \( p_0 \) denotes the surface average pressure and \( \lambda = R/C_p \). Thus the surface pressure is equal to \( p_0 h \) with \( h = (1 + \eta)^{1/\lambda} \). [The model atmosphere extends from \( z = 0 \) to \( z = (C_p \theta + c^2 \eta)^{-1} \), where \( T = p = 0 \). If, instead, a passive layer with potential temperature \( \theta' > \theta \) is on top of the active one, the function \( h(\eta) \) is more complicated.]

The relations between \( h \) and \( \eta \) for the ocean and the atmosphere models can be cast into a single expression, viz.,

\[
\begin{align*}
\lambda &= 1, \\
\lambda &= R/C_p (-2/7)
\end{align*}
\]  

(2.2a) (2.2b)

for the ocean and

(2.3a) (2.3b)

for the atmosphere. With the expression (2.2) for \( h(\eta) \), the equation of mass conservation, (2.1c), becomes

\[
(\partial_t + u \partial_x + v \partial_y) \eta + (1 + \lambda \eta) (\partial_x u + \partial_y v) = 0.
\]

The nonlinearity of the model equations is exactly quadratic in \( (u, v, \eta) \). Expression (2.3) is indeed the most general relation \( h(\eta) \) for which the nonlinearity in (2.1) is just quadratic. The system (2.1) is invariant under the transformation \( \eta \rightarrow (\eta - a)/(1 + a) \) and \( c^2 \rightarrow (1 + a)c^2 \), where \( a \) is any constant (for the atmosphere model, this means a change in the normalization of \( \theta \) and \( p_0 \)). These variables are uniquely determined by requiring

\[
h \sim 1 + \eta \quad \text{as} \quad \eta \rightarrow 0, \quad \langle h \rangle = 1 \quad (2.4a, b)
\]

where angle braces denote an average over the entire horizontal domain.

Alternatively, we might start with an arbitrarily stratified incompressible ocean, perform an expansion in vertical modes using an isopycnal coordinate, and truncate the nonlinear equations in the expansion amplitudes down to only one mode (e.g., the first baroclinic one). The horizontal velocity and pressure perturbation fields are then given by \( (u, v, \eta) \) multiplied by certain structure functions, characterized by the separation constant \( c \) with a particular normalization which makes the nonlinear coefficient in (2.1a, b) equal to unity; e.g., see R82 Section 3b. It turns out that the mass conservation law takes the form (2.1c) with \( h = 1 + \eta \). A similar procedure may be done starting with a stratified atmosphere and using an isentropic coordinate for the expansion; the relation \( h(\eta) \) for (2.1c) is, in general, not of the form (2.2).

Domaracki and Loesch (1977) pioneered the study of equatorial wave interactions, adding the advective terms to the linear model of Matsuno (1966), i.e., Eqs. (2.1a, 2.1b), and \( (\partial_t + u \partial_x + v \partial_y) \eta + \partial_x u + \partial_y v = 0 \). Matsuno's model consists of a single layer of incompressible fluid, i.e., \( \lambda = 1 \) in (2.2), but Domaracki and Loesch explicitly assumed that the term \( \eta (\partial_x u + \partial_y v) \), in (2.1c) for \( \lambda = 1 \), is negligible. In R82 we stated that this is not always the case, rather, the missing term may be of the same order of magnitude of the advective ones, particularly for gravity and Kelvin components. The height equation used by Domaracki and Loesch can be written in the form (2.1c) with \( h = \exp \eta \), which corresponds to the limit \( \lambda \rightarrow 0 \) in (2.2). The practical difference between the models of this paper and that of Domaracki and Loesch is that of using \( \lambda \) from (2.3) or \( \lambda = 0 \) in the formulas for the evaluation of the coupling coefficients, which are presented below.

The model employed by Tribbia (1979) to study the nonlinear initialization problem of global-scale numerical models corresponds to the \( \lambda = 1 \) case of (2.1a)-(2.2).

\section*{a. Conservation laws}

Having posed the physical problem, the next step is to derive the exact conservation laws, satisfied by the fully nonlinear equations. This is important at this point so that when approximations are specified, it is possible to check how the conservation laws are affected and whether or not new integrals of motion appear.

First, potential vorticity is conserved following any fluid column,

\[
(\partial_t + u \partial_x + v \partial_y) q = 0, \quad (2.5a)
\]

(2.5b)

Second, total zonal momentum is conserved (if there are no meridional boundaries),

\[
\int dxdy M = \text{constant}, \quad (2.6a)
\]

\[
M = h(u - 1/2 \beta^{-1} f^2). \quad (2.6b)
\]

Finally, total energy is also conserved. The kinetic energy density, per unit horizontal area, is \( \frac{1}{2} h(u^2 + v^2) \), and the potential energy density is equal to \( \int dh c^2 \eta \). With \( h(\eta) \) from (2.2) we get

\[
\int dxdy E = \text{constant}, \quad (2.7a)
\]

\[
E = \frac{1}{2} h(u^2 + v^2) + c^2 \eta + (h - h + 1)(\lambda + 1)^{-1}. \quad (2.7b)
\]

To lowest order total energy is quadratic in \( (u, v, \eta) \), but total momentum is only linear. As a consequence, the constraint imposed by (2.7a) is more powerful than that derived from (2.6a); at least in a
phase-space formalism as the one developed next. However, we can construct a pseudomomentum, which is quadratic, to lowest order, in \((u, v, \eta)\), in the following way: Combining the equations of mass and potential vorticity conservation, (2.1c) and (2.5), we get
\[
\int \int dxdy hF(q) = \text{constant}, \quad (2.8)
\]
where \(F\) is an arbitrary function of \(q\). We now select \(F(q)\) so that, by subtracting (2.8) from (2.6a), there are no linear terms in \(\int \int dxdy[M - hF(q)]\), and we obtain the law of pseudomomentum conservation,
\[
\int \int dxdy P = \text{constant}, \quad (2.9a)
\]
\[
P = (h - 1)u - \frac{1}{2} \beta^{-1} h(q - f)^2. \quad (2.9b)
\]
The reader can verify that \(P = M - \frac{1}{2} \beta^{-1} \eta q^2 + \{f^2 + \delta_x(\beta v) - \delta_y \beta (\beta u)\}|\beta^{-1} - \text{the integral of the term between braces is trivially conserved.}

By combining (2.8) with (2.6a) and (2.7a) it is also possible to define pseudomomentum and pseudostream function such that they are quadratic in the deviation from a reference state which represents a steady and stable zonal flow; i.e., quadratic in \([u - u_0(y), v, \eta - \eta_0(y)]\) (Ripa, 1983a). On the sphere, zonal momenta must be changed into angular momenta.

The conservation laws may be linked to the symmetries of the problem, by using Noether’s (1918) theorem. Thus, potential vorticity conservation is related to the invariance of the system under a general change of the particle labels, energy conservation to time homogeneity, and momentum and pseudomomentum conservation to invariance under translations in \(x\) for both the particles positions and labels (Ripa, 1981b). If boundaries other than zonal walls are included in the problem, the last symmetry is broken: meridional walls are sources and sinks of momentum and pseudomomentum (see R82).

It is convenient to show explicitly the lower contributions to (2.7a) and (2.9a), in the form
\[
E^{(2)} + E^{(3)} + O(u, v, \eta)^4 = \text{constant}; \quad (2.10)
\]
\[
P^{(2)} + P^{(3)} + O(u, v, \eta)^4 = \text{constant}, \quad (2.10)
\]
where
\[
E^{(2)} = \int \int dxdy \frac{1}{2}(u^2 + v^2 + c^2 \eta^2), \quad (2.11a)
\]
\[
P^{(2)} = \int \int dxdy \eta u - \frac{1}{2} \beta^{-1} \xi^2, \quad (2.11b)
\]
and
\[
E^{(3)} = \int \int dxdy
\times \frac{1}{2} \eta(u^2 + v^2) + \frac{1}{3} c^2 (1 - \lambda) \eta^3, \quad (2.12a)
\]
\[
P^{(3)} = \int \int dx dy \frac{1}{2}
\times \left[ \beta^{-1} \eta \xi^2 + (1 - \lambda) \eta^2 (u + \beta^{-1} f \xi) \right]. \quad (2.12b)
\]
In (2.11b) and (2.12b),
\[
\xi = \delta_x v - \delta_y u - f \eta \quad (2.13)
\]
is the first order deviation of potential vorticity from its equilibrium value [viz., \(q = f + \xi + O(u, v, \eta)^2\)].

Notice that \(\lambda\), which parameterizes the difference between the incompressible and isentropic models, does not appear in the quadratic term of energy and pseudomomentum, but does so in the cubic one. For the linearized problem only second order terms are relevant, but for nonlinear calculations the form and value of the cubic terms (and, presumably, higher ones) are also important. For instance, the existence of the cubic term in the kinetic energy (which is not present in the midlatitude quasi-geostrophic theory) makes it possible for equatorial jets with constant gradient of potential vorticity to be unstable (Ripa, 1983a). The nonlinear coupling coefficients satisfy certain sum rules, developed below, related to the cubic terms of total energy and pseudomomentum.

All these conservation laws, Eqs. (2.5)–(2.13), are also valid for the model of Domaracki and Loesch (1977) if the somewhat artificial relations \(h = \text{exp}(\eta)\) and \(\lambda = 0\) are used. Their statement that the quadratic part of the total energy is conserved, \(E^{(2)} = \text{constant}\), is, however, incorrect [see Eq. (24) in their paper].

b. Phase-space expansion

We now change the representation of the state of the system from the dynamical fields in physical space \((u, v, \eta)\) to the complex amplitudes in phase–space, \(Z_a(t)\), by means of the expansion
\[
Z_a(t) = \sum \alpha_a Z_a(t) \left( \begin{array}{c}
u_a(x, y) \\
\eta(x, y) \\v_a(x, y) \\
\eta_a(x, y)
\end{array} \right). \quad (2.14)
\]
The inverse transformation is
\[
Z_a(t) = \int \int dxdy(u_a u + v_a v + c^2 \eta_a \eta). \quad (2.15)
\]
The expansion modes are the eigenfunctions of the linear operator in (2.1), i.e.,
\[
-i \omega_a u_a - f v_a + c^2 \partial_x \eta_a = 0, \quad (2.16a)
\]
\[
-i \omega_a v_a + f u_a + c^2 \partial_y \eta_a = 0, \quad (2.16b)
\]
\[
-i \omega_a \eta_a + \partial_x u_a + \partial_y v_a = 0, \quad (2.16c)
\]
normalized by the quadratic energy integral (2.11a) [e.g., see (2.17) below]. The eigenvalue \(\omega_a\) corresponds to the frequency in the linear case.

For the equatorial beta-plane, the expansion modes fall into four categories: Rossby (R), mixed Rossby–

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gravity \((M)\), gravity or inertia–gravity \((G)\), and Kelvin \((K)\). In the variable \(a\) are grouped the zonal wave-number \(k\), the meridional quantum number \(n\), and another discrete index \(r\) \([-\infty < k < \infty, \ n \geq -1, -1 < r < 1]\). The vertical structure is parameterized by a single value of the separation constant \(c\); in a system (unlike the one in this paper) with a more realistic density structure, \(c\) takes different values and must be included in the label \(a\). The eigenfunctions of (2.16) are of the form \(\exp(i\kappa x)\) multiplied by a combination of Hermite functions; the normalization is such that

\[
\int dx dy (u_a^* u_b + u_b^* v_b + c^2 n_a^* n_b) = \left(\frac{1}{c}\right)^{1/2} 2\pi \delta(k_a - k_b), \quad (2.17)
\]

if \(n_a = n_b\) and \(r_a = r_b\), or is equal to zero otherwise \((a \neq b)\). Finally, the sum symbol in (2.14) represents

\[
\sum_a = (2\pi)^{-1/2} \int dk \sum_n.
\]

For each expansion mode with parameters \((\omega_a, k_a, n_a, r_a)\) and structure functions \((u_a, v_a, \eta_a)\), there is another mode, with parameters \((-\omega_a, -k_a, n_a, r_a)\) and structure functions \((-u_a^*, -v_a^*, -\eta_a^*)\); we denote the latter by \(a^*\) and call it the conjugate of \(a\). The summation (2.18) in (2.14) runs over both components of each conjugate pair \((a, a^*)\). Conjugate modes are mathematically independent, in the sense of (2.17), but physically equivalent, because reality of \((u, v, \eta)\) in (2.14) requires

\[
Z_a^* = -Z_a^* = -\omega_a, \ k_a = -k_a. \quad (2.19)
\]

The fact that \(a\) and \(a^*\) denote the same physical state is used below to write down formulas in a symmetric way.

It is sometimes more convenient to label the dispersive \((n > -1)\) expansion modes by \(n\), the sign of \(\omega\) and the slowness \((\text{Hayes, 1974})\)

\[
s = \frac{k}{\omega}. \quad (2.20)
\]

The relations \(\omega^2(s, n)\) are summarized in Table 1 and are shown in Fig. 1. Notice that \(R\), \(G\), and \(K\) modes are clearly separated in slowness-space (viz., \(sc \leq -2n - 1, -1 < sc < 1\), and \(sc = 1\), respectively); \(M\) states may be included among the \(R\) for \(sc \leq -1\) \((\omega^2 < \beta c/2)\), or among the \(G\) for \(sc > -1\) \((\omega^2 > \beta c/2)\). In order to fix ideas, the values of \(sc\) corresponding to short and long waves are presented in Table 2.

One important property of the expansion basis is that it provides a diagonal representation of the quadratic part of total energy and pseudomomentum, viz.,

\[
E^{(2)} = \frac{1}{2} \sum_a |Z_a|^2, \quad P^{(2)} = \frac{1}{2} \sum_a s_a |Z_a|^2. \quad (2.21)
\]

If the expansion (2.14) is made with an arbitrary complete basis, instead of the eigenfunctions of (2.16), then in (2.21) the sums are double and the phases of the \(Z_a\) will enter in them. Notice that every physically

| Table 1. Frequency \(\omega\) as a function of the meridional quantum number \(n\) and the zonal slowness \(s\). The zonal wave number is given by \(k = sa\). |
|---|---|---|---|
| Rosby-gravity | Inertia-gravity | Kelvin |
| \(n \geq 1\) | \(n = 0\) | \(n \geq -1\) | \(n = -1\) |
| \(sc \leq -2n - 1\) | \(-1 < sc < 1\) | \(sc = 1\) | 
| \(\omega^2 = \beta c (sc + 2n + 1)/(1 - sc^2)^2\) | \(\omega^2 = \beta c (sc + 2n + 1)/(1 - sc^2)^2\) | \(\omega = \beta c\) | 

| Table 2. Parameters of short and long waves. |
|---|---|---|---|---|
| \(sc\) | \(-\infty\) | \(-2n - 1\) | \(-1\) | \(0\) | \(+1\) |
| \(R\) | \(-\infty\) | 0 | \(-\infty\) | 0 | \(\infty\) |
| \(G\) | \(-\beta/k\) | \(-kc/(2n + 1)\) | \(-kc\) | \(\pm[(2n + 1)\beta c]^{1/2}\) | \(+kc\) |
| \(n\) | \(0\) | 1 | \(1\) | 0 | \(-1\) |
| \(R\) or \(M\) | \(R\) | \(G\) | \(G\) or \(M\) | \(G, M\) or \(K\) |
different state contributes twice to the summations in (2.21), in the form of conjugate modes (a, a*); see (2.19). The cubic part of total energy and pseudomomentum are obtained by replacing (2.14) in the right hand side of (2.12); the end result is

\[ E^{(3)} = \frac{1}{\hbar} \sum_{abc} (-i) \delta_{abc} S_{abc} Z_a Z_b Z_c, \quad (2.22a) \]

\[ P^{(3)} = \frac{1}{\hbar} \sum_{abc} (-i) \delta_{abc} U_{abc} Z_a Z_b Z_c, \quad (2.22b) \]

where

\[ \delta_{abc} S_{abc} = \int \int dx dy i \times [\eta_a v_b - v_c + c^2 (1 - \lambda) \eta_a \eta_b \eta_c], \quad (2.23a) \]

\[ \delta_{abc} U_{abc} = \int \int dx dy i \times [\beta^{-1} \eta_a \xi_b + s_a c^2 (1 - \lambda) \eta_a \eta_b \eta_c] + c.p. \quad (2.23b) \]

Here, “c.p.” means plus cyclic permutations of (a, b, c) in what is preceding. In (2.23b), \( \xi_a \) = \( \delta_a v_b - \delta_b v_a - f_{n_a} \) is the expansion function for the deviation of potential vorticity, defined in Eq. (2.13); from (2.16) one gets

\[ \xi_a = -i \beta v_a \omega_a^{-1} \quad (2.24) \]

which is the linearized counterpart of (2.5a). The term \( u + \beta^{-1} f \xi \) in (2.12b) was simplified using \( u_a + \beta^{-1} f \xi_a = s_a c^2 \eta_a \), which is Eq. (2.16a) written with \( \xi_a \) from (2.24), instead of \( v_a \).

If the model’s domain is the unbounded equatorial \( \beta \)-plane (or a zonal channel, for that matter), the double integrals in (2.23) factor into the integral of \( \exp[i(k_a x + k_b y + k_c z)] \) in \( x \), which yields Dirac’s delta, multiplied by an integral in \( y \) of a finite sum of products of three Hermite functions and constant coefficients, which is easily evaluated with the algebraic expression developed by Busbridge (1948) (see also Ripa 1983b). We absorb the integral in \( x \) and the symmetry condition for the existence of the integral in \( y \), in the factor \( \delta_{abc} \), defining

\[ \delta_{abc} = (c/\beta)^{1/2} 2 \pi b (k_a + k_b + k_c) \quad (2.25) \]

if

\[ n_a + n_b + n_c = \text{odd}, \quad (2.26) \]

or \( \delta_{abc} = 0 \) if the sum of the three meridional quantum numbers is even.

For some problems [e.g., Eq. (3.10) below], the phase-space expansion involves only a denumerable subset of components, instead of the continuum in \( k \). The \( Z_a \) may then be redefined so that the \( \Sigma \) in (2.14) and ff. are ordinary summations, the \( \delta_{abc} \) is a regular Kronecker’s delta, and the integrals in \( x \) are replaced by a zonal average. We distinguish the continuous and discrete cases by using a bold summation sign in the former one.

c. Evolution equation

The model equations (2.1) for the dynamical fields \( (u, v, \eta) \) are transformed into the evolution equation

\[ [\partial_t + i \omega_a] Z_a = \frac{1}{2} \sum_{bc} \sigma_a^{bc} \delta_{abc} Z_b Z_c^* \quad (2.27) \]

for the amplitudes \( Z_a(t) \) exactly. The \( \sigma_a^{bc} \) are coupling coefficients, with dimension of wavenumber; all the information on the nonlinear structure of the system is contained in them (a closed expression for its evaluation is developed below).

Not only is (2.27) an exact representation of the original model equations for there are no higher order terms neglected, but it is easier to derive than the accustomed perturbative multiple time-scale expansion: only the completeness and orthonormality of the basis \( (u_a, v_a, \eta_a) \) is needed, which is more than a consequence of the hermiticity of the problem (2.16) (see R81 and Ripa 1981b for a similar derivation of the evolution equation in other geophysical fluid dynamics problems). Perturbation expansions [which indeed may be derived from (2.27), if desired] are usually intractable beyond the first nonlinear term and have to be developed independently for each particular problem. Compare, for instance, the articles by Domaracki and Loesch (1977), and Boyd (1980a,b, 1983a,b).

Many approximations to the nonlinear problem (2.1) can be derived as truncations of the expansion (2.14) and, in some cases, approximations of the linear \( \omega \) and nonlinear \( \sigma_a^{bc} \) coefficients in (2.27): Tribbia (1979) uses a truncation of a discrete realization of (2.27), appropriate for spherical geometry, in order to study the nonlinear initialization problem of global-scale numerical models. [Tribbia’s model differs from (2.27) in his treatment of the zonal modes, which he does not compute from (2.16).] The nonlinear Kelvin wave evolution equation is obtained in R82 simply by arbitrarily using only \( K \) modes in (2.27); further physical insight is then gained including non-\( K \) modes in the left hand side of that equation [this is to be contrasted with the “strained coordinates” perturbation method of Boyd (1980a)]. Meiss’ (1979) test-wave model is obtained neglecting all terms in the right hand side of (2.27) corresponding to triads \( (a, b, c) \) in which a certain mode, the test wave, does not appear; the effect of including other triads without the test wave is discussed in Meiss (1981). Other examples of approximations to (2.27) are presented in this and the companion paper; in particular, it is shown how the equation for the \( (x, t) \) dependence of wave-packets (e.g., Boyd 1980b, 1983a,b) can be derived from the evolution equation in \( Z_a(t) \).

We now derive sum rules for the coupling coefficients, imposed by the laws of energy and pseudomomentum conservation. [The normalization used for the expansion functions is such that \( u_a, \eta_a, \) and
ξ and $v_\nu$ are purely imaginary, and $v_\nu$ is real; as a consequence, $S_{abc}$, $U_{abc}$, $\sigma_{ab}^{bc}$, $T_{ab}^{bc}$, and $\gamma_{abc}$ (the last two defined below) are real. We use the convention that any of those coefficients is invariant under permutation of either the subscripts or the superscripts. From (2.27) it follows

$$
\partial_r |Z_d|^2 = \frac{1}{2} \sum_{bc} \sigma_{ab}^d \frac{\partial Z_a}{\partial Z_b} = \frac{i}{1} (\omega_a + \omega_b + \omega_c) Z_a Z_b Z_c + O(Z^4)
$$

Replacing this in (2.21) and (2.22) and using the result in the time derivative of (2.10) we obtain

$$
0 = \sum_{abc} \left[ \frac{\partial}{\partial t} [\sigma_{ab}^{bc} - \frac{1}{2} S_{abc} \omega_a + \omega_b + \omega_c] \right] \delta_{abc} + \frac{\partial}{\partial t} [\omega_a \omega_b \omega_c] + O(Z^4)
$$

(2.29)

Since these equations must hold for any state of the system, $Z_a Z_b Z_c$ is completely arbitrary (besides its invariance under permutations). Therefore, the symmetric part of the expressions between square brackets in (2.29) must vanish for all interacting trios, viz.,

$$
\sigma_{ab}^{bc} + \sigma_{ba}^{cb} = S_{abc} \omega_a + \omega_b + \omega_c,
$$

(2.30a)

$$
\frac{s_{ab}^{bc} + s_{ba}^{cb}}{s_{ab}^{ac} + s_{ba}^{ca} + s_{bc}^{ab}} U_{abc} (\omega_a + \omega_b + \omega_c).
$$

(2.30b)

An immediate consequence of (2.30) is that, if (2.14) is approximated by an arbitrary truncation of the expansion basis and all corresponding nonlinear terms are retained in (2.27) even though total energy and pseudomomentum are not exactly conserved, the rate of change of $E$ and $P$ is $O(Z^4)$.

For the problems of quasi-geostrophic flow at mid-latitudes (QGF) or two-dimensional internal gravity waves (IGW), total energy and pseudomomentum are exactly quadratic integrals. As a consequence, the LHS of (2.30) vanishes for any interacting trio (not just for the resonant ones), the coupling coefficients of any triad satisfy $\sigma_{ab}^{bc}$, $\sigma_{ba}^{cb}$, $\sigma_{bc}^{ab}$, and $E$, $P$ are conserved for an arbitrary truncation of the expansion basis, to all orders in the amplitude (see R81).

d. Evaluation of the coupling coefficients

Here we derive a formula for the calculation of the nonlinear coefficient in (2.27), $\delta_{abc} \sigma_{ab}^{bc}$, which involves integrals of $\omega_a$, $v_\nu$ (or $\xi_\nu$) and $\eta_\nu$; the coordinates and derivatives do not appear in it. The formula is more general than what is needed in this paper: arbitrary rigid boundaries (with vanishing normal velocity) may be added and the $\beta$-plane need not be at the equator. (The formula is also easily generalized to spherical coordinates.) The $x$-homogeneity and $y$-symmetry implied in (2.25) and (2.26) does not need to hold.

Let us recollect the necessary background: we build the evolution Eq. (2.27) as an image of the model Eqs. (2.1) under the transformation (2.14)–(2.15), the expansion basis is the solution of (2.16) from which it is derived (2.24). We start by rewriting the set (2.1), with (2.2), as

$$
\partial_r \mu + (f + \partial_x v - \partial_y u) u + \partial_x (c^2 \eta + \frac{1}{2} \psi^2) = 0,
$$

(2.31a)

$$
\partial_r v + (f + \partial_x u - \partial_y v) u + \partial_x (c^2 \eta + \frac{1}{2} \psi^2) = 0,
$$

(2.31b)

$$
\partial_r \eta + [1 + (\lambda - 1) \eta] \nabla \cdot v + \nabla \cdot (\eta \psi) = 0.
$$

(2.31c)

We now: 1) substitute $u$, $v$, and $\eta$ by the expansion (2.14) in the linear terms, and its complex conjugate (2.16) in the nonlinear ones; 2) use (2.16) to replace the linear evolution operator by $i \omega$; 3) multiply (2.31a) by $u_\nu^a$, (2.31b) by $v_\nu^b$, and (2.31c) by $c^2 \eta_\nu^a$; 4) add the three products and integrate in $dxdy$; and 5) use the orthonormality of the eigensolutions of (2.16). The final result is the evolution equation (2.27), with

$$
\frac{\delta_{abc} \sigma_{ab}^{bc}}{a} = \int dx dy ([u_\nu^a v_\nu^b - u_\nu^b v_\nu^a] \xi^c + \eta^c) - \frac{1}{2} u_\nu^a \nabla \cdot (v_\nu^b \psi) - c^2 \eta_\nu^a \nabla \cdot (v_\nu^b \psi) + (b \leftrightarrow c).
$$

(2.32)

Integrating by parts, the term containing $v_\nu \cdot \nabla$ is changed to $\frac{1}{2} \nabla \cdot (v_\nu \psi)$, and $\cdot \nabla v_\nu = i \omega_\nu \eta_\nu$ by virtue of Eq. (2.16c). Similarly, $\cdot \nabla \cdot v_\nu = i \omega_\nu \eta_\nu$. Finally, the term with $\cdot \nabla (v_\nu \eta_\nu)$ is integrated by parts and $c^2 \nabla \eta_\nu$ is replaced by $(i \omega_\nu - f \times \gamma) \eta_\nu$, Eqs. (2.16a, b); the Coriolis term then cancels with that of the first term. The final result is

$$
\sigma_{ab}^{bc} = \gamma_{abc} \omega_a + T_{ab}^{bc} (\omega_a + \omega_b + \omega_c),
$$

(2.33)

where $\gamma_{abc}$ the interaction coefficient, is defined by

$$
\gamma_{abc} = S_{abc} - T_{ab}^{bc} - T_{cb}^{ba} - T_{bc}^{ab},
$$

(2.34)

with $S_{abc}$ given by (2.23a), and $\delta_{abc} T_{ab}^{bc}$ by

$$
= \int dx dy [\beta^{-1} u_\nu \xi^c + \eta_\nu \psi - (\lambda - 1) c^2 \eta_\nu \eta_\nu].
$$

(2.35)

Thus the evaluation of the coupling $\sigma_{ab}^{bc}$, and interaction $\gamma_{abc}$ coefficients is reduced to the computation of the integrals (2.23a) and (2.35), whose integrands are simple expressions of the linear problem eigenfunctions. Terms with non-constant coefficients (through the appearance of $f$) and derivatives have been eliminated. If the domain is invariant under zonal translations and eigenfunctions with an $exp(ikx)$ structure are used, the $x$-integral yields $\delta_{abc}$ from (2.25), and $S_{abc}$ and $T_{ab}^{bc}$ are evaluated in terms of a single $y$-integral of the product of three parabolic cylinder functions. In the case of the unbounded equa-
torial β-plane, the latter are Hermite functions and an analytic expression for their integral has been developed by Busbridge (1948).

We finish this section by showing that the coupling coefficients do indeed fulfill the sum rules (2.30a, b), necessary conditions for energy and pseudomomentum conservation [they are sufficient conditions for E-P conservation up to O(βζ)]. Recall that Sabc, Uabc, σa_b, T_a^bc, and γabc are real, invariant under permutations of either the subscripts or the superscripts and invariant under the change of (a, b, c) by (a*, b*, c*) with the exception of σa_b that changes sign with the last transformation. Quadratic energy conservation, Eq. (2.30a), is trivially satisfied for any domain using (2.34) in (2.33). Eq. (2.30b), related to pseudomomentum conservation, is applicable only if there is x-homogeneity (and thus zonal wavenumbers are defined). The first term of (2.33), γω, does not contribute to (2.30b) because s_aω_a + s_bω_b + s_cω_c is equal to k_a + k_b + k_c (by definition of slowness) and this sum must vanish as indicated by the delta function in (2.25). In order for (2.30b) to hold, then it must be

\[ s_a T_a^{bc} + s_b T_b^{bc} + s_c T_c^{bc} = U_{abc}. \]  

The difference between the lhs and rhs of this equation is proportional to the integral of \([s_a (u_a - n_a) \xi_k \xi_c + \text{c.p.}]\) but \(s_a u_a - n_a = - \beta^{-1} \xi_k \xi_c\) in virtue of (2.16c), and therefore that difference is proportional to the integral of \(\delta_k (\xi_k \xi_k)\) which vanishes (q.e.d.).

e. The limit of high meridional quantum number

Condition (2.26) is much weaker than the usual one in midlatitudes (vanishing sum of the meridional wavenumbers). For large values of \(n\), however, the expansion functions extend further away from the equator and have many oscillations; this results in negligible values of the coupling coefficients unless there are points of stationary phase of the integrand (Ripa, 1983b). Thus, if \(X_c\) represents any of the structure functions in the integrals (2.23a) or (2.35), for large enough \(n_a\) and equatorward from the turning latitudes it may be approximated by the WKBJ expression

\[ X_a \approx (l_a)^{-1/2} \cos \left( \int l_a dy \right), \]  

(\(β y^2 < (2n_a + 1)c, \quad n_a \gg 1\),  

(2.37)

where \(l_a(y)\) is the local meridional wavenumber, defined by

\[ l_a^2 = (2n_a + 1)βc^{-1} - f^2c^{-2}. \]  

(2.38)

The stationary phase points of \(X_a X_b X_c\) are such that the local meridional wavenumbers add up to zero, say at \(y = y_0\) with

\[ l_a(y_0) + l_b(y_0) + l_c(y_0) = 0, \]  

(2.39)

and must be located between the turning latitudes of the three modes. (The component with largest \(n\) is defined to have a value of \(l\) with the sign opposite to the other two.) It follows from (2.38) and (A2) that this condition is satisfied for

\[ \Delta (2n_i + 1 - \beta y_0^2 c^{-1}) = 0, \]  

(2.40)

which is a quadratic equation in \(y_0^2\) [see (A3)]. One of the solutions of (2.40) is poleward from the turning latitudes of the three modes and is, therefore, unacceptable (\(l_i^2 < 0\); the other one is given by

\[ β y_0^2 c^{-1} = 2 \langle n \rangle + 1 - [(2 \langle n \rangle + 1)^2 - y_0^2 Δ_0]^{1/2}, \]  

(2.41)

where

\[ Δ_0 = Δ(2n_i + 1), \quad \langle n \rangle = \frac{1}{3}(n_a + n_b + n_c). \]  

(2.42)

For instance, if two of the \(n_i\) are equal, we obtain

\[ n_a < n_b = n_c \to y_0^2 = \frac{(2n_a + 1)cβ^{-1}}{1}; \]

\[ l_a = 0, \quad l_b = -l_c, \]  

(2.43a)

\[ n_a = n_b < n_c \to y_0^2 = \frac{8n_a + 3 - 2n_b(c/3β)}{1}; \]

\[ l_a = l_b = -l_c. \]  

(2.43b)

It is easy to show that a necessary and sufficient condition for (2.41) to represent an acceptable solution is \(Δ_0 ≪ 0\), i.e., see Appendix,

\[ |y_0 - y_0| \ll y_c \ll y_a + y_b, \]  

\[ y_c^2 = (2n_i + 1)cβ^{-1}. \]  

(2.44)

(The case with the equal symbol, which yields \(y_0 = 0\), cannot be exactly reached with integer values of the \(n_i\).) Calculating the second derivative of the phase of \(X_a X_b X_c\) at the stationary point, it turns out that the width of the region where the three phases are locked is approximately given by

\[ |y - y_0| < c[\min[l_i]|π/2β|f]|^{1/2}, \]  

(2.45)

with \(l_i\) and \(f\) evaluated at \(y = y_0\). (This formula has to be modified if \(y_0\) is too close to the equator or to one of the turning latitudes of the component with smallest \(n\).)

In the usual midlatitudes β-plane approximation at the latitude \(y_0\), \(f\) is replaced by \(f(y_0) = \text{const}\) in (2.38), but keeping \(β \neq 0\): the expansion functions are then plane waves in a continuum of meridional wavenumbers. With this approximation: 1) only those triads whose three meridional wave numbers add to zero interact; and 2) the interaction region is virtually infinity instead of being given by (2.45). Both problems are avoided if the midlatitudes β-plane is restricted to a narrow enough zonal channel.

3. Resonant interactions

If the total energy is small, then the resonant triads (RT), i.e., those trios of components that satisfy (2.26) and

\[ k_a + k_b + k_c = 0, \]  

(3.1a)
\( \omega_a + \omega_b + \omega_c = 0, \) \hspace{1cm} (3.1b)

have an important role in the nonlinear term of (2.27). This condition can indeed be violated up to a detuning of \( O(E^{1/2}) \) and the triad still behave effectively as resonant (Bretherton, 1964. See also R81 for a quantitative result in the three-wave problem. This is used in R82, Section 3.3, for the case of \( K \) and short eastward propagating \( G \). Off-resonant interactions may also be important, even for low energies, if they are the only way to excite certain (isolated) resonant trios.

Here we study the kinematic (frequencies and wavenumbers) and dynamic (coupling coefficients) properties of all RT of equatorial waves with the same value of \( c \). Their relative importance on the nonlinear evolution of the system depends on each particular initial condition and is beyond the scope of this paper.

There are two immediate consequences of the interaction and resonance conditions:

1) Eq. (3.1a) can be rewritten as

\[
\omega_1 + \omega_2 + \omega_3 = 0, \tag{3.2}
\]

and thus, with (3.1b), it is

\[
\omega_a + \omega_b + \omega_c = 0, \tag{3.3}
\]

[The proportionality coefficient may be obtained from Eq. (4.16) below.] Consequently, the component of a resonant trio with largest absolute frequency has the intermediate slowness; it does not necessarily have the intermediate phase-speed \( \omega k^{-1} \) though. More explicitly, if

\[
\omega_a \omega_b \omega_c \gg 0, \tag{3.4}
\]

i.e., if

\[
|\omega_a| \gg |\omega_b|, |\omega_c| \Rightarrow \omega_a = \frac{[|\omega_b| \omega_3 + |\omega_c| \omega_2]}{|\omega_b| + |\omega_c|}. \tag{3.5}
\]

Henceforth, we number the components of any RT \((a, b, c)\) so that (3.4) and consequently (3.5) is satisfied.

2) Replacing (3.1b) in the sum rules (2.30), and comparing the latter with (3.1b)–(3.2) we find that the coupling coefficients associated with a resonant triad are proportional to the individual frequencies. More specifically,

\[
(\omega_a + \omega_b + \omega_c = 0) \Rightarrow 
\]

\[
\delta \omega_a \omega_b \omega_c^{-1} = \delta \omega_a \omega_b \omega_c^{-1} = \delta \omega_a \omega_c \omega_b^{-1} (= \gamma_{abc}), \tag{3.6}
\]

and consequently the dynamics of each RT is parameterized by a single coefficient \( \gamma \). This result can obviously be found directly from (2.33). However, it is interesting to see that it can be understood as a consequence of energy and pseudomomentum conservation, especially in view of the following two implications of 1) and 2):

i) Resonant interactions conserve the quadratic part of total energy and pseudomomentum; the changes of the \( O(Z^3) \) terms are due to interactions by off-resonant triads.

ii) The interaction among the members of a RT is such that the component with intermediate slowness gains energy from (or releases energy to) the other two; more explicitly,

\[
\delta E_a + \delta E_b + \delta E_c = 0, \\
\delta s_a \delta E_a + \delta s_b \delta E_b + \delta s_c \delta E_c = 0, \tag{3.7}
\]

where \( \delta \) denotes the change of energy due to interaction with the other two components. For the problems of QGF and IGW this statement is true for any triad (resonant or not); this follows from the fact that for those systems there is no \( O(Z^3) \) term in the total energy and pseudomomentum expansions (see R81).

\subsection*{a. Wave stability}

In the problems of QGF and IGW (and unlike the one of this paper), one single wave is an exact nonlinear solution; its linear stability analysis is a well posed problem for any value of the amplitude. All waves with \( \omega \neq 0 \) are unstable; \( E \) and \( P \) conservation require that a wave with slowness \( s_0 \) decay into components with values of \( s \) both larger and smaller than \( s_0 \). In particular, for QGF this means components with wavelengths larger and smaller than that of the unstable one (Gill, 1974); pseudomomentum is proportional to potential enstrophy in this case. If the amplitude of the wave is infinitesimal, the unstable decay is into pairs of waves that form a resonant triad with the former; the condition of cascade/decascade in \( s \) then translates into the unstable component having the largest of the three frequencies; [see (3.4) and (3.5), R81].

An equatorial wave [represented by \( Z_0(t) = Z_0(0) \times \exp(-i \omega_0 t), Z_0 = 0, \) for \( b \neq a \) is not an exact (nonlinear) solution of the model equations; its linear stability analysis has only a meaning in the weak interactions limit \( E^{(2)} \to 0 \). Any infinitesimal wave is unstable if it is the intermediate slowness (and thus, the maximum \(|\omega|\) component of a resonant trio. The growth rate \( \mu \) for the decay \( a \to b + c \) is given by

\[
\mu^2 = \sigma_a \sigma_b \sigma_c E^{(2)} \tag{3.8}
\]

or, using (3.6), \( \mu^2 = (\gamma_{abc})^2 \omega_a \omega_b \omega_c E^{(2)} \leq (\gamma_{abc})^2 \omega_d^2 E^{(2)}/4 \).

Given the separation of \( R, G \) and \( K \) modes in slowness space (Table 1), it follows that the only possible decays in a one-layer model are of the types:

\[
\begin{align*}
R &\to R + R, \\
R &\to R + K, \\
G &\to R + G, \\
G &\to G + K, \\
G &\to G + G, \\
G &\to R + K
\end{align*}
\]
[including the $M$ waves among the $G$ for $sc > -1$ \((\omega^2 > \frac{1}{2} \sigma \omega_c)\), or among the $R$ for $sc < -1$ \((\omega^2 < \frac{1}{2} \sigma \omega_c)\).

Duffy (1974) studied the GGR resonant triads at midlatitudes (where $G$ and $R$ modes are even more separated in $s$-space) and concluded the possibility of the decay $R \rightarrow G + G$, because he found positive values of $\sigma^2 \sigma_a^2$ in (3.8). Clearly, the coupling coefficients evaluated by Duffy (1974) are incorrect, because that decay would require the slowness of the Rossby component to be larger than the slowness of one of the inertia–gravity modes [or, equivalently, \(|\omega(R)| > |\omega(G)|\)], which is not possible (see Table 1). [The factorization (3.6) is a theorem, applicable to midlatitudes as well.]

b. System with only resonant interactions

We now study some dynamical properties of systems in which off-resonant interactions are neglected; the validity of this assumption which depends on each particular problem is not discussed here. In order to fix ideas, let us separate linear and nonlinear evolution by defining new amplitudes $X_a(t)$ in the form

$$Z_a(t) = X_a(t) \exp(-i\omega_a t).$$

(3.9)

In the linearized problem the $X_a$ are constant. The evolution equation for the $X_a$ is found substituting (3.9) in (2.27) and using the factorization (3.6) of the (resonant) coupling coefficients, viz.,

$$\frac{dX_a}{dt} = i\omega_a \sum_{bc} \gamma_{abc} X_b^* X_c^*,$$

(3.10)

where the rhs is restricted to certain resonant trios [i.e., that satisfy (3.1)]. In (3.10), unlike in (2.27), the $X_a$ must belong to a denumerable set (Breherton, 1964). The results of this subsection are not restricted to equatorial waves: they only depend on the factorization (3.6) of the coupling coefficients, which is true in any system with at least one homogeneous coordinate.

Examples of systems like in (3.10) are presented in the companion paper. Loesch and Deininger (1979) have studied systems of O(10–100) equatorial waves which are resonantly coupled to a certain mode [condition (3.1a) was somewhat relaxed and, for some reason, Kelvin-harmonic chains were not included]. Meiss (1979) discusses a test-wave model with $2M + 1$ modes, say numbered \([0, I, I', 2, 2', \ldots, M, M']\), in which all interactions coefficients $\gamma$ are set to zero with the exception of the triads type \((0, i, j)\) where the test-wave 0 appears. The nonlinear Kelvin wave problem (Boyd, 1980a) with periodic initial condition may be cast in the form (3.10) where all non-vanishing $\gamma$ have the same value (see R82). Further reference to these works is made below. The $N$-wave problem has also been used in other fields of physics (see, for instance, Armstrong et al., 1962 for the case of nonlinear optics).

First of all, notice that if $X_a(t)$ is a solution of (3.10), so is $\alpha X_a(\alpha t)$ with $\alpha$ any real constant. Mathematically, this reflects the fact that (3.10) is an asymptotic approximation of the complete evolution equation (2.27) in the limit $E^{(2)} \rightarrow 0$; practically, this means that either the nonlinear time scale or the value of the total energy are unimportant; and physically, this hints at the limitations and danger of only taking resonant interactions. [Incidentally, a similar invariance, $Z_a(t) \rightarrow \alpha Z_a(\alpha t)$, is valid in the strong interactions limit, $E^{(2)} \rightarrow \infty$, which is modeled by (2.27) without the term $i\omega_a Z_a$. This is the limit used in classical turbulence theory.]

Let us consider now the conservation laws of (3.10). They need not be the same as those of the original system, (2.1), due to the approximation of considering only resonant interactions [we have already shown that the error in total energy and pseudomomentum conservation related to an arbitrary truncation of the expansion basis is, at most, $O(Z^4)$]. The system (3.10) may be derived from Hamilton's principle, with a Lagrangian

$$L = -\frac{1}{2} \sum_a \omega_a^{-1} X_a^* \dot{X}_a + \frac{1}{\hbar} \sum_{abc} \gamma_{abc} X_a X_b X_c,$$

(3.11)

where the prime indicates summation over physically different states [i.e., only one of each conjugate pair $(a, a^*)$]. The integrals of motion can be obtained from the symmetries of $L$, using Noether's theorem (Noether, 1918; for a modern review see Sarlet and Cantrijn, 1981). Conditions (3.1a, b) imply that the Lagrangian is invariant under the transformations

$$X_a \rightarrow \exp(-i\omega_a \delta t) X_a,$$

$$X_a \rightarrow \exp(i k_a \delta x) X_a, \quad (\delta t, \delta x: \text{constant}),$$

(3.12)

which yields quadratic energy and pseudomomentum conservation, namely,

$$E^{(2)} = \sum_a |X_a|^2 = \text{constant},$$

(3.13a)

$$P^{(2)} = \sum_a \gamma_{abc} X_a X_b X_c = \text{constant}.$$}

(3.13b)

Moreover, $\partial_t L = 0$ results in the conservation of the value of the Hamiltonian

$$H = \frac{1}{\hbar} \sum_{abc} \gamma_{abc} \text{Im}(X_a X_b X_c) = \text{constant}.$$}

(3.14)

Notice that the Hamiltonian does not coincide with the cubic energy $E^{(3)}$ in (2.22a). Thus the system (3.10) has at least three integrals of motion, which may be shown to be in involution. Since the number of degrees of freedom is equal to the number of physically independent modes, the three-wave problem is integrable, but nothing can be said a priori on the integrability of a system with four or more components. Indeed the three-wave prob-
lem is integrable also in the off-resonant case and even with arbitrary coupling coefficients. For its general solution, from any initial condition, see R81.) If all modes in (3.10) have the same value of $s_a$, then conservation of quadratic energy and pseudomomentum, (3.13a, b), are not independent laws; this is the case of resonance among harmonics studied in the companion paper.

Meiss (1979) has shown that for the test-wave model, with only the coefficients $\gamma(0, j, j') \neq 0$, the difference in the actions $E/\omega$ of components $j$ and $j'$ are conserved. There are then $M + 2$ integrals of motion (there is a linear relation between $E^{(2)}$, $p^{(2)}$ and the $M$ action differences) for $2M + 1$ degrees of freedom: again only the integrability of the three-wave problem ($M = 1$) is assured. However, Meiss (1979) have performed numerical integrations of cases with $M = 2$, 3, and 4, which suggest that the test-wave model is integrable. Interestingly enough, when coefficients of triads where the test-wave is not present are set different from zero (which is a more realistic situation), the evolution becomes chaotic and thus nonintegrable. More explicitly, Meiss (1981) found this by making $\gamma(I^*, I', 2) \neq 0$ and $\gamma(I', I, 2^*) \neq 0$ in an $M = 2$ case.

The numerical solutions of many-wave problems presented by Loesch and Deininger (1979) look periodic, and not chaotic. It is not clear how much this result depends on the initial conditions chosen and on the length of the elapsed time.

Let us finish this section by discussing the validity of taking discrete modes. Consider a RT close to the $(a, b, c)$, viz., with the same $n_i$ and with wavenumbers $k_i + \delta k_i$, $(i = a, b, c)$. In order for the new trio to satisfy (3.1) it must be

$$\delta k_a; \delta k_b; \delta k_c : C_b - C_c; C_c - C_a; C_a - C_c,$$  

(3.15)

where

$$C = do/dk_c,$$  

(3.16)

the zonal group velocity, can be evaluated as

$$C = \left\{ \begin{array}{ll} c[1 + sc(4n + 2 + sc)] \\ [4n + 2 + sc(3 - s^2 c^2)] & \text{for } n > 0 \\ \frac{c}{2 - sc}, & \text{for } n = 0 \\ c, & \text{for } n = -1 \end{array} \right. \quad (3.17)$$

Now, if two of the group velocities coincide, say $C_a = C_c (\neq C_b)$, then (3.15) implies $\delta k_a = 0$, i.e., wave $a$ effectively resonates with a set of modes, with wave numbers $k_a + \delta k$ and $k_c - \delta k$, because the detuning is $O(\delta k)^2$; in such a case, (3.10) is not valid, and wave-packet interactions should probably be used.

4. Taxonomy of resonant triads

Here we consider the general solution of the interaction and resonance conditions: (2.26), (3.1a), and (3.1b). A knowledge of all possible resonant triads is helpful in both the formulation of approximations to the problem and in interpreting results of calculations with many interacting modes. For instance, McComas and Bretherton (1977) found that the energy transfer among oceanic internal gravity waves is dominated, under the assumption of weak interactions, by three types of non-local resonant triads (i.e., interactions between modes well-separated in frequency and/or wavenumber). We showed in R81 that those three are the only classes of non-local RT of IGW; two of them are also present in the system of QGF.

For the systems of QGF and IGW it is found (R81) that $\sigma_a^{bc}$ vanishes for any triad $(a, b, c)$, resonant or not, such that $s_b = s_c$ [this can be derived from the lack of the $O(Z^0)$ terms in the equivalent of (2.10) for those problems]. As a consequence, any combination of waves with the same speed and arbitrary amplitudes (in particular, one single wave) is proved to be an exact nonlinear solution. This is not true, however, for the system of this paper.

In this section we study: non-local triads; resonant triads of waves with the same speed; and the general solution of the interaction and resonance conditions.

a. Non-local resonant triads

If the frequency of one of the components of a RT is much smaller (in magnitude) than those of the other two, say

$$|\omega_a| \ll |\omega_b| \sim |\omega_c|,$$  

(4.1)

then (3.3) implies that, either $s_c$ is finite and thus it is $s_b = s_c$, or $|s_b| \rightarrow \infty$.

For $s_b = s_c$ it is $\omega_c = 0$ and $k_c = s_c \omega_c = 0$ ($s_c \sim -2n_a - 1$); $c$ is a long Rossby or Kelvin mode (see Table 2). In addition, if $n_a = n_b$ then $k_b = -k_a$ is finite; conditions (3.1) express, in this limit, that $a$ and $b$ are components of a wave packet with group velocity equal to the phase speed of mode $c$ [Eq. (4.2a) below; $n_a, n_b \neq 1$]. If $n_a \neq n_b$, on the other hand, then $|\omega_a| \approx |\omega_b|$ is attained only in the limit $|k_a| \rightarrow \infty$ [Eq. (4.2b); $n_c \neq 0$].

For $s_c \rightarrow -\infty$ [$c$ is a short Rossby or mixed Rossby-gravity mode; see Table 2] it is $|k_c| \rightarrow \infty$ and thus $|\omega_b|, |k_c| \rightarrow \infty$ also. The three components cannot be short Rossby waves because if they were, $a$ should have the largest value of $|k|$ (the $s_i$ have all the same sign) and thus it could not have the intermediate value of $s$ [see (3.4) and (3.5)] since in this limit it is $s_i \sim -b^{-1}k_i^2$. Consequently, it has to be $s_a, s_b \rightarrow -1$ (a is short westward propagating $G$) and $s_c \rightarrow 1$ (a is short eastward propagating $G$, $M$, or $K$). See Table 2. Finally, since $\omega_a \sim -\omega_b$ and $s_a \sim s_b$ it is $k_a \sim k_b \sim -s_c k_c$ [Eq. (4.2c); $n_c \neq 0$].

Therefore, non-local RT fall, for finite values of the $n_i$ in one of the following classes:
\[ |s_3| < \infty, \quad s_a = s_b, \]
\[ k_c = \omega_c = 0 \]
\[ n_a = n_b; \quad C_o = \omega_c/k_c, \quad (4.2a) \]
\[ n_a \neq n_b; \quad |k_o| \rightarrow \infty, \quad (4.2b) \]
\[ s_c \rightarrow -\infty, \quad s_a \neq s_b; \quad (4.2c) \]

Notice that in cases (4.2a) and (4.2b) the triads are also non-local in wavenumber space. It can be shown (using the formulas of Section 2) that the interaction coefficients of non-local triads are finite for these cases; for case (4.2c) it is found that \( \gamma \rightarrow 0 \) in such a way that \( \gamma \omega_a \) is finite.

Summarizing, the types of non-local phenomena are: 1) \( G \) or \( R \) wave packets traveling in a constant phase line of a long \( R \), just like groups of capillary waves ride on the crests of long gravity waves [see also R81, Eq. (4.4b), for the problems of QGF and IGW; there is a typographical error in that equation: \( C_o \) should read \( C_d \)]; 2) Interactions between short \( G \) or \( R \) with different meridional quantum numbers and a long \( R \) with a much longer period; 3) elastic scattering of a short westward propagating \( G \) into a short eastward propagating \( G, M \) or \( K \) (or vice versa) by a short \( R \) with twice the wavenumber.

We use the name elastic scattering in the last case in analogy with the classification of McComas and Bretherton (1977) for IGW. These triads could indeed be responsible for the Bragg scattering of a short gravity wave into a similar wave, with equal wavelength and meridional structure but opposite propagation direction, due to the interaction with a shorter Rossby wave, which is essentially unaffected by the process. It is interesting to note that short gravity waves suffer a similar change, only the sign of the speed, by reflection in a meridional wall (Clarke, 1983).

Case 3) represents the centroid of a degenerate case, because waves \( a \) and \( b \) are the same (and thus \( C_a = C_b \)) and, furthermore, it is \( C_c = C_a \), which implies that for any interacting trio in the neighborhood of \( (a, b, c) \) the detuning is \( O(\delta k_i)^2 \). This completely modifies the behavior of the system; e.g., a plane-wave solution of the three-mode problem is found to be always unstable to side-band perturbations due to this degeneracy (Ma and Redekopp, 1979).

The non-local triads in (4.2) are limiting cases of the local ones (which have comparable frequencies and wavenumbers). In addition, there are the trivial solutions of (2.6) and (3.1) given by
\[ n_c = \text{odd}, \quad k_c = \omega_c = 0, \]
\[ n_a = n_b, \quad \text{and} \quad s_a = s_b = \text{anything}, \quad (4.3) \]
that enter in the elementary interaction between a wave and a geostrophic steady current.

b. Triads of waves with the same speed

Let us search for solutions of (2.6) and (3.1), such that
\[ s_a = s_b = s_c = s, \quad \text{(4.4)} \]
and \( \gamma \neq 0 \). [For a RT, if \( s_a = s_b = s \) then \( s_c = s \) also, unless \( \omega_c = k_c = 0 \), as in (4.3).] Recall that \( E^{(2)} \) and \( P^{(2)} \) conservation do not impose independent constraints for the evolution of an isolated resonant trio (or larger set) of waves with the same speed.

One example of the implications that the existence of such triads may have is provided by the constant energies solution of the three-wave problem [R81, Eq. (5.21) and (5.22)]: If only one triad is used in (3.10), there is a solution such that
\[ |Z_a| = |Z_b| = |Z_c| = \text{constant}, \]
\[ \text{Re}(Z_aZ_bZ_c) = 0. \quad (4.5) \]
The three frequencies are shifted to
\[ \omega_i = (1 \pm \gamma|Z_a|)\omega_i, \quad (i = a, b, c), \quad (4.6) \]
where the \( \pm \) sign is that of \( \text{Im}(Z_aZ_bZ_c) \). Since the three components have the same speed and their frequencies suffer the same relative change, this (approximate) solution represents fields that propagate without changing shape and that are not periodic in space or time (because the \( k_i \) are incommensurable, and so are the \( \omega_i \)).

For solution of the resonance condition: given three non-negative \( n_i \) that satisfy (2.26) [\( K \) modes are excluded because (4.4) implies \( s_c = 1 \), i.e., there is only the trivial solution \( n_a = n_b = n_c = -1 \)], then the solution of (3.1) is given by that of [see Table 1 and (A2)] \( \Delta(2n_i + 1 + sc) = 0; \) (3.1b) is an immediate consequence of (3.1a) and (4.4). This equation is quadratic in \( sc \) [see (A3)] with solutions
\[ sc = -2\langle n \rangle - 1 \pm [(2\langle n \rangle + 1)^2 - \delta \Delta_0]^{1/2}, \quad (4.7) \]
where \( \langle n \rangle \) and \( \Delta_0 \) are given in (2.42).

It is easy to see that the negative solution is always acceptable \((sc \leq -2n_i - 1; \) the equal sign happens only if two of the \( n_i \) are equal) and corresponds to a trio of Rossby components. Condition (3.4) is satisfied by
\[ n_a \leq n_b, \quad n_c. \quad (4.8) \]

For the positive solution to be valid \((-1 < sc < 1) \) the function \( \Delta(2n_i + 1 + sc) \), which is a parabola in \( sc \) [see (A3)], must be negative at \( sc = -1 \) and positive at \( sc = 1 \), i.e., the conditions
\[ n_a^{1/2} > n_b^{1/2} + n_c^{1/2}, \]
\[ (n_a + 1)^{1/2} < (n_b + 1)^{1/2} + (n_c + 1)^{1/2}, \quad (4.9a, b) \]
must be satisfied, in which case it corresponds to a trio of inertia-gravity modes.

Several examples of RT of waves with the same speed are presented in Table 3; unlike the systems studied in R81, the interaction coefficient \( \gamma \) is different from zero.
Table 3. Examples of resonant triads of waves with the same phase-speed. The first (last) seven cases are composed of Rossby (gravity) modes. The interaction coefficient $\gamma$ is calculated for the ocean model: $\lambda = 1$ in (2.2).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$\omega R/c$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$sc$</th>
<th>$\gamma c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>-2.99</td>
<td>2.19</td>
<td>0.80</td>
<td>0.56</td>
<td>-0.41</td>
<td>-0.15</td>
<td>-5.31</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>3</td>
<td>-3.73</td>
<td>2.41</td>
<td>1.32</td>
<td>0.48</td>
<td>-0.31</td>
<td>-0.17</td>
<td>-7.86</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>4</td>
<td>-4.06</td>
<td>3.53</td>
<td>0.53</td>
<td>0.44</td>
<td>-0.39</td>
<td>-0.06</td>
<td>-9.14</td>
</tr>
<tr>
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<td>2</td>
<td>4</td>
<td>-3.54</td>
<td>2.91</td>
<td>0.63</td>
<td>0.39</td>
<td>-0.32</td>
<td>-0.07</td>
<td>-9.19</td>
</tr>
<tr>
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<td>-4.37</td>
<td>2.65</td>
<td>1.72</td>
<td>0.42</td>
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<td>-0.16</td>
<td>-10.47</td>
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<td>0.98</td>
<td>0.40</td>
<td>-0.31</td>
<td>-0.09</td>
<td>-11.48</td>
</tr>
<tr>
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<td>3</td>
<td>5</td>
<td>-4.17</td>
<td>3.05</td>
<td>1.12</td>
<td>0.36</td>
<td>-0.26</td>
<td>-0.10</td>
<td>-11.62</td>
</tr>
<tr>
<td>2</td>
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<td>2.05</td>
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<td>4.06</td>
<td>-2.97</td>
<td>-1.09</td>
<td>-0.69</td>
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<td>3</td>
<td>2</td>
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<td>-4.79</td>
<td>3.94</td>
<td>0.85</td>
<td>5.94</td>
<td>-4.89</td>
<td>-1.05</td>
<td>-0.81</td>
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<tr>
<td>4</td>
<td>1</td>
<td>0</td>
<td>2.34</td>
<td>-1.42</td>
<td>-0.92</td>
<td>4.94</td>
<td>-2.99</td>
<td>-1.95</td>
<td>0.47</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>0</td>
<td>-6.78</td>
<td>5.89</td>
<td>0.89</td>
<td>7.89</td>
<td>-6.85</td>
<td>-1.04</td>
<td>-0.86</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>0</td>
<td>0.69</td>
<td>-0.47</td>
<td>-0.22</td>
<td>4.77</td>
<td>-3.24</td>
<td>-1.53</td>
<td>0.15</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>0</td>
<td>-8.77</td>
<td>7.86</td>
<td>0.92</td>
<td>9.86</td>
<td>-8.83</td>
<td>-1.03</td>
<td>-0.89</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>1</td>
<td>-9.60</td>
<td>5.59</td>
<td>4.01</td>
<td>10.79</td>
<td>-6.29</td>
<td>-4.50</td>
<td>-0.89</td>
</tr>
</tbody>
</table>

c. General solution of the resonance condition

The solutions of (3.1), for fixed values of the $n_j$, $r_j$, and $k_a$, are the roots of a transcendental function of $k_b$. The solutions can be found, for instance, by Newton's method iterating $k_b \rightarrow k_b - (\omega_a + \omega_b + \omega_c)/(C_a - C_b)$; however, it is difficult to predict the number and types of roots in this way. It is convenient, therefore, to reduce (3.1), for fixed values of the $n_j$, to an expression of the form $F(s_a, s_b) = 0$ (since $\omega^2$ is a simpler function of $s$ and $n$).

Let us consider first the case of $R$ and $G$ modes ($n_j \geq 1$). Defining

$$N = 2n + 1 + S, \quad D = 1 - S^2, \quad S = sc, \quad (4.10)$$

the frequency is given by

$$\omega^2 = \beta c ND^{-1}, \quad n > 0. \quad (4.11)$$

(see Table 1). We now: 1) write the last equation as $\omega^2 - \omega^2 S^2 = \beta c N^2$ for component c; 2) replace $\omega = -\omega_a - \omega_b$ and $\omega, S = -\omega_a S_a - \omega_b S_b$ in it; and 3) use the first equation for components $a$ and $b$, to obtain

$$2(1 - S_a S_b)\omega_a \omega_b = \beta c (N_c - N_a - N_b). \quad (4.12)$$

Substituting $S_c = (S_a \omega_a + S_b \omega_b)/(\omega_a + \omega_b)$ in $N_c$ and multiplying by $D_a D_b(\omega_a + \omega_b)$, we get

$$\omega_a D_a B_{ba} = -\omega_b D_b B_{ab}, \quad (4.13)$$

where

$$B_{ba} = 2(n_a - n_b)D_b + (3 - 2S_a S_b - S_b^2)N_b, \quad (4.14)$$

and similarly for $B_{ab}$. Finally, squaring (4.13) and using (4.11) we obtain

$$F(s_a, s_b) = N_a D_a B_{ba}^2 - N_b D_b B_{ab}^2 = 0. \quad (4.15)$$

This is polynomial of sixth degree in $s_b$ which has, for a fixed value of $s_a$, at most six real roots if $a$ is a $G$-mode ($N_a > 0, D_a > 0$) or four real roots if $a$ is a $R$-mode ($N_a \leq 0, D_a < 0$). If $n_a = n_b$, then the trivial solution $s_a = s_b$, Eq. (4.3), can be factored out from (4.15).

A useful relation is obtained squaring (4.12) and using (4.11),

$$\Delta(N_i) = -4\omega_a^2 \omega_b^2 (s_a - s_b)^2 \beta^2, \quad (4.16)$$

or the same with any permutation of $a$, $b$, and $c$.

There are three groups of RT for $n_i > 0$, namely, $RRR$, $GGR$ and $GGG$; there are no $GRR$ RT because the $G$ component, that has the largest value of $|\omega|$, should have the intermediate value of $s$, [cf. (3.4)–(3.5)], which is impossible (see Table 1). The three groups are analyzed separately in the following subsections. Recall that we number the components so that $a$ has the intermediate value of $s$ and, therefore, the maximum value of $|\omega|$ (unless stated otherwise). The extreme cases are summarized in Table 4.

1) TRIADS OF ROSSBY MODES

For $R$ modes, the slowness has always the same sign (negative) and it is a decreasing function of both $k^2$ and $n$. Therefore component $a$, that has the maximum value of $|\omega|$ (namely $\omega_{a|\omega_a} \geq 0$), must also have the maximum $|k|$ (since $k_a k_c \geq 0$) and also

$$n_d < \max(n_b, n_c) \quad (4.17)$$

[because $s_a \leq \max(s_b, s_c)$]. Choosing, without any loss of generality, $n_b \leq n_c$, it follows that there are four types of $RRR$ resonant triads (Numbers 1–4 in Table 4), classified according to the values of the three $n_j$.

If $n_a = n_b = n_c$, then the solution is the trivial one (4.3).

Examples of the four types of $RRR$ resonant triads are presented in Fig. 2. $\gamma$, $s_a$ and $s_c$ are shown as functions of $s_b$ (for fixed values of the $n_j$). The states are labeled by $-1/sc$, instead of by the slowness, in order to be able to plot the whole curve. [Recall that
Table 4. Kinematic properties of all resonant trios of dispersive waves. The first component has the intermediate slowness (and thus, the maximum absolute frequency). Superscripts (a), (b), and (c) refer to the three types of non-local triads in (4.2), (4.3) is \( \omega_i \) as given by (3.3), (4.7), \( s \) given by (4.7) and (4.8) applies if (4.9b) is satisfied.

<table>
<thead>
<tr>
<th>No.</th>
<th>Triad</th>
<th>Limits ((s_a, s_b, s_c)^c)</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>RRR</td>
<td>((-\infty, -2n_b - 1, -\infty)^c) to ((-2n_a - 1, -2n_b - 1, -2n_c - 1)^c)</td>
<td>(n_b &lt; n_a &lt; n_c)</td>
</tr>
<tr>
<td>2</td>
<td>MRR</td>
<td>((-\infty, -2n_b - 1, -\infty)^c) to ((-2n_a - 1, -2n_b - 1, -2n_c - 1)^c)</td>
<td>(n_b &lt; n_a &lt; n_c)</td>
</tr>
<tr>
<td>3</td>
<td>GMR</td>
<td>((-\infty, -2n_b - 1, -\infty)^c) to ((-2n_a - 1, -2n_b - 1, -2n_c - 1)^c)</td>
<td>(n_a &lt; n_b &lt; n_c)</td>
</tr>
<tr>
<td>4</td>
<td>GGR</td>
<td>((-\infty, -2n_b - 1, -\infty)^c) to ((-2n_a - 1, -2n_b - 1, -2n_c - 1)^c)</td>
<td>(n_a &lt; n_b &lt; n_c)</td>
</tr>
<tr>
<td>5</td>
<td>GGM</td>
<td>((-\infty, -2n_b - 1, -\infty)^c) to ((-2n_a - 1, -2n_b - 1, -2n_c - 1)^c)</td>
<td>(n_a &lt; n_b &lt; n_c)</td>
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<td>6</td>
<td>GMM</td>
<td>((-\infty, -2n_b - 1, -\infty)^c) to ((-2n_a - 1, -2n_b - 1, -2n_c - 1)^c)</td>
<td>(n_a &lt; n_b &lt; n_c)</td>
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<td>7</td>
<td>GGM</td>
<td>((-\infty, -2n_b - 1, -\infty)^c) to ((-2n_a - 1, -2n_b - 1, -2n_c - 1)^c)</td>
<td>(n_a &lt; n_b &lt; n_c)</td>
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<td>8</td>
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<td>(n_a &lt; n_b &lt; n_c)</td>
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<td>9</td>
<td>GGM</td>
<td>((-\infty, -2n_b - 1, -\infty)^c) to ((-2n_a - 1, -2n_b - 1, -2n_c - 1)^c)</td>
<td>(n_a &lt; n_b &lt; n_c)</td>
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<td>GGM</td>
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<td>11</td>
<td>GGM</td>
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<td>(n_a &lt; n_b &lt; n_c)</td>
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<tr>
<td>12</td>
<td>GGM</td>
<td>((-\infty, -2n_b - 1, -\infty)^c) to ((-2n_a - 1, -2n_b - 1, -2n_c - 1)^c)</td>
<td>(n_a &lt; n_b &lt; n_c)</td>
</tr>
<tr>
<td>13</td>
<td>GGM</td>
<td>((-\infty, -2n_b - 1, -\infty)^c) to ((-2n_a - 1, -2n_b - 1, -2n_c - 1)^c)</td>
<td>(n_a &lt; n_b &lt; n_c)</td>
</tr>
<tr>
<td>14</td>
<td>GGM</td>
<td>((-\infty, -2n_b - 1, -\infty)^c) to ((-2n_a - 1, -2n_b - 1, -2n_c - 1)^c)</td>
<td>(n_a &lt; n_b &lt; n_c)</td>
</tr>
</tbody>
</table>

\(-1/(sc) = 0+ \) means \(|k|R \to \infty, and \(-1/(sc) = 1/(2n + 1)\) corresponds to \(k = 0\). See Table 2.] As \(|k|R \to \infty\), it is found that \(|\gamma| \to \infty\), but \(\beta a^2\) remains finite; more specifically, it is

\[
\gamma = O(s_a),
\]

(4.18)

for the whole curve. The same result holds for the systems of non-divergent QGF and IGW [cf., R81, Eq. (4.5)].

In one of the extremes of the curves in Fig. 2 (and for the four cases) \(a\) and \(c\) are short waves and \(b\) is a long one [cf., (4.2b) with \(b \leftrightarrow c\); the other extreme represents the following:

**Case 1** \((n_b < n_a < n_c)\). Three long waves; their frequencies are in the proportion \(\omega_a:\omega_b:\omega_c = n_b - n_c : n_c - n_a : n_a - n_c\) [cf., (3.3)] with \(s_c \sim -2n_c - 1\). This is the only RT of three long-R which is a limiting case of RT waves with finite wavelength. The self-interaction of a long-R \([n_b = n_c = n_c = o]d, \ |k|R \ll (2n + 1)^{1/2}\) studied by Boyd (1980b), is strictly resonant in the trivial case (4.3).

**Case 2** \((n_b = n_a < n_c)\). Here \(a\) and \(b\) are (physically) the same component and have a group velocity equal to the phase speed of \(c\), which is a long wave [cf., (4.2a)]. For high values of the \(n_i\), the (local) meridional wavenumbers of the three components are related, in the interaction region, as in (2.43b) \(l_a = l_b = -l_c\) (i.e., it corresponds to the elastic scattering triads of R81).

**Case 3** \((n_a < n_b = n_c)\). The \(b\) and \(c\) are physically the same component and \(a\) is the harmonic. Eq. (4.7) yields for this limit

\[
|s_a = s_b = (2n_a - 8n_b - 3)(3c)^{-1}.
\]

(4.19)

For high values of the \(n_i\), the (local) meridional wavenumbers of the three components are related, in the interaction region, as in (2.43a)

\[
l_a = 0, \quad l_b = l_c.
\]

**Case 4** \((n_a < n_b \neq n_c)\). The \(a\) and \(b\) are short waves and \(c\) is a long one [cf., (4.2b)]. The intersection of the curves of \(s_a\) and \(s_c\) corresponds to the triad of waves with the same speed, Eq. (4.7) with the negative sign. There is a maximum value of \(s_a\) (or a minimum of \(|k_a|\), reached at the point where [cf., Eq. (3.15)]

\[
C_b = C_c.
\]

(4.20)

2) ROBBY AND INERTIA-GRAVITY MODES

The unstable component (maximum \(|\omega|\)) of a RT made up of two G-modes and a R-wave is the G-component with smaller slowness (not necessarily the one with larger phase speed, though), because \(s(G) > s(R)\). There are three types of GGR-RT (Numbers 7, 8, and 9 in Table 4), classified by the relative values of \(n_a\) and \(n_b\). A first approximation to this solution of (4.15) may be obtained making \(\omega_a + \omega_b = 0\); this yields \(s_a(s_a)\) as the root of a quadratic equation (cf., Table 1). [This simplification is more accurate for large values of the \(n_j\), i.e., in the midlatitudes case studied by Duffy (1974).] It follows from this approximation that the G-component with smaller value of \(n\) (if \(n_a \neq n_b\)) cannot change the sign of \(s\) (i.e., the \(k = 0\) point is not included, for that component).

Examples of the three types of GGR RT are presented in Fig. 3; \(s_c, s_c\), and \(-1/(sc)\) are shown as functions of \(s_c\) (for fixed values of the \(n_i\)). [Recall that \(sc \sim -1 + 0\) (1 - 0) represents short westward (eastward) propagating \(G\), and \(sc \sim 0\) corresponds to a long \(G\).] The interaction coefficient for these triads is found to be

\[
\gamma = O(c^{-1}).\]

(4.21)

In one extreme of the curves in Fig. 3 (and for the three types) \(a\) and \(b\) are short G, propagating in opposite directions and with the same wavelength, and \(c\) is a short \(R\) with half their length [cf., (4.3c)]. In this limit it is \(\gamma \to 0\), but \(\gamma a\) (and \(\gamma \omega_q\)) is finite. The other extreme corresponds, if \(n_a > n_b\) (n_a < n_b), to
two eastward (westward) propagating short $G$ interacting with a long $R$; this is type 7 (9) in Table 4 and the limit corresponds to Eq. (4.3b). If $n_a = n_b$ (type 8 in Table 4), however, the curves finish with the triad in which $a$ and $b$ are physically the same component (a westward propagating $G$), and have a group velocity equal to the phase speed of $c$, which is a long $R$ wave [cf., (4.2a)].

3) INERTIA-GRAVITY MODES
The third major group of RT is that of those made up of three $G$ components. Since $|S| < 1$ for $G$ modes,
is the necessary condition for the existence of RT in the limit of short eastward propagating $G$ [$N_f \rightarrow 2(n_f + 1)$]. Clearly, (4.23) is more restrictive than (4.22). Therefore, (4.22) is the necessary condition for the existence of GGG resonant triads, given the three $n_f$. Resonant GGG triads extend from the $s_{c}c = -1$ limit to the $s_{c}c = 1$ limit if (4.23) is also satisfied, or to a maximum value of $s_{b}$ (corresponding to condition (4.20)) if it is not, i.e., in the case of Eq. (4.9b). There are two different branches, corresponding to $s_{b} > s_{c}$ and $s_{b} < s_{c}$, unless $n_{b} = n_{c}$, in which case there is only one branch. One example of this type of RT ($n_f = 5, 1, 1$) is presented in Fig. 4; notice that the interaction coefficient is given as in (4.23).

Unlike for the other two classes, RRR and GGR, there is no midlatitudes counterpart of equatorial GGG RT because of the following reason: The dispersion relation of $G$ waves at midlatitudes is approximately given by

$$\omega^2 = f^2 + k^2 c^2, \quad k^2 = k^2 + l^2 \quad (f\text{-plane}), \quad (4.24)$$

where $l$ is the meridional wavenumber. The interaction condition is given by $k_a + k_b + k_c = 0$, which implies (see Appendix) $\Delta(k^2) > 0$. It then follows from (4.24) and (A3) that

$$\Delta(\omega_f^2) = \Delta(k^2) + 2k^2 c^2(k_a^2 + k_b^2 + k_c^2) + 3l^4 \neq 0,$$

i.e., the resonance condition, (3.1b), cannot be satisfied.

Nevertheless, there are GGG RT of equatorial waves for arbitrarily large values of the $n_f$, which thus extend arbitrarily away from the equator. In order to explain this apparent paradox let us consider the limits of short $G$ waves ($s_{c}c \sim -1$ or $+1$): The relevant

both sides of (4.12) with $a \leftrightarrow c$ are positive ($N_a > N_b + N_c$); moreover, it must be $N_f^{1/2} > N_b^{1/2} + N_c^{1/2}$ because (4.16) implies $\Delta(N_f) < 0$. A necessary condition for the existence of RT in the limit of short westward propagating $G$ ($N_f \rightarrow 2n_f$) is

$$n_{b}^{1/2} > n_{c}^{1/2} + n_{c}^{1/2}. \quad (4.22)$$

Similarly,

$$(n_{a} + 1)^{1/2} > (n_{b} + 1)^{1/2} + (n_{c} + 1)^{1/2} \quad (4.23)$$

FIG. 3. As in Fig. 2, except for one Rossby ($c$) and two inertia–gravity ($a$ and $b$) waves. The limits $s_{c}c \rightarrow -1$ ($s_{c}c \rightarrow 1$) correspond to short westward (eastward) propagating gravity waves, whereas $s_{c}c \sim 0$ represents long gravity waves (see Table 2).

FIG. 4. As in Figs. 2 and 3, except for three inertia–gravity waves.
structure functions in these limits have a WKBJ structure given by \( v_j \sim 0 \) and

\[
\varphi_{ij} \sim \pm u_j \sim l_j^{-1/2} \cos \int l_j dy ,
\]

\[
\beta y_c c^{-1} < 2(n_j + 1) + 1 , \quad n_j \gg 1,
\]

where \( l_j(y) \) is the local meridional wavenumber, defined by

\[
l_j^2 = [2(n_j + 1)] \beta c - 1 - f^2 c^2.
\]

It follows from the analysis of Section 2e that the interaction coefficient will be negligible unless \( \Delta [2(n_j + 1) + 1] \geq 0 \) (see also Ripa, 1983b).

For westward propagating waves (\( s_c \sim -1 \)) the necessary condition for resonance, (4.22) or \( \Delta (n_j) < 0 \), implies \( \Delta (2n_j - 1) < 0 \), and thus the interaction coefficient is negligible small for large \( n_j \) [In fact, even for moderate \( n_j \), \( \gamma \) takes its smallest value at this limit; see for instance Fig. 4.]

For eastward propagating waves (\( s_c \sim 1 \)), though, the necessary condition for resonance, (4.23) or \( \Delta (n_j + 1) < 0 \), and of finite interaction coefficient, \( \Delta (2n_j + 3) \geq 0 \), may both be satisfied with arbitrarily large \( n_j \). However, since the latitudes of stationary phase are solutions of \( \Delta l_j = \Delta (2n_j + 3 - \beta y_c c^{-1}) = 0 \), they are within one deformation radius from the equator (\( y_c^0 < 2 \beta y_c c^{-1} \)). The \( \beta \) correction to the dispersion relation [neglected in (4.26)] is crucial to obtain this result.

In summary, RT of \( G \) waves with high meridional quantum numbers either have a negligible value of \( \gamma \) (and thus there is no interaction) or interact within one deformation radius from the equator; this explains the absence of \( GGG \) RT at midlatitudes.

4) TRIADS WITH MIXED ROSSBY–GRAVITY MODES

In the same way that \( M \) waves have the characteristics of \( R \) or \( G \) modes (say, for \( sc \leq -1 \) or \( sc > 1 \) respectively), RT with one or two \( M \) components have the properties of \( RRR \), \( GGR \) or \( GGG \) resonant trios (see Table 4). The parameters of RT with, say, \( n_a = 0 \), may be obtained from (4.15), factoring out the spurious \( s_c \sim -1 \) solution first. (Momentarily we do not assume that \( a \) has the maximum frequency). However, special (and simpler) formulas may be obtained for the case of RT with \( M \) components.

For instance, if \( n_a = 0 \) and \( n_b, n_c > 0 \), then i) replacing \( \omega_a = -\omega_b - \omega_c \) and \( k_a = -k_b - k_c \) in \( \omega_a^2 = (\beta + k_b \alpha c)^2 \) and ii) using \( \omega^2 = (2n + 1) \beta c + (k^2 + \beta k_\omega c^2) c^2 \) for \( b \) and \( c \), we get

\[
[(1 + S_{nh})(1 + S_{sh})(2 - S_{bh} - S_{hkh}) \beta c)^{-1} - 4n_b n_c - N_b N_c . \quad (4.25)
\]

Squaring this equation and using (4.11) for \( \omega_b^2 \) and \( \omega_c^2 \), we obtain a quartic in \( s_h \) (for fixed \( s_b \)), which has four (two) physical solutions if \( c \) is a \( G \) (\( R \)).

Moreover if, say, \( n_b \neq 0 \) and \( n_h = n_c = 0 \), then using \( S_j = 1 - \beta c \omega_j^{-2} \) \((j = b, c) \) in \( S_{bh}(\omega_b + \omega_c) = (S_{bh} + S_{hc}\omega_c) \) we get

\[
(1 - S_{bh}) = \beta c , \quad (4.26)
\]

which shows that \( a \) has the intermediate value of \( s \) in this case because \( S_j < 1 \) implies \( \omega_j \omega_c > 0 \) [cf., (3.4)–(3.5)]. Finally, replacing \( \omega_b = -\omega_a - \omega_c \) in this equation, we obtain

\[
\omega_b, \omega_c = -\frac{1}{2} \omega_a \pm \frac{1}{2} [\beta c (N_a - N_c)]^{-1/2} . \quad (4.27)
\]

From the analysis of (4.25) and (4.27) it follows that the RT with one or two \( M \) components fall into the following types (cf., Table 4):

Case 5 (MRR). It is similar to \( RRR \) (Case 4); notice that \( s_c < -\max(2n_a + 1, 2n_c + 1) \) < 0, and thus the \( M \) component behaves like a \( R \) (see Table 1).

Case 6 (MGR). It is similar to \( GGR \) (Case 7); since \( s_c > -1 \), the \( M \) component behaves like a \( G \).

Case 10 (GGM). It is similar to \( GGR \) (Case 9); since \( s_c < -1 \), the \( M \) component behaves like a \( R \). Notice, however, that at the rightmost limit in Table 4 (\( s_c = -1 \)), the \( M \) wave does not have vanishing \( \omega \) and \( k \) (unlike Case 9), although there is a large separation of space-time scales as in (4.2b). [In fact, this type of triads continues into Case 13 below.]

Case 11 (GMM): It goes continuously from the properties of a \( GGR \) (Case 9) to those of a \( GGG \) (Case 14); one of the \( M \) behaves always like a \( G \), and the other \( M \) changes from \( R \) to \( G \) behavior.

Case 12 (GGM): It goes continuously from the properties of a \( GGR \) (Case 7) to those of a \( GGG \) (Case 14); the \( M \) changes from \( R \) to \( G \) behavior.

Case 13 (GGM): It is similar to \( GGG \) (Case 14); the \( M \) satisfies \( s_c > -1 \), and thus it has the properties of a \( G \). Notice, however, that at the leftmost limit in Table 4 (\( s_c = -1 \)), the \( M \) wave is not a short \( G \) (as in case 14).

5) TRIADS OF KELVIN AND DISPERSIVE MODES

The classification of all possible RT of equatorial waves is completed with the inclusion of \( K \) components. Any three interacting \( K \) waves are in resonance (\( K \) modes self interaction is discussed in R82); two \( K \) modes cannot resonate with a non-\( K \) (because the latter should have \( sc = 1 \)); therefore we need only to consider the case of two non-\( K \) and one \( K \).

If \( n_a = -1 \) (the \( K \) component cannot have the intermediate slowness obviously) and replacing \( \omega_c \) and \( k_c \) from (3.1) in \( \omega_c = \omega_c, c \), we get

\[
\omega_a(1 - S_{nh}) = -\omega_b(1 - S_{bh}) . \quad (4.28)
\]

Squaring this equation and using (4.11) we obtain

\[
(2n_a + 1 + S_{nh}) (1 - S_{nh})(1 + S_{bh}) = (2n_b + 1 + S_{bh})(1 - S_{bh})(1 + S_{ah}) , \quad (4.29)
\]

which is a quadratic in \( s_b \) (for fixed \( s_a \)). If one of the
non-\( K \) is an \( M \), e.g., \( n_a = 0, n_b > 0, n_c = -1 \), (4.29) reduces to

\[
(1 - S_a)(1 + S_b) = (2n_b + 1 + S_0)(1 - S_b); \tag{4.30}
\]

[If both non-\( K \) are \( M \), the only solution is the trivial one, (4.3).] The kinematical properties of RT with a \( K \) mode are summarized in Table 5; their characteristics are the following:

Case 15 (RRK): The \( K \) component is always long; this is like Case 1 of RRR (exchanging \( b \) and \( c \) in Table 4).

Case 16 (MRK): It goes continuously from the properties of \( RRR \), for \( s_c < -1 \), to those of a \( GRG \), for \( s_c > -1 \), (i.e., Cases 1 and 7 in Table 4, exchanging \( b \) and \( c \)).

Case 17 (GRK): The \( K \) component is always short; this is like Case 7 of \( GRG \) (exchanging \( b \) and \( c \) in Table 4).

Case 18 (GMR): It goes continuously from the properties of \( GRG \), for \( s_c < -1 \), to those of a \( GGG \), for \( s_c > -1 \), (i.e., Cases 7 and 14 in Table 4, exchanging \( b \) and \( c \)).

Case 19 (GKG): This is similar to \( GGG \) (case 14).

However, in the leftmost limit it is more like the interaction of two short \( G \) and a long \( R \) (rightmost limit of Case 9).

Finally, notice that the cases with an \( M \) (16 and 18) may also be seen as interpolations between those without \( M \) (15, 17, and 19).

5. Summary

The zonal slowness \( s = (k/\omega) \) is a variable more useful than the zonal wavenumber \( k \) in order to deal with equatorial dispersive waves. For instance, the different types of modes are separated in \( s \)-space and the relations \( \omega^2(s, n) \) are simpler than the \( \omega(k, n) \) ones (see Table 1 and Fig. 1). Also, the unstable component of a resonant triad is that with intermediate value of \( s \), as required by energy and pseudomomentum conservation. Last but not least, the interaction and resonance conditions, (3.1), are reduced to finding the zeroes of a sixth-order polynomial, (4.15) [or fourth-order if one of the components is an \( M \), second-order if one is a \( K \), etc.].

Limiting cases of resonant triads are the non-local ones which represent one of the following:

1) A packet of \( R \) or \( G \) waves with a finite wavelength interacting with a long \( R \) (see Table 3). The group velocity of the packet matches the phase speed of the long wave. This system has different types of solutions with solitons and solitary waves (Ma and Redekopp, 1979; Ma, 1981).

2) The interaction of short waves of similar speed and different meridional quantum numbers, with a long \( R \) or \( K \) (namely, with a quasi-steady-geostrophic zonal flow). The short waves may be \( R \) (including the \( M \)) or \( G \) (including the \( M \) and \( K \)).

3) The scattering of short westward propagating \( G \) into a short eastward propagating \( G \) (including the \( M \) and \( K \)), by a short \( R \) (or \( M \)) of twice the wavenumber.

Unlike the cases of midlatitude Rossby waves or two-dimensional internal gravity waves (Ripa, 1981), there are resonant trios of equatorial waves with the same speed and with a finite interaction coefficient (e.g., those of Table 4).

There are nineteen different types of resonant triads; see Tables 4 and 5. However, these may be seen as particular cases of three general categories, RRR, GGR and GGG, in the sense that \( M \) and \( K \) are considered as a \( R \) for \( \omega^2 < \beta c/2 \), or as a \( G \) for \( \omega^2 > \beta c/2 \). Main characteristics of these classes are:

\textbf{RRR} (Fig. 2): Limiting cases are non-local triads of types (a) and (b) in (4.2), triads of waves with the same speed, and a triad of long (almost nondispersive) \( R \). The interaction coefficient is of the order of magnitude of the slowness of the unstable component, i.e., the same scaling found for resonant triads of "mid-latitudes" non-divergent planetary waves (Ripa, 1981).

\textbf{GGR} (Fig. 3): Limiting cases are non-local triads of the three types in (4.2). The interaction coefficient is of the order of \( 1/c \), where \( c \) is the separation constant.

\textbf{GGG} (Fig. 4): Limiting cases are three short waves of similar speed. The interaction coefficient is of the order \( 1/c \). Unlike the other two cases they exist only if the square root of the three meridional quantum numbers cannot be the sides of a triangle, (4.22). Inertia-gravity waves that extend far away from the equator (large \( n_t \)) either have a negligible interaction coefficient or the interaction takes place at low latitudes; this is consistent with the fact that there are no resonant trios of \( G \) modes in the (midlatitudes) \( f \)-plane.

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APPENDIX

Triangular Function

The triangular function, defined by

$$\Delta(x_1, x_2, x_3) = 2(x_1x_2 + x_2x_3 + x_3x_1) - x_1^2 - x_2^2 - x_3^2,$$  \hspace{1cm} (A1)

is such that 1) If the $x_i$ are not all of the same sign, $\Delta$ is negative as can be seen from the identity $\Delta = 4x_1x_2 - (x_3 - x_1 - x_2)^2$, or any permutation of $(1, 2, 3)$. 2) If the $x_i$ are all of the same sign, say $x_i = a_i^2$ or $x_i = -a_i^2$, it is

$$\Delta(a_i^2) = (a_1 + a_2 + a_3)(a_1 + a_2 - a_3),$$  \hspace{1cm} (A2)

which shows that $\Delta$ is positive (negative) if the $a_i$ can (cannot) be the sides of a triangle, and $\Delta$ vanishes if one of the $a_i$ is equal to the sum of the other two. A useful property is

$$\Delta(x_1 + a) = \Delta(x_1) + 2a(x_1 + x_2 + x_3) + 3a^2.$$  \hspace{1cm} (A3)

Longuet-Higgins and Gill (1967) have used this function in a context similar to that of this paper.

Notation

- $E$: energy
- $P$: pseudomomentum
- $k$: zonal wavenumber
- $n$: meridional quantum number
- $c$: vertical separation constant $\sim O(1 \text{ m s}^{-1})$ for the ocean, $\sim O(100 \text{ m s}^{-1})$ for the atmosphere
- $\omega$: frequency
- $s$: slowness $= k/\omega$
- $G$: group velocity $= d\omega/dk$
- $\beta$: meridional gradient of the Coriolis parameter ($= 2.29 \times 10^{-11} \text{ m}^{-1} \text{s}^{-1}$)
- $R$: deformation radius $= (2c\beta^{-1})^{1/2}$
- $\mathcal{R}$: Rossby (expansion mode)
- $M$: mixed Rossby-gravity (expansion mode)
- $G$: inertia-gravity (expansion mode)
- $K$: Kelvin (expansion mode)
- $\sigma^G$: coupling coefficient
- $\gamma_{abc}$: interaction coefficient
- $RT$: resonant triad
- $\mu$: inverse of nonlinear time scale
- $QGF$: quasi-geostrophic flow
- $IGW$: two-dimensional internal gravity waves.

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