A Simple Model for Deep Equatorial Zonal Currents Forc ed at Lateral Boundaries*

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ABSTRACT

Deep lateral boundary processes (e.g., western boundary currents) are hypothesized as an alternative energy source exciting the equatorial wave guide at long time scales. A linear, continuously stratified model is used to study the equatorial zonal currents generated by a time dependent, short vertical scale deep zonal jet located at the meridional walls and centered at the equator. Examples of solutions with periodic, transient and spectral forcing are presented. For low frequency forcing at the western or eastern boundaries, energy travels from the source along ray paths associated with Kelvin and long Rossby waves, respectively. Linearly damped solutions look similar in both cases.

Solutions show in general a rich baroclinic structure and a complex time dependence (e.g., periodic solutions can exhibit both upward and downward phase propagation and standing mode oscillations at different depths in the water column), with the vertical structure depending, among other factors, on the vertical scale and frequency composition assumed for the boundary jet. Results suggest the potential importance of deep forcing mechanisms to the existence of long time scale, deep baroclinic currents in the equatorial ocean. Solutions are qualitatively similar to observations of the equatorial deep jets, but any detailed comparison between model results and data is premature, given the lack of observational knowledge about the time scales, strength and spatial distribution of deep energy sources.

1. Introduction

Observational studies of the deep equatorial circulation have revealed the presence of strong zonal currents of long time and short vertical scales (e.g., O’Neill and Luyten 1984; Eriksen 1981; Firing 1987; Ponte and Luyten 1989). The energy source associated with these flows, generically called equatorial deep jets in the literature, remains a challenging issue. The interpretation of the jets in terms of low-frequency surface-forced linear equatorial waves has been advocated in the past (e.g., Wunsch 1977; McCarey 1984). Wunsch’s model jets consist of long Rossby waves forced at the annual period by a surface vertical velocity pattern of a particular zonal wavenumber and unbounded in x. As pointed out later by McCarey (1984), for reasonably long time scales, equatorial waves propagate energy into the deep ocean at very shallow angles to the surface. Wunsch’s ability to generate currents in the deep ocean was related to the infinite zonal extent of his forcing function. McCarey (1984) was able to inject considerable energy into the ocean’s interior with a wind patch of finite zonal extent, by considering reflections at oceanic boundaries.

Recent observational results in the central Pacific (Ponte and Luyten 1989) suggest time scales of several years for the energetic deep flows. Waves of such long periods and short vertical wavelengths have extremely small vertical group velocities. Moderate amounts of damping would most likely cause them to strongly decay in the vertical, away from the forcing region. Thus, a surface energy source for these waves seems problematic. As an alternative approach, we consider the hypothesis of having a deep energy source exciting the equatorial wave guide. Kawase (1987) used a shallow water model to study the linear response of the deep ocean to an off-equatorial mass source, located at the western boundary and mimicking deep water formation processes. His steady, moderately damped solutions exhibit strong equatorial zonal flows. Spinup calculations illustrate how these flows are set up by equatorial Kelvin waves, which connect to the coastal Kelvin waves carrying the signal from the source to the equator along the western boundary. Some indication that these processes could be at work is found in the equatorial Atlantic tracer fields, especially in chlorofluorocarbons (Weiss et al., 1985).

Although observational knowledge about deep boundary currents is relatively scarce (e.g., Warren 1981), these currents probably fluctuate at interannual time scales, reflecting changes occurring at the source.

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These fluctuations would in turn drive a response in the equatorial region, as illustrated by Kawase (1987). It is the character of this response that is considered, in very simple terms, in the rest of this paper. In section 2, we find the general mathematical form of solutions to a linear model forced at the meridional wall by a time dependent deep zonal jet. Both western and eastern boundary forcing cases are treated, although strong deep energy sources are more likely to exist at the western walls (e.g., Stommel 1958). We do not address the details on how the forcing jet came to existence, but instead concentrate our effort in learning about the character of the forced solutions, particularly the zonal velocity fields. Sections 3, 4 and 5 present examples of solutions for periodic, transient and spectral forcing, respectively. The final section discusses the relevant features of the model in the context of the observations.

2. The model and its mathematical treatment

Consider an equatorial $\beta$-plane, constant buoyancy frequency $N$ ocean, with top and bottom boundaries at $z = \pm H/2$, and unbounded to the west or east of a meridional wall placed at $x = 0$, depending on whether we want to study the ocean response to eastern or western boundary forcing, respectively. The model ocean is forced at the meridional wall by a time-dependent zonal jet, confined to a certain depth and centered at the equator. The assumption of a zonal jet seems reasonable (e.g., Kawase 1987) and should be the most efficient in forcing strong zonal flows at the equator. A convenient form for the boundary condition at $x = 0$ which simplifies the analytical solutions is

$$u = \left\{ \begin{array}{ll} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{U}_0(\omega) e^{i\omega t} d\omega \cos(m_0z) e^{-\mu z^2}, & |z| \leq \frac{l}{2} \\
0, & |z| > \frac{l}{2} \end{array} \right.$$ \hspace{1cm} (1)

where $m_0 = \pi/l$, $\hat{U}_0(\omega)$ is the Fourier transform in time, $\mu$ gives the latitudinal decay scale of the jet and $l$ its vertical extent. The forcing jet consists of a half-cosine centered at mid-depth.

Forcing at lateral boundaries is significantly different than the more usual surface forcing, in which the frequency and zonal wavenumber spectra are specified and the ocean picks the vertical scales accordingly. Wunsch (1977) was able to select a dominant vertical scale in his solutions by choosing a forcing function of a particular period and zonal wavelength. This is not so in our case, since the forcing function is confined to a certain depth and has a well defined vertical wavenumber spectrum. Instead, the vertical extent $l$ of the boundary jet will play a more significant role in selecting a basic scale for the response.

The forced solutions will consist of equatorial waves propagating energy away from the source region (see Appendix A or Pedlosky 1979 for the theoretical treatment of these waves). The complete set of mutually orthogonal modes includes Kelvin, Rossby, Yanai and inertio-gravity waves. In the low frequency limit studied here, only Kelvin and long Rossby waves are efficient carriers of energy eastward and westward, respectively. Dissipation causes short Rossby waves to decay rapidly away from the forcing region. Therefore, only Kelvin and long Rossby waves will be included in the model for western and eastern boundary forcing, respectively.

a. Western boundary forcing

In general, we can express the forced solution as a sum of an infinite number of Kelvin wave vertical normal modes (e.g., McCreary 1984), i.e.,

$$u = \frac{1}{2\pi} \sum_{n=1}^{\infty} A_n \int_{-\infty}^{+\infty} \hat{u}(\omega) e^{-i(kx-\omega t)} d\omega$$

$$\times \exp \left\{ -\frac{\beta y^2}{2c_n} \right\} \cos \frac{N}{c_n} z \hspace{1cm} (2)$$

where $A_n$ is a constant, $\hat{u}(\omega)$ is the Fourier transform in time, $\omega$ and $k$ are dimensional frequency and zonal wavenumber respectively, related by the dispersion relation $\omega = c_n k$, and $c_n$ is given by

$$c_n = \frac{NH}{2\pi n} \hspace{1cm} (3)$$

where $H$ is the total ocean depth. The eigenvalues $c_n$ are the phase speeds of each vertical mode in an ocean with rigid boundaries at $z = \pm H/2$. The barotropic mode $n = 0$ is not included in the sum in (2), since it is not relevant when looking for jetlike flows.

To determine the forced solutions, one has to match expression (2) to the boundary condition (1) at $x = 0$ to find the coefficients $A_n$ and the form of the Fourier transform $\hat{u}(\omega)$. In general, the meridional structure chosen in (1) will force both Kelvin and short Rossby waves and determination of their relative amplitudes is not trivial (e.g., McCreary 1985). But in the low frequency limit, since the short Rossby waves carry no net mass flux, all the transport associated with the boundary jet has to be carried by the Kelvin waves (e.g., Cane and Sarachik 1979), allowing the amplitudes in (2) to be determined independently. Expanding the forcing in (1) as a sum over the complete set of free vertical modes and equating the expansion coefficients of the forcing to those of the solution (2) evaluated at $x = 0$, we arrive at
\[ \hat{u}(\omega) = \hat{U}_0(\omega) \quad (4a) \]

\[ A_n = \cos \left( \frac{n\pi l}{H} \right) \left( \frac{4m_0 H}{(Hm_0)^2 - (2\pi)^2} \right)^{1/2} \quad (4b) \]

where we have used the orthogonality of the vertical structure functions. The square root term comes from matching the Kelvin wave transport to the mass flux at the boundary. The above expression implies that \( A_n \sim n^{-3/2} \). This is helpful when we numerically evaluate the sum over vertical modes in (2), assuring for rapid convergence of the solutions.

The effect of linear friction on these solutions can be easily accommodated by letting the Kelvin wave dispersion relation become

\[ \omega = c_n k + ir \quad (5) \]

where \( r \) is the friction coefficient. This is equivalent to having Rayleigh friction terms as well as Newtonian cooling in the Kelvin wave equations, with the additional assumption of these coefficients being equal (see McCreary 1985). The solutions are then modified by the inclusion of an exponential term of the form

\[ \exp \left( -\frac{r}{c_n} x \right) \quad (6) \]

which simply results from solving the dispersion relation (5) for \( k \) and substituting it in (2). Each vertical mode decays in \( x \), as it propagates away from the forcing region. The decay scale \( c_n/r \) is inversely proportional to \( n \) (high modes travel slower and therefore do not penetrate as far into the interior as lower modes, before being dissipated).

b. Eastern boundary forcing

If the forcing is placed at the eastern wall, only long Rossby waves are available to carry energy westward away from the source, at low frequencies. The meridional structure of the zonal velocity associated with equatorial Rossby waves is given by expression (A6b). Using dispersion relation (A7) in (A6b), a general field of long Rossby waves can be written as

\[ u = \frac{1}{2\pi} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} A_{nj} \int_{-\infty}^{+\infty} \hat{u}(\omega) e^{-i(kx - \omega t)} d\omega \]

\[ \times \cos \left( \frac{N}{c_n} z \right) \left[ \frac{\psi_{j-1}}{\sqrt{2j}} - \frac{\psi_{j+1}}{\sqrt{2(j+1)}} \right] \quad (7) \]

involving a double sum over vertical and meridional mode numbers \( n \) and \( j \), respectively.

The problem of determining the long Rossby wave response to the forcing prescribed in (1) is similar to the problem of finding the Kelvin wave reflection from an eastern boundary, treated originally by Moore (1968). In his case, the “forcing” Kelvin wave field has a meridional structure given by \( \psi_0 \). To cancel this flow at the boundary, a \( j = 1 \) long Rossby wave is needed. The \( \psi_2 \) term generated this way is in turn canceled by adding a \( j = 3 \) wave and so on, leading to an infinite sum of westward and poleward propagating wave modes (for large enough \( j \), the zonal wavenumber becomes complex and the modes are trapped to the coast). Here, the flow at the boundary has a more general latitudinal structure. We expand the forcing (1) as a double sum over the complete set of vertical and meridional modes of Appendix A and solve for the constants \( A_{nj} \) by matching (7) to the boundary condition at \( x = 0 \). Using the orthonormality of the Hermite functions yields

\[ \hat{u}(\omega) = \hat{U}_0(\omega) \quad (8a) \]

\[ \sum_{j=1}^{\infty} A_{nj} \left[ \frac{\psi_{j-1}}{\sqrt{2j}} - \frac{\psi_{j+1}}{\sqrt{2(j+1)}} \right] = B_n \sum_{j=1}^{\infty} \psi_j \int_{-\infty}^{+\infty} \exp \left\{ -\left( \frac{\mu c_n}{\beta} \right)^2 \eta^2 \right\} \psi_j(\eta) d\eta \]

where \( B_n \) is given by

\[ B_n = \cos \left( \frac{n\pi l}{H} \right) \left( \frac{4m_0 H}{(Hm_0)^2 - (2\pi)^2} \right) \quad (9) \]

and the nondimensionalization (A3) was used to express the integral in terms of the nondimensional variable \( \eta \). This procedure is equivalent to Moore’s method. However, solution (7) is only representative of the far-field response because it does not contemplate the possibility of coastally trapped modes being forced. In addition, it is only strictly valid for values of \( j \) for which \( k \) is real, as determined by the dispersion relation (A5). In any case, for the frequency and friction values considered here, only the lowest meridional modes contribute to the solutions and the upper limit on the summation over \( j \) is not important.

Since the forcing function is even in \( \eta \), only Hermite functions of even \( j \) will be part of the expansion. Then, by equating coefficients in (8b), we obtain

\[ \frac{1}{\sqrt{2}} \left[ \frac{A_{2m+1}}{V_{2m+1}} - \frac{A_{2m-1}}{V_{2m}} \right] = B \int_{-\infty}^{+\infty} \exp \left\{ -\left( \frac{\mu c_n}{\beta} \right)^2 \eta^2 \right\} \psi_{2m}(\eta) d\eta \quad (10) \]

for \( m = 1, 2, \cdots \), where for simplicity we have dropped the subscript \( n \) pertaining to the vertical mode number under consideration (the only subscript refers to meridional mode number). Similarly, for the special case of \( m = 0 \), one easily arrives at

\[ \frac{A_1}{V_2} = B \int_{-\infty}^{+\infty} \exp \left\{ -\left( \frac{\mu c_n}{\beta} \right)^2 \eta^2 \right\} \psi_0(\eta) d\eta \quad (11) \]
The integrals in expressions (10) and (11) can be evaluated with the help of a table (Gradsteyn and Ryzhik 1965), yielding

\[ \frac{A_j}{V_2} = B \pi^{1/4} \left( q + \frac{1}{2} \right)^{-1/2} \]

(12a)

\[ \frac{1}{V_2} \left[ \frac{A_{2m+1}}{V_{2m+1}} - \frac{A_{2m-1}}{V_{2m}} \right] = B \frac{(\sqrt{\pi}(2m)!)}{2^m m!} \left( q + \frac{1}{2} \right)^{-1/2} \left( \frac{1/2 - q}{1/2 + q} \right)^m \]

(12b)

for \( m = 1, 2, \ldots, \) and \( q = \mu c_n / \beta. \) Finally, putting \( 2m + 1 = j \) in (12b) leads to the following recursion relation

\[ \frac{A_j}{V_2} = B \frac{(\sqrt{\pi} j)!(j-1)!}{2^{(j-1)/2} \left( \frac{j-1}{2} \right)!} \left( q + \frac{1}{2} \right)^{-1/2} \]

\[ \times \left( \frac{1/2 - q}{1/2 + q} \right)^{(j-1)/2} + \left( \frac{j}{2(j-1)} \right)^{1/2} A_{j-2} \]

(13)

for \( j = 3, 5, \ldots, 2m + 1. \) Only odd \( j \) will enter when actually computing the sums in (7).

Introducing linear friction in these solutions amounts again to letting the dispersion relation (A7) become, in dimensional terms,

\[ \omega = - \frac{c_n k}{2j + 1} + ir. \]

(14)

Solutions will have an extra exponential decay term of the form

\[ \exp \left\{ \frac{r}{c_n} (2j + 1) x \right\}. \]

(15)

The zonal decay scale still decreases with vertical mode number \( n, \) but there is an additional strong dependence on meridional mode number \( j \) as well (higher meridional modes have slower zonal group speeds, therefore they are more closely trapped to the eastern boundary). The inclusion of linear damping improves the convergence rate of the sums over \( n \) and \( j \) in the solutions (7).

3. Solutions with periodic forcing

In this section, the response of the model to periodic forcing in time is studied. The Fourier transform \( U_0(\omega) \) in (1) is a Dirac delta function and the time dependence of the forcing takes the simple \( \cos(\omega t) \) form. The Kelvin wave solution (2) becomes

\[ u = U_0 \sum_{n=1}^{\infty} A_n \cos(k x - \omega t) \exp \left\{ - \frac{\beta y^2}{2 c_n} \right\} \cos \frac{N}{c_n} z \]

(16)

while the Rossby wave solution (7) becomes

\[ u = U_0 \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} A_{n,j} \cos(k x - \omega t) \times \cos \frac{N}{c_n} z \left[ \psi_{j+1} - \frac{\psi_{j-1}}{\sqrt{2(j+1)}} \right] \]

(17)

with the coefficients \( A_n \) and \( A_{n,j} \) still given by (4b) and (12), respectively. Here, \( U_0 \) is the amplitude of the forcing.

Before calculating any solutions, we choose the following values for the various parameters appearing in the model:

\[ H = 3500 \text{ m} \quad (18a) \]
\[ l = 200 \text{ m} \quad (18b) \]
\[ \mu = 1.6 \times 10^{-10} \text{ m}^{-2} \quad (18c) \]
\[ U_0 = 10 \text{ cm s}^{-1} \quad (18d) \]
\[ T = \frac{2 \pi}{\omega} = 4 \text{ yr} \quad (18e) \]
\[ N = 2 \times 10^{-3} \text{ s}^{-1}. \quad (18f) \]

The mid-depth forcing jet has a maximum amplitude of 10 cm s\(^{-1}\), oscillates at a period of 4 years and extends over 200 m in the vertical, with a latitudinal e-folding decay scale of approximately 80 km. These values are more or less arbitrary, since knowledge about the forcing boundary processes we are trying to model here is scarce at best. The rather long forcing period is intended to mimic interannual variability of processes such as deep water formation, and to replicate roughly the time scales inferred from observations in the central Pacific by Ponte and Luyten (1989). A typical average value was chosen for the buoyancy frequency (e.g., Ponte and Luyten 1989). The dependence of solutions on the choice of parameters will be discussed as we go along.

a. The Kelvin wave response

Solutions (16) for western boundary forcing were evaluated numerically, by summing the first 50 vertical modes [i.e., \( n \leq 50 \) in (16)]. This number was sufficient to assure convergence. Unless otherwise stated, figures presented here relate to solutions calculated with no friction.

Figure 1 shows contours of zonal current in the \( y-z \) plane, at a distance of 5000 km from the western boundary located at \( x = 0. \) The structure of the flow is dominated by two main eastward jetlike features with maximum amplitudes of about 3 cm s\(^{-1}\), centered on the equator at depths of roughly ±125 m, with weaker westward flow between them. No energy is seen in the remaining part of the water column. In addition, currents are strongly trapped to the equator, with decay
The upward (downward) going rays of Fig. 2 have positive (negative) vertical group velocities associated with them. Linear equatorial theory then implies that vertical phase speeds be in the opposite sense. This is illustrated in Fig. 3, which shows a time sequence over approximately 2 years of vertical profiles of zonal velocity, obtained on the equator at $x = 6000$ km. If we take for example the top ray, as time goes on we basically see the gradual appearance of a new westward jet, which emerges from the upper edge of the packet and propagates downward increasing in amplitude, while the strong eastward jet, which is present at the first time step in the sequence, slowly migrates down and decays in amplitude as it disappears on the lower edge of the packet. Notice also the apparent standing oscillation pattern underlying the evolution of the flow in the vicinity of $z = 0$. The energy seen in these weaker currents travels along both upward and downward going rays which emanate from off-center points in the source and cross each other over depths $|z| < 100$ m. In this region, waves with both upward and downward phase propagation coexist and the standing mode pattern of Fig. 3 results.

The maximum amplitudes attained by the main jets in these solutions are smaller than that of the forcing jet ($U_0 = 10$ cm s$^{-1}$) reflecting in part the dispersion of the energy available at the source between two diverging ray paths. In addition, amplitudes depend on how well the meridional structure of the forcing projects on each vertical Kelvin mode making up the solutions. If we computed a vertical wavenumber spectrum of the forcing (1), we would find most of the energy at the low wavenumbers. Thus, the amplitude of the solutions generally depends on how strongly these large vertical scales are forced. The factor $(\beta/2\mu_c)^{1/2}$ in (4b) suggests that the shorter the meridional extent of the forcing jet (i.e., the larger the value of $\mu$), the smaller the amplitudes of the response, simply because the low vertical modes are less strongly excited. Frictional processes can also affect the amplitudes of the response, but for spindown times $r^{-1}$ on the order of the forcing period, the zonal decay scales $r/c_b$ given in (6) are very large, especially for the lowest vertical modes which have the faster zonal phase and group speeds (e.g., for $n = 10$, the decay scale is 14 000 km). Thus, the effects of friction are relatively unimportant in these solutions (Ponte 1988).

b. The long Rossby wave response

Using the insight gained in the previous section, we again expect energy available at the eastern wall to propagate along rays, but now even for periodic forcing, an infinite number of paths are possible with slopes given by $\pm \omega(2j + 1)/N$. Each meridional mode propagates energy at different angles to the horizontal, with the source at shallow angles due to the very low frequency chosen for the forcing.
the steepest paths occurring for the highest modes. Surface and bottom reflections become more important in these solutions than in the previous section, where the very small slopes prevented the rays from hitting the top and bottom ocean boundaries within reasonable distances from $x = 0$. Friction will also have a more significant role since it will affect each of these rays and its reflections differently, according to (15).

The sums over meridional and vertical modes in the mathematical solutions defined by (17) and (12) were evaluated numerically. We carried the summation to $n = 50$ and $j = 69$, with only odd $j$ entering in the calculations. Figure 4 shows zonal velocity at the equator for a distance of 3000 km west of the forcing, and calculated with a linear damping coefficient equivalent to a spin-down time of 8 years. The vertical structure of these flows is similar to the previous Kelvin wave flows, but energy is not confined so closely to the depth at which the source is located. An alternating pattern is now visible throughout the water column, with a 4 cm s$^{-1}$ eastward jet at 250 m and weaker flows above. The lowest meridional modes are the most strongly excited. In fact, the maximum eastward and westward velocities in Fig. 4 occur at the same depths predicted by tracing the different ray paths associated with the first six odd meridional modes, and emanating from the point $(x, z) = (0, 0)$ (Ponte 1988). The divergence of ray paths associated with the more strongly forced gravest meridional modes accounts for the difference in vertical structure between solutions calculated at closely separated zonal distances (Ponte 1988).

It is useful to give the $e$-folding decay scales $c_n/[r(2j + 1)]$ for some meridional modes, for the case of $n = 1$. Table 1 shows the disparity in the scales between different meridional modes, resulting essentially from the large difference between their zonal group velocities. The presence of small amounts of friction causes energy traveling along the steeper ray paths to be severely damped before penetrating too far into the ocean interior. Linearly damped solutions are primarily determined by the first few meridional modes, especially in the far field. The reflection processes at the top and bottom boundaries become irrelevant.

The meridional structure of the solution in Fig. 4 is shown in Fig. 5. The latitudinal decay scale of the jet at 250 m closely follows that of the forcing ($\sim$80 km). All the other flows have magnitudes smaller than 1 cm s$^{-1}$, with the exception of the weak westward jet at 500 m, associated with meridional mode number $j = 3$. Although not computed, the time evolution of the flow field should follow the basic pattern of upward (downward) vertical phase propagation across each downward (upward) going ray, as observed for the Kelvin wave solutions.

4. Solutions for transient forcing

The periodic solutions analyzed in the previous section showed the importance of the frequency composition of the forcing in determining the vertical structure of the flow fields. An interesting situation may arise if the forcing is transient in character rather than
Consider a boundary jet with a time dependence of the form

$$U(t) = U_0 \exp\left(-\frac{t}{\Delta T}\right)^2$$

(19)

where the characteristic scale $\Delta T$ is assumed long enough so that no significant energy is present at high frequencies and hence, only Kelvin and Rossby waves are important in the oceanic response, as in the previous section. The simple exponential form is convenient when inverting the Fourier transforms in (2) and (7).

Using the Kelvin wave dispersion $\omega = c_n k$ to express $k$ in terms of $\omega$ in (2) and evaluating the Fourier integral, one obtains solutions of the form

$$u = U_0 \sum_{n=1}^{\infty} A_n \exp\left(-\frac{x}{c_n} - t\right)/\Delta T\right)^2$$

$$\times \exp\left(-\frac{\beta y^2}{2c_n}\right) \cos \frac{N}{c_n} z.$$  

(20)

To illustrate the basic differences between oscillatory and transient solutions, it suffices to consider the response to western boundary forcing (as suggested by previous results, Kelvin and Rossby solutions should be similar in character). The amplitude $U_0$ and the vertical and meridional scales for the boundary jet are given in (18). Again, only the first fifty vertical Kelvin modes were used in the sum in (20) and frictional effects were neglected.

It is helpful to consider what would happen if the model were forced with a Dirac delta function $\delta(t)$ ($\Delta T \to 0$). All vertical modes would be excited simultaneously at $t = 0$. As these modes propagated from the boundary, they would become separated due to their different propagation speeds. At a point far from the source, we would observe first the passage of the lowest mode which has the faster phase speed. All the

<table>
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other modes would follow sequentially. Thus, a continuously changing, dominant vertical scale would be observed, in contrast with the periodic solutions of section 3 (Ponte 1988). For finite values of $\Delta T$, more superposition of modes may occur, leading to stronger flows and slower changes of the dominant vertical scale with time.

Figure 6 shows a time sequence of the zonal flows calculated at a distance of 5000 km east of the forcing region and at $y = 0$, for a value of $\Delta T = 6$ months. Zonal flows do not exhibit an individual mode character, but are confined closely to the depth of the forcing jet. Several modes coexist at each time in the solutions. The absence of significant flows away from $z = 0$ is caused by destructive interference between modes. There is a clear increase in “baroclinicity” (i.e., number of wiggles) of the currents with time. At $t = 2$ months, a rich mixture of modes combine to give a flow pattern closely resembling what we would expect from a steady forcing, with one main jet centered at $z = 0$. At later times, the energy associated with the lowest modes has propagated faster to the east, leaving behind the higher baroclinic modes, which gradually gain more prominence in the solutions.

5. Forcing the model with a frequency spectrum

In some situations, a realistic description of the forcing involves defining its frequency spectrum. To obtain the response spectrum, we make the usual stochastic assumption of random phases between different frequency components of the forcing (e.g., Ponte 1986). In mathematical terms,

$$\langle \hat{U}_0(\omega) \hat{U}_0^*(\omega') \rangle = \Phi_f(\omega) \delta(\omega - \omega')$$  \hspace{1cm} (21)

where the angular brackets denote an ensemble average, $\hat{U}_0(\omega)$ is the Fourier transform of the forcing, the asterisk stands for complex conjugate, $\Phi_f(\omega)$ is the forcing spectrum and $\delta$ is the Dirac delta function.

Consider the case of western boundary forcing. We redefine the Fourier transforms of solution (2) to be

$$\hat{u}(x, y, z) = \sum_{n=1}^{\infty} A_n \hat{U}_0(\omega) e^{-i(x\nu/c_n)} \exp \left( - \frac{\beta y^2}{2c_n} \right) \cos \frac{N}{c_n} \hat{z}. \hspace{1cm} (22)$$

FIG. 5. Contours of zonal velocity in the $y-z$ plane for linearly damped Rossby wave solution computed at $x = -3000$ km, with a spindown time of 8 years. Contouring interval was 1 cm s$^{-1}$. Only the upper 1750 m are shown since solution is symmetric about the line $z = 0$.

FIG. 6. Time sequence of Kelvin wave solutions showing the zonal velocity field 5000 km east of the source region, at $t = 2$ months (solid), $t = 6$ months (dashed), $t = 12$ months (dotted) and $t = 18$ months (chaindotted). Transient forcing of the form (19) was used, with $\Delta T = 6$ months and $U_0 = 10$ cm s$^{-1}$. 
Then, multiplying (22) by its complex conjugate, taking the ensemble average on both sides and using assumption (21) yields

$$\Phi(\omega, x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_n A_m \Phi_f(\omega) Q(y, z) \exp\left(i \omega \left(\frac{1}{c_n} - \frac{1}{c_m}\right)\right)$$

(23)

where

$$Q(y, z) = \exp\left(-\frac{\beta y^2}{2} \left(\frac{1}{c_n} - \frac{1}{c_m}\right)\right) \cos \frac{N}{c_n} z \cos \frac{N}{c_m} z.$$  

(24)

Relation (23) gives the frequency power spectrum of the Kelvin wave response in terms of the forcing spectrum $\Phi_f(\omega)$, at each spatial point $(x, y, z)$. Notice that the frequency dependence of $\Phi(\omega, x, y, z)$ is not simply given by the spectral shape of $\Phi_f(\omega)$ but includes another factor dependent on the particular dynamics of the ocean response. Frictional effects can be accommodated by carrying the exponential decaying terms discussed in section 2 in the calculations.

One can compute using (23) the time variance associated with a particular frequency as a function of depth, at some point $(x, y)$. Such calculations (not shown here) reveal a peak in variance along the predicted energy ray paths associated with each frequency, just as expected from the discussion of the periodic solutions in section 3. Energy associated with different frequencies appears at separate depths in the water column. Alternatively, given some form for $\Phi_f(\omega)$, one can integrate expressions (23) over $\omega$ to obtain the total time variance (i.e., $\int \Phi_f d\omega$) as a function of depth. For illustrative purposes, one can assume a simple forcing function of the form

$$\Phi_f(\omega) = \begin{cases} 
\Phi_0, & 0 < \omega < \omega_0 \\
0, & \omega > \omega_0
\end{cases}$$

(25)

where $\Phi_0$ is a constant. Solutions were computed with $\omega_0 = 2 \times 10^{-7}$ s$^{-1}$, equivalent to a period of 1 year. Plots in Fig. 7a, b represent the total time variance associated with undamped Kelvin wave solutions evaluated on the equator at $x = 5000$ km and $x = 10000$ km, respectively. Amplitudes are normalized by the values at $z = 0$. Energy is expected within the cone traced by the steepest rays emanating from the source. This is why energy is seen further away from $z = 0$ at $x = 10000$ km than at $x = 5000$ km. Variance at mid-depth (i.e., near the depth of the forcing) is dominated by the lowest frequency motions while as one moves away from $z = 0$, higher frequencies become more and more important.

The relative absence of structure in the vertical distribution of energy depicted in Fig. 7 partly reflects the choice of a flat spectrum for the forcing. More complicated spectral shapes would change this result. For instance, the presence of peaks at some frequencies would show up as maxima in energy at depths coinciding with ray paths associated with those frequencies.

6. Discussing model results

The results of previous sections illustrate the basic nature of the linear response of the equatorial wave guide to low frequency forcing at lateral boundaries. Energy propagates away from the source along available ray paths predicted by linear wave theory. Frictional solutions for eastern and western boundary forcing are very similar in the far field. Their behavior in time can be complicated, even for the case of simple periodic forcing. The vertical scale of the response is set by the scale of the forcing. The baroclinic structure of the flow field is ultimately defined by the possible patterns of energy propagation, and thus sensitive to the frequency composition of the forcing jet. Other factors affecting that structure are the bending of rays caused by a variable buoyancy frequency profile and the actual depth of the energy source (Ponte 1988). In addition, the existence of more than one source at different depths (not contemplated here) may give rise to solutions with very complicated interference patterns.

Some of the model assumptions need a brief discussion. The absence of a second meridional wall is not realistic, but the flows associated with energy reflections off the coasts would be weaker than the ones
directly forced, due to additional damping and the possible partial reflectivity of the boundaries, and therefore would not contribute strongly to the solutions. The scales of the forcing were arbitrarily chosen. Although deep energy sources at the equator may have short vertical scales, as the observations of Weiss et al. (1985) seem to suggest, currents associated with them most probably do not reverse direction as implied in the examples with a periodic boundary jet. A more reasonable representation of the forcing would include a mean plus a time dependent flow. The linear wave solutions would then be superposed on the mean circulation resulting from the steady part of the forcing. The geostrophic mean flow component, corresponding to solutions with $\omega = 0$, simply consists of a strong jet centered at the depth of the source and decaying away from the boundaries in the zonal direction due to frictional effects.

The dependence of the solutions on the detailed characteristics of the forcing about which there is virtually no knowledge prevents a meaningful comparison between model results and data. However, simple periodic and transient forcing of the form assumed here give rise to baroclinic zonal flow fields with some qualitative resemblance to the sheared flows in the observations (e.g., Firing 1987; Ponte and Luyten 1989). The general complex patterns of vertical phase propagation in the solutions could account for the difficulty in measuring it in the records. The model suggests the general importance of deep boundary energy sources in forcing subthermocline flows in the equatorial band. More observations are needed to reveal the connections (if any) between boundary processes and the interior circulation. In particular, monitoring of deep water mass signals as they enter the equatorial regions would be needed to understand how much of these cross-equatorial flows actually turn along the equator and on what time scales do they vary.

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APPENDIX A

Linear Equatorial Wave Theory

The theory of linear equatorial waves has been treated in detail in the literature (e.g., Pedlosky 1979; McCreary 1985). The linearized, Boussinesq, equatorial $\beta$-plane equations can be separated into a horizontal and vertical problem. The vertical structure equation takes the form

$$G''(z) + \frac{N^2(z)}{c_n^2} G(z) = 0 \quad (A1)$$

where $N$ is the buoyancy frequency and $c_n$ is the separation constant. Solutions $G(z)$ are subject to the boundary conditions $G = 0$ at the bottom, and $G = 0$ (rigid surface) or $G'(z) - (g/c_n^2)G(z) = 0$ (free surface) at $z = 0$. For oscillatory solutions in $x$ and $t$, the horizontal structure equation for meridional velocity becomes

$$V_{\eta} + \left[ \sigma^2 - s^2 - \left( \frac{s}{\sigma} \right) - \eta^2 \right] V = 0 \quad (A2)$$

where

$$\eta = \left( \frac{\beta}{c_n} \right)^{1/2} y, \quad \sigma = \frac{\omega}{\sqrt{\beta c_n}}, \quad s = \left( \frac{c_n}{\beta} \right)^{1/2} k \quad (A3)$$

are nondimensional variables ($\omega$ and $k$ are the dimensional frequency and zonal wavenumber, respectively). Solutions to (A2) which decay away from the equator are the Hermite functions

$$\psi_j(\eta) = e^{-\eta^2/2} H_j(\eta) \frac{1}{(2j!)\sqrt{\pi}}^{1/2} \quad j = 0, 1, 2, \ldots \quad (A4)$$

where $H_j(\eta)$ are Hermite polynomials. The free modes obey the dispersion relation

$$\sigma^2 - s^2 - \frac{s}{\sigma} = 2j + 1 \quad (A5)$$

[for a plot of (A5) see for example Pedlosky 1979]. The general latitudinal structure of the meridional and zonal velocity fields takes the form

$$V_j(\eta) = A_j \psi_j(\eta) \quad (A6a)$$

$$U_j(\eta) = - A_j \left[ \frac{(j/2)^{1/2} \psi_{j-1}}{\sigma + s} + \frac{(j + 1/2)^{1/2} \psi_{j+1}}{\sigma - s} \right] \quad (A6b)$$

for each meridional mode. For $j \geq 1$, these solutions represent Rossby waves (small $\sigma$) and gravity waves (large $\sigma$). Long Rossby waves ($s \ll 1$) have the approximate dispersion relation

$$\sigma = - \frac{s}{2j + 1} \quad (A7)$$

A special solution of the equatorial $\beta$-plane equations which has no meridional flow field is the Kelvin wave.
Its zonal velocity field has a latitudinal structure given by

\[ U_{-1}(\eta) = A_{-1}\psi_0 \]  

(A8)

and dispersion relation

\[ \sigma = s. \]  

(A9)

REFERENCES


