Normal Modes of the World Ocean. Part III: A Procedure for Tidal Synthesis

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ABSTRACT

In preceding parts of this study a set of normal modes was constructed as a basis for synthesizing diurnal and semidiurnal solutions of Laplace's tidal equations. The present part describes a procedure by which such solutions can be computed as eigenfunction expansions. Since the calculated normal modes are nondissipative, it is necessary to incorporate dissipation into the synthesis procedure. This is done by a variational treatment of the tidal equations.

1. Introduction

Free modes of Laplace's tidal equations were calculated recently in the period range 8 to 80 h for a barotropic world ocean without dissipation (Platzman, 1978, and Platzman et al., 1981, hereafter cited respectively as Part I and Part II). In the range 8 to 30 h the model contains 41 modes, all predominantly of gravity-wave type, namely, ocean-basin modes, Kelvin waves, and quarter-wave resonances. In the range 30 to 80 h there are 15 modes which, with two exceptions, are topographically-trapped vorticity waves. For the present study of tidal synthesis the period range was extended from 80 to 96 h. This added 4 topographic vorticity waves and thus provided a total of 60 modes as a basis for synthesis.

The motivation for seeking a tidally-forced solution of Laplace's tidal equations by modal synthesis is not computational efficiency or accuracy. On the contrary, numerical computation of the eigenfunctions is itself a task at least as large as that of directly solving the forced tidal equations, and must be followed by the additional work of synthesis. Moreover, the result may not be as accurate as solutions attainable by other methods. It is reasonable to expect, however, that a synthesis consisting of a relatively small number of modes should suffice to give an approximate solution that, while perhaps not correct in all details, should capture the main features of the global tide. If this is so, we may be able to gain physical insight by viewing the tide as a mixture of forced modes of oscillation which have spatial configurations of well-defined free modes.

The basis for this expectation is the fact that the tide potential has a scale not much different from that of the principal ocean basins, and therefore tends preferentially to excite the lowest basin modes, the number of which is not large. Additional selectivity is provided by the fact that some modes have periods not far from tidal resonance. These circumstances have been well known since the time of Laplace, and have often been adduced to account for large-scale tidal morphology. Indeed, numerical solutions of the tidal equations have generally been found to be sensitive to details of model formulation, a fact usually attributed to quasi-resonance. Until recently, however, little progress was possible in implementing the spectral representation of the tidal operator, because realistic normal modes were unknown. Now that tentative catalogues of modes are available (Part II; also Gottlieb and Kagan, 1982), it seems opportune to make a serious attempt to apply this knowledge to the tide problem. Recently I gave a preliminary account of some results of this endeavor (Platzman, 1983).

The present paper specifies the design of a model for tidal synthesis. (It is followed by Part IV, a description of synthesized diurnal and semidiurnal tides.) Sections 2–6 summarize the main features of the finite-element discretization used in this work. The model incorporates dissipation by means of a radiation boundary condition, and elastic body-tide effects by means of the Love reduction factor. Ocean self-attraction and load tide are not included. Sections 7 and 8 describe some general properties of the eigenelements of the tidal operator, and give details of the enumeration of modes over the frequency range 0.25 to 3.0 cpd, to which this study is confined.

The problem of tidal synthesis is formulated in Sections 9–11. Since the available modes are nondissipative, it is necessary to introduce dissipation as part of the process of synthesis. This is done by a variational treatment of the tidal equations, set forth in Sections 12–14. Finally, the computation procedure
for tidal synthesis is summarized in Section 15, and some limitations of the model are noted in Section 16.

2. Elevation and velocity

The grid is formed from quasi-uniform triangular elements, the average area of which is equal to that of a 4.5-degree equatorial square. It does not resolve the Bering Strait and thus combines the Eurasian and American continents as one land mass. It also excludes all marginal seas. It does, however, include the Drake passage as well as Madagascar and New Zealand (all at the limit of resolution). This produces a triply-connected grid. Bathymetry and potential vorticity are resolved by specifying average values of depth, inverse depth, and Coriolis parameter for each of the grid triangles. The grid characteristics are summarized in Table 1. (For more information see Part I, Section 2, and Part II, Sections 2 and 3.)

Let $\xi, \phi, \psi$ denote sea-surface elevation and Stokes/Helmholtz velocity potentials interpolated continuously over the ocean domain from nodal values by means of finite-element basis functions (Part I, Sections 3 and 6). Since linear interpolation is used, the gradients of these functions are uniform over each grid triangle, as is the volume transport (depth times velocity)

$$u = -h \nabla \phi + k \times \nabla \psi. \quad (2.1)$$

Here $h$ is the ocean depth (averaged over the triangle) and $k$ the upward unit vector.

Nodal values of $\xi, \phi, \psi$ are arranged in a matrix column $x$—the "sea-state vector"—that consists of three subcolumns, the first containing 743 values of $\xi$, the second 742 values of $\phi$, the third 557 values of $\psi$. The length (number of rows) of $x$ is equal to 2042, the total number of degrees of freedom in the discretized variables. This is less than three times the number of nodes ($3 \times 743 = 2229$), because additive constants are irrelevant in $\phi$ and $\psi$ and because $\psi$ must be uniform on the boundary of each of the four land masses of the model. For these reasons, one degree of freedom was removed from $\phi$ by assigning $\phi = 0$ at an arbitrarily selected node. Similarly, $\psi = 0$ was assigned at all 154 nodes of the mainland boundary, while on the boundaries of Antarctica (23 nodes), Madagascar (6 nodes) and New Zealand (6 nodes) $\psi$ was constrained to be uniform but nonzero. This removes $154 + 22 + 5 + 5 = 186$ degrees of freedom from $\psi$. (See Table 2.)

3. Laplace’s tidal equations

An energy-consistent, Galerkin-type approximation leads to the following finite-element discretization of Laplace’s tidal equations:

$$\mathbf{B} \partial x / \partial t = (\mathbf{A} - \mathbf{D}) x - \mathbf{A} \dot{x}. \quad (3.1)$$

(Part I, Section 6) where $x$ is the equilibrium-tide sea-state vector—a column with nodal equilibrium-tide elevation in the first subcolumn and zeros in the other two subcolumns. The operators $\mathbf{B}, \mathbf{A}$ and $\mathbf{D}$ are square matrices, each of order 2042, whose elements are integrals that depend upon the basis functions used for interpolation on the triangular grid. Each has a natural partition into nine submatrix blocks in accord with the partition (743, 742, 557) of column $x$.

The following display shows the block structure schematically, in particular the locations of blocks of formal zeros, and the way each nonzero block depends on assigned physical parameters, namely, gravity $g$, depth $h$, Coriolis parameter $f$, radiation parameter $c$ (denoted $hm$ in Part I), and bottom-friction parameter $r$:

$$\mathbf{B} = \begin{bmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h^{-1} \end{bmatrix};$$

$$\mathbf{A} = \begin{bmatrix} gh & 0 \\ hf & -f \\ 0 & h^{-1}f \end{bmatrix};$$

$$\mathbf{D} = \begin{bmatrix} gc & 0 & 0 \\ 0 & hr & r \\ 0 & r & h^{-1}r \end{bmatrix}.$$
In this arrangement the first block-row corresponds to the mass equation (multiplied by \( g \)), the second to the divergence of \( h \) times the momentum equation, and the third to the vertical component of the curl of the momentum equation. The block-diagonal “mass” matrix \( \mathbf{B} \) is symmetric and positive, the “stiffness” matrix \( \mathbf{A} \) is skew-symmetric, and the dissipation matrix \( \mathbf{D} \) is symmetric and non-negative.

Parameter \( c \) (dimension: velocity) arises from Proudman’s (1941) radiation boundary condition

\[
\mathbf{u}_n = c \zeta,
\]

where \( \mathbf{u}_n \) is the outward component of volume transport normal to the boundary. This condition provides for dissipation by allowing a portion of the energy incident on the continental slope to be transmitted onto the shelf, and transferred toward the coast by pure gravity waves at speed \( c \), into regions of complete absorption. A value of \( c \) or of the corresponding effective shelf depth \( (=c^2/g) \), must be specified as an input datum. The symbol \( \frac{c}{S} \) in (3.2) calls attention to the fact that the governing-equation discretization (3.1) makes this constraint a natural rather than an essential boundary condition (Part I, Section 4). Consequently, (3.2) is satisfied only approximately, the size of the error being that of truncation errors associated with grid resolution.

The bottom-friction parameter \( r \) (dimension: inverse time) is a resistance coefficient used to express bottom stress \( \rho \mathbf{u} \) (where \( \rho \) is sea-water density). Since the grid used in the present calculations is too coarse to permit adequate resolution of continental shelves or marginal seas, where bottom friction is most effective, I ignore the bottom-friction parts of \( \mathbf{D} \) and require dissipation to take place entirely by radiation through the boundary.

4. Matrix elements

It is conceptually and computationally desirable to modify the Stokes/Helmholtz potentials so that all entries in the sea-state vector \( \mathbf{x} \) have the dimension and scale of sea-surface elevation. Let

\[
\phi' = (gHS)^{-1/2} \phi; \quad \psi' = (gHS)^{-1/2} \psi,
\]

where \( S \) is the total area of the model domain and \( H \) is a scale of depth. Now re-define \( \mathbf{x} \) as the column formed from nodal values of \( \zeta, \phi', \psi' \). Then after the factor \( gS \) is removed, matrix \( \mathbf{B} \) in (3.1) is nondimensional while \( \mathbf{A} \) and \( \mathbf{D} \) have dimensions of inverse time.

Thus modified, the elements of the relevant sub-matrix blocks are:

\[
\begin{align*}
(B11)_{ij} &= \int S^{-1} \alpha' \alpha' dS \\
(B22)_{ij} &= H^{-1} \int h \nabla \beta' \cdot \nabla \beta' dS
\end{align*}
\]

\[
\begin{align*}
(A12)_{ij} &= -(g/HS)^{1/2} \int h \nabla \alpha' \cdot \nabla \beta' dS \\
(A22)_{ij} &= H^{-1} \int h f \nabla \beta' \times \nabla \beta' \cdot k dS \\
(A23)_{ij} &= -\int \mathbf{f} \nabla \beta' \cdot \nabla \gamma' dS \\
(A33)_{ij} &= H \int h^{-1} f \nabla \gamma' \times \nabla \gamma' \cdot k dS \\
(D11)_{ij} &= P^{-1} \int \Theta \alpha' \alpha' dP.
\end{align*}
\]

Here \( dS \) and \( dP \) are differential elements of, respectively, ocean-surface area and ocean-boundary perimeter; and \( \alpha', \beta', \gamma' \) are finite-element basis functions respectively for \( \zeta, \phi, \psi \) at node \( i \). Further

\[
\theta = cPS,
\]

(dimension: inverse time) is a proxy for the radiation parameter \( c \). The model values of \( P \) and \( S \) are given in Table 1. The value to be adopted (in Part IV) for \( c \) corresponds to an effective shelf depth of 17 m \((=c^2/g)\), and this makes \( \theta = 5.33 \times 10^{-6} \) rad s\(^{-1} \) \( = 0.0733 \) cpd.

The integrals in (4.2) were evaluated by taking the basis functions to be piecewise linear, thus causing \( \nabla \alpha, \nabla \beta, \nabla \gamma \) to be uniform within each grid triangle. The evaluations were further simplified by using mean values of \( h, h^{-1} \), and \( f \) within each triangle (Part II, Section 3). Note that, because of the skew-symmetry of \( \mathbf{A} \), we have \((A21)_{ij} = -(A12)_{ji}\) and \((A32)_{ij} = -(A23)_{ji}\).

5. Energy equation

Since \( \mathbf{B} \) is symmetric and \( \mathbf{A} \) skew-symmetric, the result of premultiplying (3.1) by \( \mathbf{x}' \) (the transpose of \( \mathbf{x} \)) is the following discretized energy equation:

\[
\frac{d}{dt} \left( \frac{1}{2} \mathbf{x}'^{\mathbf{T}} \mathbf{B} \mathbf{x} \right) = -\mathbf{x}'^{\mathbf{T}} \mathbf{A} \ddot{\mathbf{x}} - \mathbf{x}'^{\mathbf{T}} \mathbf{D} \mathbf{x}.
\]

In view of (2.1) and (4.2), the dimensional interpretations of the three quadratic forms here are

\[
\begin{align*}
\frac{1}{2} \mathbf{x}'^{\mathbf{T}} \mathbf{B} \mathbf{x} \cdot \rho g S &= \frac{1}{2} \rho \int (g \zeta^2 + h^{-1} u^2) dS, \\
-\mathbf{x}'^{\mathbf{T}} \mathbf{A} \ddot{\mathbf{x}} \cdot \rho g S &= \rho \int \mathbf{u} \cdot g \nabla \zeta dS, \\
\mathbf{x}'^{\mathbf{T}} \mathbf{D} \mathbf{x} \cdot \rho g S &= \rho \int g c \xi^2 dP,
\end{align*}
\]

namely, respectively the instantaneous total energy, the instantaneous rate of accession of energy from...
the tide force, and the instantaneous rate of dissipation of energy by transmission across the boundary.

In (5.2) matrix squares \( B, A, D \) and column \( x \) are dimensionally modified as explained in the preceding section. Consequently, \( x^T B x \) has dimensions (length\(^2\)) while \( x^T A x \) and \( x^T D x \) have dimensions (length\(^2\)/time). The elements of column \( x \) consist of nodal values of sea-surface elevation and of the dimensionally-modified Stokes/Helmholtz potentials, while \( \xi, \eta, u \) on the right-hand sides in (5.2) are continuous functions obtained by linear interpolation from the elements of \( \bar{x} \) and \( x \) (and restoration of full dimensions to \( u \)).

In the first integral of (5.2) the term \( h^{-1} u \) was introduced in place of \( h (\nabla \phi)^2 + h^{-1} (\nabla \psi)^2 \), for on forming \( h^{-1} u \) from (2.1) we find that the surface integral of the product term \(-2 \nabla \phi \cdot k \times \nabla \psi \) can be expressed as a boundary integral that is zero because \( \psi \) is uniform on all boundary arcs. The kinetic energy in the whole domain is therefore the sum of the kinetic energies associated respectively with the divergent part \((-h \nabla \phi)\) and the nondivergent part \((k \times \nabla \psi)\) of \( u \). Similarly, in the second integral \( u_\mathcal{H} \) has replaced \(-h \nabla \phi \) because the other part of \( u \) gives rise to a term whose surface integral vanishes for the reason given above. In other words, when integrated over the whole domain, the work done by the tide force on the non-divergent part of \( u \) is zero.

We will represent the equilibrium tide and dependent variables as periodic functions for a single tide constituent of frequency \( \omega \):

\[
(\bar{x}, \bar{\psi}) = \text{Re}([\bar{Z}, \bar{Z}, \bar{U}] e^{i\omega t})
\]  

(5.3)

(and likewise for \( \phi, \psi \)). Similarly, the constituent sea-state vectors are

\[
(\bar{x}, x) = \text{Re}([\bar{X}, X] e^{i\omega t}),
\]  

(5.4)

where \( \bar{X} \) and \( X \) are columns containing complex amplitudes at the nodes.

When averaged over the tide period the balance equation (5.1) is

\[
0 = -\frac{1}{2} \text{Re}(X^H B \bar{X}) - \frac{1}{2} X^H D X,
\]  

(5.5)

where \( X^H \) is the complex conjugate (Hermitian) transpose of \( X \). The time average of (5.2) is

\[
\frac{1}{4} X^H B X \cdot \rho g S = \frac{1}{4} \rho \int \left( g |Z|^2 + h^{-1} |U|^2 \right) dS,
\]

\[
- \frac{1}{2} \text{Re}(X^H A \bar{X}) \cdot \rho g S = \frac{1}{2} \rho \int \text{Re}(U^* \cdot g \nabla \bar{Z}) dS,
\]

\[
\frac{1}{2} (X^H D X) \cdot \rho g S = \frac{1}{2} \rho \int g c |Z|^2 dP.
\]  

(5.6)

Thus, after multiplication by \( \rho g S \), (5.5) states that the time-average rate of accession of energy from the tide force is equal to the time-average rate of dissipation. The first equation in (5.6) gives the time-average total energy ("stored" energy); the third equation gives the time-average rate of dissipation.

It is convenient to define energy norms of the sea state \( x \) and of its complex amplitude \( X \) as

\[
\|x\| = (x^T B x)^{1/2},
\]

\[
\|X\| = (X^H B X)^{1/2}.
\]  

(5.7)

The conventions adopted in the preceding section give these norms the dimension and scale of sea-surface elevation. According to the first equations in (5.2) and (5.6) the expressions \( \frac{1}{2} \|x\|^2 \rho g S \) and \( \frac{1}{2} \|X\|^2 \rho g S \) give respectively the instantaneous and the time-average total energy.

6. Mass equation

The radiation boundary condition permits a net flux of mass between the model domain and the marginal seas (plus continental shelf regions) that are excluded from the model. The total mass within the model domain therefore may fluctuate. Although this fluctuation should be small, clarity obliges us to account for it.

A convenient measure of mass is the area-mean sea-surface elevation. This can be expressed in terms of sea-state vector \( x \) by means of the "mass vector"

\[
e = \begin{cases} 
1 & \text{in all rows of the first} \\
(\text{elevation}) \text{ subcolumn} & \text{subcolumn} \\
0 & \text{in all other rows}.
\end{cases}
\]

(6.1)

In view of the fact that at any point of the domain, the sum of basis functions \( \alpha' \) over all nodes is = 1, it is apparent from B11 in (4.2) that \( e^T B \) is a row with \( S^{-1} \int \alpha' dS \) in the "elevation" columns and zero in all other columns, so the scalar

\[
e^T B x = S^{-1} \int \xi dS
\]

(6.2)

is the area-mean sea-surface elevation.

Similarly, it follows from D11 in (4.2) that \( e^T D \) is a row with \( P^{-1} \int \theta \xi dP \) in the elevation columns and zero in all other columns. Hence the scalar

\[
e^T D x = P^{-1} \int \theta \xi dP
\]  

(6.3)

is the arc-mean of \( \theta \xi \) over the complete boundary of the domain. On the other hand \( e^T A = 0 \), because the only elements here not formally zero are integrals that contain \( \Sigma \nabla \alpha' \) as a factor of the integrand; but this factor is the same as \( \nabla \Sigma \alpha' = \nabla 1 = 0 \).

We can now obtain the discretized mass-balance equation by premultiplying (3.1) by \( e^T \rho S \). In view of (6.2, 3) the result is

\[
(\partial/\partial t) \rho \int \xi dS = -\rho \int c \xi dP
\]

(6.4)
after $\theta$ is replaced from (4.3). Since the left-hand side
is the rate of increase of model-domain mass (relative
to mean sea level), this equation enables us to interpret
$\rho \int \mathcal{C} \mathcal{P} dP$ as the net flux of mass out of the domain,
as should be expected from the radiation condition
(3.2). Note that, because it is a global condition, (6.4)
is exact—unlike (3.2), a local condition.

7. Eigenelements

Before a procedure for tidal synthesis is formulated,
it is useful to summarize some general features of the
spectrum of the tidal operator, namely of the matrix
$A - D$ which, in association with $B$, determines the
spatial transformation of the sea state $x$ in the tidal
equation (3.1). These are largely textbook matters
(for example, see Friedman, 1956, Chapters 2 and 4).

As in previous studies, I define the tidal operator
as the imaginary matrix

$$ L = -i (A - D) = \hat{L} + iD. \quad (7.1) $$

Here $\hat{L} = -iA$ is the non-dissipative operator; it is
Hermitian ($L^H = L$) in view of the skew-symmetry
of $A$. Consequently, when there is no dissipation ($D = 0$)
the eigenvalues of the tidal operator are real
and in fact are simply the frequencies of the free
modes of oscillation. The dimension of $L$ is inverse
time.

Let $\sigma_k$ denote an eigenvalue of $L$, and $X_k$ the
associated eigenfunction (both of which are in general
complex):

$$ (L - \sigma_k B)X_k = 0. \quad (7.2) $$

By taking the complex conjugate of (7.2) we see that,
since $L$ is imaginary and $B$ is real, the eigenelements
occur in pairs; that is, to each $(\sigma_k, X_k)$ there
corresponds a "conjugate" $(-\sigma_k, \bar{X}_k)$. The members of a
pair are distinct unless $\text{Re} \sigma_k = 0$ (considered below).

In indexing them it is convenient to assign to the one with $\text{Re} \sigma_k > 0$ a positive index $k$, and to the other
the corresponding negative index:

$$ (\sigma_{-k}, X_{-k}) = (-\sigma_k, X_k). \quad (7.3) $$

Thus, computationally, (7.2) need be solved only for
one member of each pair: the other can be found
merely by taking complex conjugates.

In view of the symmetry of $D$, the adjoint tidal
operator is $L^H = L - iD$ (and so is obtained from $L$
by reversing the sign of the dissipation matrix).

Since the eigenvalues of $L^H$ are the complex conjugates of
those of $L$, the adjoint eigenproblem is

$$ (L^H - \sigma_k^* B)Y_k = 0, \quad (7.4) $$

where $Y_k$ is the eigenfunction adjoint to $X_k$. Again,
unlike $\text{Re} \sigma_k = 0$, the eigenelements of (7.4) occur in
conjugate pairs $(\sigma_k, Y_k)$ and

$$ (\sigma_{-k}, Y_{-k}) = (-\sigma_k, Y_k). \quad (7.5) $$

Thus $Y_k$ is adjoint to $X_k$.

As is well known, Eqs. (7.2) and (7.4) imply the
bi-orthogonality

$$ Y_k^H B X_k = 0 \quad \text{if} \quad \sigma_j \neq \sigma_k. \quad (7.6) $$

Either $j$ or $k$ may be negative here, in the sense of
(7.3) and (7.5). When expressed as an integral in
terms of elevation and volume transport, as in (5.6),
this orthogonality condition will be recognized as the
one first stated by Proudman (1928).

A quotient formula for $\sigma_k$ can be obtained by
 premultiplying (7.2) by $Y_k^H$:

$$ \sigma_k = Y_k^H L X_k / Y_k^H B X_k. $$

This is a Rayleigh quotient: when $L$ and $B$ are fixed,
it is stationary with respect to arbitrary variations of
the eigenfunctions from their states determined by
(7.2) and (7.4). Another quotient (but not a Rayleigh
quotient in the variational sense) is

$$ \sigma_k = X_k^H L X_k / X_k^H B X_k $$

from premultiplication of (7.2) by $X_k^H$. By means of
(7.1) this can be separated into real and imaginary
parts

$$ \text{Re} \sigma_k = X_k^H L X_k / \|X_k\|^2 $$

$$ \text{Im} \sigma_k = X_k^H D X_k / \|X_k\|^2, \quad (7.7) $$

since $\hat{L}$ is Hermitian and $D$ symmetric. A similar
operation performed on (7.4) shows that the right-hand
sides of (7.7) are not changed in value by
replacing $\bar{X}_k$ with $Y_k$. Note that $\text{Im} \sigma_k \geq 0$, since $D$ is
non-negative.

8. Mode enumeration

If $\text{Re} \sigma_k = 0$, the real and imaginary parts of $X_k$
satisfy (7.2) separately, and the conjugate element is
redundant. One such non-oscillatory mode is needed
in the synthesis to provide for the slight change of
mass that can accompany radiative dissipation. It has
an especially simple structure when there is no dissipation,
namely zero velocity and uniform nonzero
elevation. (In fact, this structure is a solution of the
continuum as well as of the discrete equations.) The
effect of weak dissipation on this "mass" mode is to
cause the elevation to be slightly nonuniform and to
call for weak currents. The mode is then damped,
but not oscillatory.

If any other non oscillatory mode is present, it is
probably an unimportant consequence of the
discretization. In particular, when the number of degrees
of freedom is even (2042 in the present model), there
must be an even number of non oscillatory eigenelements,
because the oscillatory elements occur in
pairs, so there must be an odd number of non-
oscillatory eigenelements in addition to the mass
mode. Referring again to the case with no dissipation,
we know that there are special types of idealized
basins that admit a large number (infinite before

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discretization) of steady vorticity modes (geostrophic circulations). A simple example is zonal geostrophic flow in an ocean covering the complete sphere with depth at most a function of latitude. In a natural basin, however, such modes should not be expected, because they require the potential vorticity \( f/\lambda \) to be uniform on the coastal boundaries, a circumstance not found anywhere in the real oceans. It follows that, if the discretized model calls for an odd number of non-oscillatory modes in addition to the mass mode, there is likely to be only one, and that one presumably is a pathological artifact of the discretization, since it arises only when the number of degrees of freedom is even.

In addition to strictly non-oscillatory modes there is of course a large number of oscillatory vorticity modes, which in the present model have periods greater than 30 h. Some of these quasigeostrophic modes have large-scale features and therefore, despite long periods, may have a beneficial effect on the synthesis, at least for diurnal tides.

Figure 1 shows the spectral distribution of the 60 oscillatory modes available for tide synthesis, which occupy the frequency range 0.25 to 3.00 cpd (96 to 8 h). In addition, the model has 261 vorticity modes slower than 0.25 cpd, and 699 gravity modes faster than 3 cpd. The latter probably are distributed with number density increasing roughly linearly out to about 40 cpd. Most are too severely affected by truncation error to be of physical interest. The same is true of the very slow vorticity modes. There is a total of 1020 oscillatory modes present in the model, of which 742 are gravity modes and 278 are vorticity modes (Part II, Section 4).

A remarkable feature of the distribution shown in Fig. 1 is the minimum number density in the “diurnal” band 0.75 to 1.25 cpd (32 to 19.2 h), which has a total of only five modes. Of these the slowest (28.7 h) is a fundamental (half-wave) basin oscillation of the Pacific, augmented by the fundamental Antarctic Kelvin wave. We will see that this mode dominates the global diurnal tide (Part IV). The next two are North Atlantic modes (half wave, 25.7 h and three-quarter wave, 23.7 h), the fourth is a full-wave Pacific mode (21.2 h) and the fifth (20.2 h) a nondescript mode that originates in the Arctic Ocean. The sparseness of modes in the diurnal band is partly a consequence of the present-day configurations of the main ocean basins. It also represents the gap associated with the transition between two limit points, one produced by low-frequency vorticity modes with number density \( \sim \sigma^{-2} \), the other by high-frequency gravity modes with number density \( \sim \sigma \).

9. Constituent tidal equation

We turn attention now to a single tide constituent with frequency \( \omega \). The result of substituting the periodic expressions (5.4) into (3.1) is the tidal equation in “harmonic” form

\[
(L - \omega B)X = \tilde{L}\tilde{X}
\]  

(9.1)

with \( L \) and \( \tilde{L} \) as in (7.1). To fix the phases of \( \tilde{X} \) and \( X \) unambiguously, I will adhere to the convention that \( \omega > 0 \).

For a single constituent the time-average energy and rate of dissipation are given by the left-hand sides of the first and third equations in (5.6). It follows that the response quality of a constituent is

\[
Q = \frac{1}{2} \omega ||X||^2/\omega dX,
\]  

(9.2)

if defined as (frequency) \( \times \) (average energy)/(average rate of dissipation).

By analogy with (9.2) the quality of an individual mode is

\[
Q_k = \frac{1}{2} |\text{Re} \sigma_k| ||X_k||^2/\omega dX_k.
\]  

(9.3)

In view of (7.7), this is equal to \( \frac{1}{2}|\text{Re} \sigma_k|/(|\text{Im} \sigma_k|) \), a relation that can be used to estimate \( \text{Im} \sigma_k \) when \( Q_k \) is known independently.

10. Synthesis

We seek to approximate the complex solution \( X \) of (9.1) as a linear combination of some of the eigenfunctions \( X_k \) of \( L \):

\[
S(X) = ||\tilde{X}|| \sum a_k X_k.
\]  

(10.1)
The notation \( S(X) \) ("synthesized \( X \)") is used to emphasize the fact that the sum does not span the complete set of \( X_k \) and therefore cannot in general represent the exact solution of (9.1). The factor \( \| \tilde{X} \| \) is inserted to make the synthesis coefficients \( a_k \) independent of scale factors in the tide potential. It has the same dimension as \( S(X) \), namely that of sea-surface elevation, so the sum in (10.1) is non-dimensional. Further, the normalization to be adopted makes \( X_k \) nondimensional, so the \( a_k \) are also non-dimensional.

Each eigenelement \( (\sigma_k, X_k) \) of \( L \) corresponds to a “free mode” of Laplace’s tidal equations, in the sense that \( \text{Re}[X_k \exp(i\sigma_k t)] \) is a solution of (3.1) with \( \bar{x} = 0 \). The conjugate element \( (-\sigma^*_k, X^*_k) \) corresponds to the same free mode, that is,

\[
\text{Re}[X_k \exp(i\sigma_k t)] = \text{Re}[X^*_k \exp(-i\sigma_k^* t)]. \tag{10.2}
\]

However, the members of a pair of conjugate eigenfunctions in (10.1) contribute to the time-dependent synthesized tide

\[
S(X) = \text{Re}[S(X)e^{i\omega t}] \tag{10.3}
\]

in characteristically different ways, because in contrast to (10.2), the time factor in (10.3) is the same for both members (and indeed for all eigenvalues in the synthesis).

In particular, let \( \xi \) be a component of \( X_k \); for example, the complex elevation of mode \( k \) at a particular node of the grid. The contributions to \( S(X) \) at this node for \( X_k \) and \( X_{-k} \) \((-X^*_k\)) are respectively, according to (10.3)

\[
\text{Re}(a_k \xi e^{i\omega t}) = |\xi| \cos(\arg\xi + \omega t + \arg a_k) \cdot |a_k|,
\]

\[
\text{Re}(a_{-k} \xi^* e^{i\omega t}) = |\xi| \cos(\arg\xi - \omega t - \arg a_{-k}) \cdot |a_{-k}|.
\]

Bearing in mind that \( \xi \) varies spatially while \( a_k \) and \( a_{-k} \) do not, we see that these two forced modes have the same spatial structure, apart from a uniform scale factor in amplitude and a uniform shift in phase. However, their phases propagate in opposite directions. On the understanding that \( \omega > 0 \), we can say further that the forced mode corresponding to the eigenelement with \( \text{Re} \sigma > 0 \) propagates in the same direction as the free mode. Since its \( \text{Re} \sigma \) has the same sign as \( \omega \), this “prograde” forced mode is potentially resonant, while the “retrograde” forced mode \( (\text{Re} \sigma < 0) \) is not. Indeed, we will see that, although roughly half the energy of diurnal and semidiurnal tide potentials comes from retrograde modes, the energy of the response is virtually all in the form of prograde modes, because of resonance magnification.

The indexing convention adopted in (7.3) places the prograde and retrograde forced modes respectively in the \( k > 0 \) and \( k < 0 \) parts of the sum in (10.1). Also included is the mass mode \( (\text{Re} \sigma = 0) \), indexed \( k = 0 \). Since 60 oscillatory free modes are available, and thus 60 pairs of conjugate eigenelements, the syntheses made in this study have 121 terms, indexed \( k = -60 \) to +60 in (10.1). It should be added that, although in principle the synthesis can be restricted to any subset of the available eigenelements, each included term tends to increase the accuracy of the result; moreover, the computational cost per term is relatively small.

11. Least squares and spectral estimation

The coefficients in (10.1) must be determined in a way designed to make \( S(X) \) a “best” approximation to the hypothetically unknown \( X \) according to an adopted criterion.

As in the method of “weighted residuals,” criteria for estimating the \( a_k \) can be formulated in terms of the residual that results from substitution of \( S(X) \) in place of \( X \) in the governing equation (9.1). It is convenient to normalize this residual as

\[
R = B^{-1}[(L - \omega B)S(X) - \tilde{L}X]/\| \tilde{X} \| \tag{11.1}
\]

On inserting (10.1) and making use of (7.2), we get

\[
R = \sum a_k(\sigma_k - \omega)X_k - W,
\]

\[
W = B^{-1}\tilde{L}X/\| \tilde{X} \|. \tag{11.2}
\]

The inhomogeneous term \( W \) is a scaled form of the tide force.

One method of estimation—the method of least squares (“approximation in the mean” in the terminology of Courant and Hilbert)—is to minimize the integral, over the ocean domain, of the square of the absolute value of the residual, that is, of the continuous function obtained from \( R \) by finite-element interpolation. According to the first equation in (5.2) this integral is proportional to \( R^HBR \), the minimum of which with respect to the real and imaginary parts of a particular \( a_k \) is attained when \( (\partial R^H/\partial a_k^*)B = 0 \). This gives

\[
X_k^HBR = 0, \tag{11.3}
\]

and consequently by (11.2), the system

\[
\sum_k a_k(\sigma_k - \omega)X_k^HBX_k = X_k^HBW \tag{11.4}
\]

of linear inhomogeneous equations for the \( a_k \). Since the \( X_k \) are not mutually orthogonal, (11.4) is not a diagonal system and its solution requires inversion of the matrix whose elements are \( X_k^HBX_k \).

The weighted-residual equations (11.3) impose on \( R \) the Galerkin-type constraint that the integrals, over the ocean domain, of the product of \( R \) by each of the eigenelements selected for synthesis should be zero. An alternative is to take the adjoint eigenelements as weights:

\[
Y_k^HBR = 0. \tag{11.5}
\]

In view of the bi-orthogonality (7.6), and using (11.2)
for \( R \), we now get a diagonal system, the solution of which is well known:

\[
A_k = (\sigma_k - \omega)^{-1} Y_k^H B W^j Y_k^H B X_k. \tag{11.6}
\]

This "spectral" value of \( A_k \) is unaffected by the presence of any other eigenvalue in the synthesis, in contrast to the least-squares values that arise from (11.4). In this sense it is an intrinsic property of the exact solution of (9.1).

To describe the least-squares and spectral estimates somewhat more formally, let \([L]\) denote the complete set of eigenfunctions \( X_k \), and \( S[L] \) the subset used in (10.1) for synthesis. Further, let \([L^H]\) denote the complete set of adjoint eigenfunctions, and \( S[L^H] \) the particular subset that is adjoint to \( S[L] \). In geometric language (11.3) asserts that the residual \( R \), regarded as expanded over \([L^H]\), has no component along any \( Y_k \) in \( S[L^H] \). Conversely, (11.5) asserts that \( R \), regarded as expanded over \([L]\), has no component along any \( X_k \) in \( S[L] \).

For reasons explained in the next two sections, the approximate method of synthesis used in this study makes the least-squares and spectral estimates virtually equivalent.

12. Variational treatment of dissipation

The only modes available for this investigation are those calculated (in Part II) without dissipation. The formalism based on either least-squares or spectral estimation outlined above must therefore be adapted to the use of nondissipative modes.

One way to do this is to assume each eigenelement to be a departure of order \( D \) (the dissipation matrix) from its nondissipative state:

\[
(\sigma_k, X_k) = (\hat{\sigma}_k, \hat{X}_k) + (\Delta \sigma_k, \Delta X_k),
\]

\[
(\sigma_k^*, Y_k) = (\hat{\sigma}_k, \hat{X}_k) + (\Delta \sigma_k^*, \Delta Y_k), \tag{12.1}
\]

where \((\hat{\sigma}_k, \hat{X}_k)\) denotes an eigenelement of the nondissipative tidal operator \( L \), and \( \Delta \) a variation whose magnitude is of the first order in \( D \). By inserting (12.1) into the eigenproblems (7.2) and (7.4), we get the variational equations

\[
(\hat{L} - \hat{\sigma}_k B) \Delta X_k = - (\Delta L - B \Delta \sigma_k) \hat{X}_k + O(D^2),
\]

\[
(\hat{L} - \hat{\sigma}_k B) \Delta Y_k = - (\Delta L^H - B \Delta \sigma_k^*) \hat{X}_k + O(D^2). \tag{12.2}
\]

In view of (7.1) we have \( \Delta L = iD \) and \( \Delta L^H = -iD \).

It is convenient to normalize the nondissipative eigenfunctions to \( \|X\|^2 = 1 \), so

\[
\hat{X}_j^H B \hat{X}_k = \delta_{jk} \tag{12.3}
\]

in view of their mutual orthogonality. The dissipative eigenfunctions are best normalized by

\[
\hat{X}_k^H B Y_k = 1; \quad \hat{X}_k^H B X_k = 1 \tag{12.4}
\]

which means that in the expansions of \( X_k \) and \( Y_k \) over \([L]\) the coefficient of \( \hat{X}_k \) is \( i = 1 \). It follows that \( \Delta X_k \) and \( \Delta Y_k \) are orthogonal to \( \hat{X}_k \).

The variational problem (12.2) is well known. On premultiplying either equation by \( \hat{X}_k^H \), we get zero on the left-hand side, in view of (7.2), so the right-hand side leads to the solvability condition

\[
\Delta \sigma_k = i \hat{X}_k^H D \hat{X}_k + O(D^2). \tag{12.5}
\]

Note that to first order \( \Delta \sigma_k \) is imaginary, because \( D \) is symmetric. An alternative statement of (12.5) is

\[
\Delta \sigma_k = \frac{1}{2} i \sigma_k^2 / Q_k \tag{12.6}
\]

in terms of mode quality as defined in (9.3).

The solution of (12.2) can be completed by premultiplication with \( \hat{X}_j^H (j \neq k) \). From the first equation we get

\[
\hat{X}_j^H B \Delta X_k = - (\hat{\sigma}_j - \hat{\sigma}_k)^{-1} i \hat{X}_j^H D \hat{X}_k + O(D^2) \tag{12.7}
\]

which is the coefficient of \( \hat{X}_j \) in expansion of \( \Delta X_k \) over \([L]\). The second equation makes

\[
\Delta Y_k = - \Delta X_k + O(D^2) \tag{12.8}
\]

because the right-hand sides of the two equations are negatives of each other, to \( O(D) \). Hence it suffices to solve (12.2) only for \( \Delta X_k \).

The physical interpretation of (12.5) can be seen by introducing the expression given in (4.2) for the matrix elements of \( D \):

\[
\Delta \sigma_k = i P^{-1} \int \theta |Z_0|^2 dP + O(\theta^2). \tag{12.9}
\]

Here \( \hat{Z}_0 \) is sea-surface elevation of nondissipative mode \( k \), distributed continuously by finite-element interpolation from nodal values normalized in accordance with (12.3). On comparing the integral in (12.9) with the third in (5.2), we see that it is proportional to the outward flux of energy in dissipative mode \( k \).

The integral in (12.9) can be evaluated explicitly in one special case, namely for the mass mode \( k = 0 \), which has uniform elevation \( \hat{Z}_0 \) in its nondissipative state. In fact, the normalization (12.3) makes \( \hat{Z}_0 = 1 \) so (if \( \theta \) is assigned to be uniform on the boundary)

\[
\Delta \sigma_0 = i \theta + O(\theta^2). \tag{12.10}
\]

The value \( \theta = 0.0733 \) cpd cited previously (Section 4) gives the dissipative mass mode an exponential decay time of about 14 days.

13. Secondary completeness error

Having thus found the means to determine \( \sigma_k, X_k, \)

\( Y_k \) (to lowest order in \( D \)), we can complete the synthesis (10.1), in principle, by evaluating either the least-squares \( A_k \) from (11.4) or the spectral \( A_k \) from (11.6). In practice, however, the representation of
\( \Delta X_k \) over \([L]\) must be limited to the available subset \( S[L] \) of nondissipative eigenfunctions. In other words, in addition to a second-order variational error, \( \Delta X_k \) (and \( \Delta Y_k \)) will have a completeness error (of first order).

This means that after substitution of \( \hat{X}_k + \Delta X_k \) for \( X_k \) and replacement of \( \Delta X_k \) by its incomplete expansion, the sum in (10.1) will span only \( S[L] \):

\[
S(X) = \hat{S}(X) + CO(D) + VO(D^2),
\]
\[
\hat{S}(X) = \|\hat{X}\| \sum a_k \hat{X}_k.
\]  

(13.1)

(Variational and completeness errors are indicated by \( VO \) and \( CO \).) The coefficients in \( \hat{S}(X) \) are not the same as those in (10.1) but can be determined from the latter by implementing the solution of the variational problem (next section).

We can summarize by saying that the variational treatment of dissipation by means of nondissipative eigenfunctions introduces a second completeness error in the synthesis of the tide response \( X \). The primary error is in the restriction of \( S(X) \) to the span of \( S[L] \). The secondary error is in the restriction of all \( X_k \) in \( S[L] \) and \( Y_k \) in \( S[L^H] \) to the span of \( S[L] \). While the primary error is formally moderated by least-squares or spectral estimation, the only remedy for the secondary error is to put an adequately large number of modes into \( S[L] \).

A beneficial consequence of the secondary completeness error is that the least-squares and spectral constraints are effectively equivalent. This can be seen by writing (11.3) and (11.5) as

\[
\hat{X}_j^H B R = \pm \Delta X_j^H B R.
\]  

(13.2)

The lower sign is for (11.3), the upper for (11.5) in view of (12.8). Now, the complete expansion of \( \Delta X_j \) can be partitioned into one part over \( S[L] \) and a second part over the rest of \([L]\). An individual term in the expansion contributes on the right-hand side of (13.2) the product of the complex conjugate of its expansion coefficient and the factor \( \hat{X}_j^H B R \). The former factor is of \( O(D) \) by (12.7) and the latter is of \( O(D) \) if \( \hat{X}_j \) is in \( S[L] \), by (13.2). The first part of \( \Delta X_j \) therefore contributes only at \( O(D^2) \) on the right-hand side of (13.2), so we can write

\[
\hat{X}_j^H B R \approx 0 + O(D^2),
\]  

(13.3)

where \( \approx \) signifies incidence of the secondary completeness error, of \( O(D) \), that arises from ignoring the part of \( \Delta X_j \) outside of \( S[L] \). (The symbol \( \approx \) is used exclusively in this sense in the sequel.)

Since (13.3) applies for either sign in (13.2), we can say that, apart from the secondary completeness error (which is unavoidable), least squares and spectral estimation must lead to synthesis coefficients that differ respectively only to \( O(D^2) \).

14. Variational estimation

The coefficients \( a_k \) in (13.1) can be determined from those of (10.1) by using the solution of the variational problem to evaluate the right-hand side of (11.6) and to convert the span of the sum in (10.1) from \( S[L^H] \) to \( S[L] \).

We can get the same result in a simpler way, which avoids explicit reference to the \( \Delta X_k \), by applying the residual constraint (13.3) directly to the residual obtained by substituting (13.1) into (11.1):

\[
R = \sum a_k \hat{\sigma}_k + iB^{-1}D - \omega \hat{X}_k - W + O(D^2)
\]

with \( W \) as in (11.2). On applying (13.3) and making use of (12.5), we get, for the \( a \)'s of (13.1):

\[
a_j(\sigma_j - \omega) + i\hat{X}_j^H B \sum_{k \neq j} a_k \hat{X}_k \approx \hat{X}_j^H B W + O(D^2).
\]  

(14.1)

We must solve this nondiagonal system.

A first-order solution of (14.1) can be found without matrix inversion by noting that in the sum, an error of \( O(D) \) is permitted in \( a_k \) because of the \( D \)-factor in front, so the estimate of \( a_k \) provided by the inhomogeneous term suffices. The latter term can be written

\[
\hat{X}_j^H B W = \hat{\sigma}_j \hat{a}_j,
\]  

\[
\hat{a}_j = \hat{X}_j^H B \hat{X}/\|\hat{X}\|.
\]  

(14.2)

Thus, by ignoring the sum in (14.1) we find the estimate in question to be (writing \( k \) for \( j \))

\[
a_k \approx (\sigma_k - \omega)^{-1}\hat{\sigma}_k \hat{a}_k + O(D^2).
\]  

(14.3)

Insert this for the \( a \)'s in the sum in (14.1) and define the auxiliary synthesis

\[
X' = \|\hat{X}\| \sum (\sigma_j - \omega)^{-1}\hat{\sigma}_j \hat{a}_j \hat{X}_j.
\]  

(14.4)

Then the solution of (14.1) is (writing \( k \) for \( j \))

\[
a_k \approx (\sigma_k - \omega)^{-1}\left[1 + (\sigma_k - \omega)^{-1}\Delta \hat{\sigma}_k \hat{a}_kight]
\]

\[
- i\hat{X}_k^H D X'/\|\hat{X}\| + O(D^2).
\]  

(14.5)

This is the working formula I have used in (13.1) as the computational basis for tidal synthesis. Unlike the coefficients (11.6) derived from true spectral estimation, the \( a_k \) of (14.5) are not strictly intrinsic properties of the solution, because the presence of \( X' \) makes them depend upon the span of the synthesis.

The reader may have noticed an inconsistency in one step leading to (14.5), namely, the use of \( \hat{\sigma}_k \) rather than \( \hat{\sigma}_k \) in the resonance factor \((\sigma_k - \omega)^{-1}\) of (14.3) (and thus in \( X' \)). Although \( \hat{\sigma}_k \) would suffice formally to give a zero-order estimate of \( a_k \), it would have the undesirable effect of permitting an arbitrarily large response if \( \omega \) is close to resonance. By retaining \( \sigma_k \) with its imaginary part, we obtain a response limited by the \( Q \) of the resonant mode. The same
remark applies to the factor that multiplies $\Delta \sigma_k$ in (14.5).

The structure of $D$ is such that in the $X'$-term in (14.5) only the elevation part of $X'$ is needed, and moreover, only the boundary values of elevation. However, there is some merit in looking at the whole $X'$ because, if there were no dissipation, (14.5) would reduce to $(\delta_k - \omega) \delta_k \tilde{\sigma}_k$, and so if $\sigma_k$ in (14.4) were replaced by $\tilde{\sigma}_k$, the result would be exactly the response for a model without dissipation. The $X'$ of (14.4) can therefore be described as a synthesis in which dissipation is provided for in the resonance magnification factor $(\sigma_k - \omega)^{-1} \tilde{\sigma}_k$ but not in its effect on the shapes of the eigenfunctions. This “quasi-dissipative” synthesis is sometimes used (for example, Garrett and Greenberg, 1977) to approximate the “fully-dissipative” synthesis determined by (14.5). Its merit is that it requires only a value of $\Delta \sigma_k$, and this can be inferred from (12.6) without postulating a specific mechanism for dissipation, provided an acceptable value of $Q_k$ is available.

15. Computation procedure

The first step in implementing (14.5) is to compute all $\tilde{\alpha}_j$ in $S[L]$ from (14.2). In view of the structure of $\check{X}$, only the $B11$ block of $B$ is needed there, so we find from (4.2) that the numerator and denominator of $\tilde{\alpha}_j$ can be evaluated as

$$\check{X}^H B \check{X} = S^{-1} \int \check{Z}^H \check{Z} dS,$$

$$|\check{X}| = (S^{-1} \int |\check{Z}|^2 dS)^{1/2},$$

(15.1)

where $\check{Z}_j$ (nondimensional) and $\check{Z}$ (dimensional) are sea-surface elevations respectively in nondissipative mode $j$ and in the equilibrium tide, distributed continuously by finite-element interpolation from the nodal values of $\check{X}_j$ and $\check{X}$. The integrals needed in (15.1) are evaluated over each grid triangle by interpolating $\check{Z}_j$ and $\check{Z}$ linearly from their respective values at the three vertices.

Next, all $\Delta \sigma_k$ are computed by means of (12.9). There the integral is evaluated over each segment of the boundary by linear interpolation of $\check{Z}_k$ between the end points of the segment. In the computation whose results are described in the next part of this study, I took the radiation parameter $c$ (hence also $\theta$) to be uniform on the entire boundary.

The quasi-dissipative tide $X'$ (dimensional) defined in (14.4), with $\sigma_k = \tilde{\sigma}_k + \Delta \sigma_k$, can now be evaluated by summation over the span of $S[L]$. Finally, the expression that involves $X'$ in (14.5) is evaluated from

$$\check{X}_k^H D X' = \rho^{-1} \int \theta \check{Z}_k Z' dP$$

(15.2)
in accordance with the matrix elements of $D11$ given in (4.2). The integral here is expressed in terms of nodal values by linear interpolation of $\check{Z}_k$ and $Z'$ within each boundary segment.

This completes the information needed to calculate $a_k$ from (14.5). The synthesis (13.1) can then proceed.

16. Discussion

The main weakness of the synthesis procedure described here is that it is based on nondissipative modes. Although useful results can be expected as long as the tidal $Q$ is at least of order 10 (and that appears to be the case), it would be more satisfactory to use dissipative modes calculated ab initio from a dissipative tidal operator. Another defect of the procedure is that it does not take account of tidal loading and self-attraction.

At the outset the investigation of oceanic normal modes seemed to me to pose severe computational difficulties, and indeed, it was not until the Lanczos method of tridiagonalization was adopted that steady progress began. For this reason I chose to deal with the simpler case of nondissipative modes, which presents the method with a symmetric matrix. The computational cost of the dissipative case must, of course, be greater, and moreover, the loss of orthogonality of the Lanczos functions may be more troublesome. Nevertheless, computation of the spectrum of the fully-dissipative tidal operator may now be a reasonable task to undertake. The spectrum of the adjoint operator would also be needed; in fact, the Lanczos method produces the two spectra simultaneously.

Apart from computational matters, it could fairly be argued that a substantial effort aimed at the dissipative operator is not warranted until a wider consensus is reached on the proper modeling of tidal dissipation. It also should be said that a synthesis based on dissipative modes is attended by a conceptual disadvantage, namely, that it does not admit an energy spectrum. In other words, the total energy is not expressible as a sum of squares of spectral components. This is simply a consequence of the nonorthogonality of eigenfunctions of a non-Hermitian operator. An appropriate analogy is to the use of oblique coordinates to express the square of the length of a vector.

A final point concerns the choice of operator whose spectrum is to be used in the synthesis. I retained the “primitive” operator—that is, the operator formed from first-order differential equations in which velocities and sea-surface elevation are dependent variables. The eigenvalues of this operator are equal to the frequencies of the normal modes. Expansion of arbitrary sea states in eigenfunctions of the primitive operator was investigated by Proudman in a series of papers over the years 1917–31. The spectral value (11.6) of the expansion coefficient, when expressed as an integral in terms of sea-surface elevation, is equivalent to equation (2.61) in Proudman (1928).

In principle, it is possible to use a second-order operator, obtained by eliminating either the elevation or the velocity, as is commonly done in the calculation.
of tides by direct numerical integration. For example, if the velocity is eliminated, the result is Laplace’s elevation equation

\[ M(\omega)(Z - \tilde{Z}) = Z, \]

\[ M(\omega) = -\nabla \cdot \left( \frac{gs(\omega + i\beta k x)}{\omega(\omega^2 - f^2)} \right) \nabla, \] (16.1)

where \( \beta \) is the Coriolis parameter.

The solution of the tidal problem (16.1) can be synthesized from the spectrum of \( M(\omega) \)—that is, by solving the eigenproblem

\[ M(\omega)Z_k = \lambda_k Z_k. \] (16.2)

The eigenfunctions \( Z_k \) are mutually orthogonal (when there is no dissipation); in an ocean of uniform depth they are the familiar Hough functions. The eigenvalues \( \lambda_k \) are parametrically dependent on \( \omega \). From a computational standpoint there is an economy in (16.2) gained by the fact that the number of degrees of freedom in this problem is about one-third that of the primitive eigenproblem (because the velocity has been eliminated). On the other hand, this economy is offset by the increase in bandwidth of the matrix image of the operator, caused by the doubling of differential order that attends the elimination. I am not aware of any comparison between the two methods (primitive and derivative) on the basis of arithmetic operation count.

Although a synthesis based on the spectrum of \( M(\omega) \) might be computationally more efficient than that derived from the primitive operator, it has some relative disadvantages. Foremost, from the standpoint of the present study, is the fact that the eigenelement \( (\lambda_k, Z_k) \) of \( M(\omega) \) corresponds to a normal mode (with frequency \( \omega \)) not of the real ocean with depth \( h \) but rather of a fictitious ocean with “equivalent” depth \( h/\lambda_k \). In other words, the spectrum of \( M(\omega) \) does not represent normal modes of the actual ocean, and thus we lose an important means of giving a realistic physical interpretation to the components of the synthesis.

An alternative point of view is to suppress \( \tilde{Z} \) in (16.1) and regard \( \omega \) as an eigenfrequency, then solve \( M(\omega)Z = Z \) for these frequencies and for the elevation functions \( Z \) that belong to them. This is equivalent to seeking solutions of (16.2) with \( \lambda_k = 1 \). However, such solutions are not in themselves orthogonal: they are, in fact, the elevation parts of the sea-state eigenfunctions of the primitive eigenproblem. Hence, they must be supplemented by velocities in order to express orthogonality. The synthesis would therefore be essentially the same as that of the present study.

17. Addendum

It will be useful, finally, to mention three comments made by a referee. One concerns the primary completeness error—that is, restriction of the synthesis to only about 6 percent of the modes admitted by the discretization. Can an a priori estimate of this error be given? I do not know of a way to do so. In principle, the question could be answered a posteriori by comparing the energy of the synthesized tide with that of a solution obtained by direct integration without synthesis (provided the latter solution is subject to the same variational treatment as is discussed in Section 12). Regrettably, I have not made that calculation.

Another question refers to the “quasi-dissipative” tide defined in (14.4); is it a reasonably good approximation to the “fully-dissipative” tide? (The latter takes account of effects of dissipation on shapes of the modes, whereas the former does not.) I have not mapped the quasi-dissipative tides, and therefore cannot answer this question directly. However, energy and dissipation diagnostics indicate that the quasi-dissipative tide may be a good approximation. For example, the potential energy of the quasi-dissipative \( M_2 \), in percent of total energy, is 43.4, while for the fully-dissipative \( M_2 \) it is 42.7. The difference in these values is not larger than modeling uncertainties.

The third comment is that, by permitting the radiation parameter \( c \) to be complex, we can account for the phase shift that occurs when some of the energy received by the shelf is reflected; and by permitting \( c \) to vary along the ocean boundary, we can account to some extent for nonuniformities in shelf morphology. The restrictions I adopted, that \( c \) be real and uniform, are indeed severe. Any attempt to refine the present calculations should be accompanied by a more realistic treatment of radiation, such as has been examined in recent work by Gotlib and Kagan, cited in Part IV of this study.

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