The Stationary Frontal Wave Packet Spontaneously Generated in Mesoscale Dipoles

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(Manuscript received 1 September 2006, in final form 26 December 2006)

ABSTRACT

Three-dimensional numerical simulations of rotating, statically and inertially stable, mesoscale flows show that wave packets, with vertical velocity comparable to that of the balanced flow, can be spontaneously generated and amplified in the frontal part of translating vortical structures. These frontal wave packets remain stationary relative to the vortical structure (e.g., in the baroclinic dipole, tripole, and quadrupole) and are due to inertia–gravity oscillations, near the inertial frequency, experienced by the fluid particles as they decelerate when leaving the large speed regions. The ratio between the horizontal and vertical wavenumbers depends on the ratio between the horizontal and vertical shears of the background velocity. Theoretical solutions of plane waves with varying wavenumbers in background flow confirm these results. Using the material description of the fields it is shown that, among the particles simultaneously located in the vertical column in the dipole’s center, the first ones to experience the inertia–gravity oscillations are those in the upper layer, in the region of the maximum vertical shear. The wave packet propagates afterward to the fluid particles located below.

1. Introduction

Coherent vortical structures, like the vortex dipole or tripole, are possible features in mesoscale dynamics (e.g., Ikeda et al. 1984; Ginzburg and Fedorov 1984; Carton 2001). Mesoscale dipoles have been repeatedly observed from satellite imagery (Millot 1985; Fedorov and Ginzburg 1986; Ahlnäs et al. 1987; Johannessen et al. 1989), and recently from hydrographic data south of Madagascar (de Ruijter et al. 2004). Theoretically, Stern (1975) derived an exact dipolar solution, for non-divergent barotropic flow on the β plane, called the modon (e.g., Berson and Kizner 2002, and references therein). Laboratory experiments in rotating tanks show that barotropic dipoles can be generated from an impulsive jet (Kloosterziel et al. 1993), become important transporters of fluid (Eames and Flór 1998), and finally decay, for example, due to bottom friction effects (Sansón et al. 2001). Numerically, the generation of oceanic dipoles has been investigated within the two-layer, f-plane, shallow-water theory (Mied et al. 1991).

Unsteady dipoles in a balanced flow may also produce the spontaneous generation of inertia–gravity waves (IGWs) as observed experimentally in unsteady vortex collisions (Afanasyev 2003). The velocity and density perturbations of the IGWs spontaneously generated by these unsteady mesoscale flows are usually of small magnitude compared to the initially balanced (void of waves), generating flow, and often this IGW emission occurs sporadically, when wave packets are emitted during a finite period of time. However, in a recent numerical study of the small-amplitude spiral wave patterns caused by the IGWs spreading away from the vortical flow in the baroclinic dipole (Viúdez 2006) it was noticed that there is a wave packet, of large vertical velocity, that remains stationary, trapped in the frontal part of the translating dipole. The characteristics of this wave packet, named here the frontal wave packet because it amplifies in the frontal part of translating vortical structures, are numerically and theoretically investigated in this paper.

The numerical models used (described in section 2a) simulate the three-dimensional flow in a rotating reference frame. The initial conditions of the main simulation (section 2b) correspond to a baroclinic dipole composed by two potential vorticity (PV) vortices. The dipolar total flow (section 3a) remains statically and inertially stable and spontaneously generates the frontal wave packet. The unbalanced flow (section 3b) is

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extracted from the total flow, and the stationary frontal wave packet (section 3c) and other transient waves (section 3d) are then analyzed. It is found that the wave packet (a phenomenon primarily observed in the spatial description) is, in fact, due to inertia-gravity oscillations, close to the inertial frequency, experienced by the fluid particles (thus, a phenomenon better explained in the material description). The frontal wave is a robust phenomenon occurring in different dipole configurations (section 3e), is highly dependent on the balanced flow (section 3f), and occurs also in other coherent vortex structures, as in the baroclinic tripole and quadrupole (section 3g).

2. Numerical models and initial conditions

a. The numerical models

The three-dimensional baroclinic, stably stratified, volume-preserving, nonhydrostatic flow, under the f-plane and Boussinesq approximations, is simulated using a triply periodic numerical model (Dritschel and Viúdez 2003) initialized with the PV initialization approach (Viúdez and Dritschel 2003). This initialization approach largely avoids the initial generation of IGWs due to the initial velocity and density conditions and makes it easier to observe the spontaneous generation due to the initial velocity and density conditions and approaches largely avoids the initial generation of IGWs.

The PV is represented by contours on isopycnals. We use, unless otherwise specified, a 128³ grid, with 128 isopycnals, in a domain of vertical extent \( L_z = 2\pi \) (which defines the unit of length) and horizontal extents \( L_x = L_y = cL_z \), where \( c = 10 \) is the ratio of the mean Brunt–Väisälä to the Coriolis frequency \( c = \frac{1}{f} = Nf \). The mean buoyancy period \( (T_{\text{bp}} = 2\pi/N) \) is set as the unit of time (thus \( N = 2\pi \)). One inertial period \( (T_I = 2\pi f) \) equals 10\( T_{\text{bp}} \). The initialization time \( t_I = 5T_{\text{bp}} \) and the time step \( \delta t = 0.01 \). The initialization time is the minimum time required for the fluid to reach its initial perturbed state with minimal generation of IGWs.

The vertical displacement \( \mathcal{D} \) of isopycnals is \( \mathcal{D}(x, t) = z - d(x, t) \), where \( d(x, t) = (\rho(x, t) - \rho_0)/\rho_z \) is the depth that an isopycnal located at \( x \) at time \( t \) has in the reference density configuration defined by \( \rho_0 + \rho_z \), where \( \rho \) is the mass density, and \( \rho_0 > 0 \) and \( \rho_z < 0 \) are constants that do not need to be specified in the Boussinesq approximation. The mean Brunt–Väisälä frequency \( \mathcal{N} = -\alpha_0 g\rho_z \), where \( \alpha_0 = 1/\rho_0 \). Static instability occurs when the stratification number \( \mathcal{N}_x = \partial \mathcal{D}/\partial z > 1 \). The Rossby number \( \mathcal{R} = \zeta/f \), and the Froude number \( \mathcal{F} = \omega_0/\mathcal{N} \), where \( \omega_0 \) and \( \zeta \) are the horizontal and vertical components of the relative vorticity \( \omega \), respectively, and \( \mathcal{N} \) is the total Brunt–Väisälä frequency. The PV anomaly \( \sigma = \Pi - 1 \), where \( \Pi = (\omega_f + \mathbf{k}) \cdot \nabla d \) is the dimensionless total PV. The state variables are the components of the vector potential \( \varphi = (\varphi, \psi, \phi) \) from which both the three-dimensional velocity \( \mathbf{u} = -f \nabla \times \varphi \) and \( \mathcal{D} = -c^2 \nabla \cdot \varphi \) are obtained.

The primary model \( (A\beta\sigma) \) integrates the two equations for the rate of change of the dimensionless horizontal ageostrophic vorticity \( \mathcal{A}_h = (A, B) = \omega_0/f - c^2 \nabla \mathcal{D} = (\omega_0/\mathcal{F})/f = \omega_0/f \), where \( \omega_0 \) is the horizontal geostrophic vorticity, and a third equation for the explicit material conservation of \( \sigma \) on isopycnals, \( d\sigma/dt = 0 \). The horizontal potentials \( \varphi_h = (\varphi, \psi) \) are recovered from the inversion of \( \mathcal{A}_h = \nabla^2 \varphi_h \), while the vertical potential \( \phi \) is obtained from the inversion of the definition of the PV anomaly \( \sigma = 1 - \Pi = 1 - (\omega_0/f + \mathbf{k}) \cdot (\mathbf{k} - \nabla \mathcal{D}) \).

To avoid the generation of grid-scale noise, a biharmonic hyperdiffusion term \( -\mu \nabla^4 \varphi_h \), where \( \nabla^4 \varphi_h = (\partial^4 \varphi_h/\partial x^4, \partial^4 \varphi_h/\partial y^4, \partial^4 \varphi_h/\partial z^4) \) is the gradient operator in the vertically stretched space, is added to the equation for \( d\mathcal{A}_h/dt \). The coefficient \( \mu \) is chosen by specifying the damping rate \( \epsilon_f = 50 \) of the largest wave-number in spectral space per \( T_{\omega_f} \).

The secondary numerical model \( (A\beta C) \) is the full pseudospectral version of the hybrid \( A\beta\sigma \) algorithm and uses the same grid-based procedures except that it omits those involving the PV contours. The prognostic variables are \( \mathcal{A}_h = (A, B, C) = \omega_0/f - c^2 \nabla \mathcal{D} \). Hence, there is no PV inversion, but only a Poisson equation for all three components of the vector potential, \( \nabla^2 \varphi = \mathcal{A}_h \), which is inverted spectrally. All of the parameter settings are the same except for \( \epsilon_f \). The results of the \( A\beta C \) model are used to verify that the frontal wave packet is robust and independent of the explicit conservation of PV inherent to the \( A\beta\sigma \) model.

b. Initial conditions

The baroclinic dipole is simulated as two ellipsoids of oppositely signed \( \sigma \) (Fig. 1). The number of initial PV contours in the middle isopycnal \( (t_I = 65) \) of each vortex is \( n_c = 20 \), \( \sigma \) varying from \( \sigma = 0 \) (outermost surface) to extrema \( \sigma^\pm = \pm 0.85 \) (vortex cores). The \( \sigma \) jump is fixed for all contours \( \delta \sigma = \sigma^\pm/\ell_c \). The outermost \( \sigma \) ellipsoidal layer has horizontal major and minor semiaxes \( (A_x^+, A_y^+, A_z^+) = (1.4, 1.2, c) \), and vertical semiaxes \( (A_x^+, A_y^+, A_z^+) = (1.2, 0.52) \). Further details on the PV configuration of an ellipsoid are given in Viúdez and Dritschel (2003). The initial distance between vortex centers is \( 2\sigma^+ + 0.01c \). The \( \sigma \) ellipsoids are defined on the isopycnal space (or physical space with flat isopycnals). During the initialization period \( (0 \leq t \leq t_I) \) the isopycnals of the anticyclone (cyclone) stretch (shrink) so that at \( t = t_I \) the vortices have similar vertical extents.
3. Numerical results and theoretical development

a. The total flow

The PV anomalies cause a moderately large ageostrophic flow both static and inertially stable. The time average, and standard deviations, from $t = 5T_{ip}$ to $50T_{ip}$, of the extreme $\mathcal{R}$ in the domain $[\mathcal{R}_{\text{min}}(t), \mathcal{R}_{\text{max}}(t)]$ are $[\langle \mathcal{R}_{\text{min}} \rangle, \langle \mathcal{R}_{\text{max}} \rangle] = [-0.73, 0.48] \pm [0.6, 6] \times 10^{-3}$, while $\langle D_{\text{max}} \rangle = 0.47 \pm [0.7 \times 10^{-3}]$. The vortices soon deform from their initial elliptical PV configuration and start their eastward propagation as a dipole (Fig. 1).

We focus on the vertical velocity $w$ since, as is typical of mesoscale balanced flows, $w$ is $10^{-3}$–$10^{-4}$ times smaller than the horizontal velocity $u_h$, which is here $O(1)$. The horizontal distribution of $w$ (Fig. 2) displays features of three different spatial scales. First, on the larger dipole scale, $w$ has the quadrupolar, time-independent pattern typical of a stationary translating balanced dipole, with $w < 0$ ($w > 0$) on the right (left) of the dipole’s head. Second, on the smaller vortex scale this quadrupolar pattern is modified by several time-dependent relative extrema, of smaller magnitude, produced by the phase oscillations, or dipole’s heading, of the vortices, in this case, predominantly by the anticyclone (Pallás-Sanz and Viúdez 2007). These two features are quasigeostrophic (QG) balanced phenomena that can be diagnosed obtaining the QG $w$ from the density field by solving the QG omega equation. Third, there is a finer, wave-scale, non-QG phenomenon located on the dipole’s head that, despite being of smaller magnitude, is clearly noticeable in the total $w$. This wave, here called the *frontal wave*, is the subject of this

![Fig. 1: Potential vorticity jumps (PV jump value $\delta \mathcal{R} \approx 0.0436$) on isopycnal $l_i = 65 (z = 0)$ at (a) $t = 0$, (b) $t = 5T_{ip}$, (c) $t = 6T_{ip}$, and (d) $t = 7T_{ip}$. An asterisk marks the $\mathcal{R}$ center of each vortex. The thick line, normal to the line joining the vortices (thin line) and located at half the distance between them, has a length $\delta l = 2c$. This line, hereinafter the along-dipole line, and its midpoint (black square symbol) are included for reference. The horizontal extent is $x, y \in [-\pi, \pi]$.](image)

![Fig. 2: Horizontal distribution of $w$ on the plane $l_i = 54 (z = -0.54); w \in [-2.1, 2.2] \times 10^{-3}$. Contour interval is $\Delta w = 0.25 \times 10^{-3}$; zero contour omitted and time in $T_{ip}$. The vortex locations and the along-dipole line are included as defined in Fig. 1. The horizontal extent is $x, y \in [-2.5, 2.5]$.](image)
study and its characteristics are described in the next sections, where the unbalanced flow is extracted from the total flow. The phase oscillations of the baroclinic vortices mentioned above seem to be the oscillation mode proper of a stable state of a physical system having internal structure. Though a complete explanation of the perturbation and restoring force that keeps the dipole stable (i.e., as a coherent pair of vortices) is not provided here, we notice that these oscillations imply interchange between the kinetic $E_K$ and potential $E_P$ energy of the total flow, where

$$E_K = u^2 \quad \text{and} \quad E_P = N^2 g^2.$$ (1)

The time series of the domain average of the kinetic $\langle E_K \rangle$, potential $\langle E_P \rangle$, and total energy $\langle E_T \rangle = \langle E_K \rangle + \langle E_P \rangle$ (Fig. 3) show that the dipole’s heading implies a cyclic conversion between kinetic and potential energy with a period of about $5T_{wp}$. The dipole’s heading has also an effect in the dipole’s velocity $U(t)$ (Fig. 4), which has a time mean $\bar{U} \approx 0.22$, but experiences small oscillations of amplitude 0.1 approximately every $5T_{wp}$.

b. Diagnosis of the unbalanced flow

The balanced vector potential $\varphi_b = (\varphi_b, \psi_b, \phi_b)$ is diagnosed using the optimal PV balance (OPVB) approach (Viúdez and Dritschel 2004), and the balanced quantities are derived from this. From a given PV field, the OPVB approach diagnoses a flow having only those IGWs that have been spontaneously generated during the process of acquiring its own PV (during a time interval set equal to $t_f = 5T_{wp}$). Thus, if the frontal wave is not spontaneously generated locally at the dipole’s head in a short time period (i.e., shorter than $t_f$), but either is due to IGWs spontaneously generated elsewhere that travel to the dipole’s head from where does not propagate or is due to the amplification during a time scale larger that $t_f$ of IGWs generated locally at the dipole’s head, then the OPVB flow will not contain most of the frontal wave, which will remain, almost entirely, in the unbalanced vector potential $\varphi = \varphi - \varphi_b$. The unbalanced velocity and vertical displacement of isopycnals, obtained directly from $\varphi$, through the usual relations $u_i = -f\nabla \times \varphi$ and $\delta_i = -\varepsilon^2 \nabla \cdot \varphi$, are described below.

c. The frontal wave packet

1) UNBALANCED VELOCITY AND VERTICAL DISPLACEMENT

The horizontal distributions of the unbalanced vertical velocity $w_i$ (Fig. 5) show the frontal waves as a wave packet localized at the dipole’s head with $w_{i,\max} = 5 \times 10^{-4}$, that is, about 4 times smaller than $w_{i,\max}$ at this depth. Here $|w_i|$ is larger in the anticyclonic side, suggesting that a large part frontal wave in this case is generated in the anticyclone. The wave amplitude grows during the first inertial periods after initialization (from $t = 5T_{wp}$ to $= 8T_{wp}$) until it becomes approximately stationary relative to the translating dipole.

Let the tangent and normal unit vectors relative to the along-dipole line be $\mathbf{t}$ and $\mathbf{n} = \mathbf{k} \times \mathbf{t}$, respectively. The tangent and normal components of $u_i$ are $u_t = u_i \cdot \mathbf{t}$ and $u_n = u_i \cdot \mathbf{n}$, respectively. The stationarity of the frontal wave and the relations between the velocity components and density anomaly can be appreciated from the distributions of $u_{it}$, $u_{in}$, $w_i$, and $\delta_i$ as functions of $(r, i)$, and $(z, i)$, where $r$ is the distance on the along-dipole line from the dipole’s center (Figs. 6 and 7). The frontal wave packet remains mostly stationary with no appreciable decay during the time shown. The time oscillations of these wave amplitudes have a period of
FIG. 5. As in Fig. 2 but for $w_i$: $w_i \in [-4.7, 5.0] \times 10^{-4}$ and $\Delta w_i = 10^{-4}$.

FIG. 6. Space–time ($r, t$) distributions of (a) $u_{i}(r, t)$ ($\Delta u_i = 5 \times 10^{-4}$), (b) $u_{m}(r, t)$ ($\Delta u_m = 5 \times 10^{-4}$), (c) $w_i(r, t)$ ($\Delta w_i = 5 \times 10^{-4}$), and (d) $\varphi(r, t)$ ($\Delta \varphi = 5 \times 10^{-3}$). The vertical axis is the distance $r/c$ on the along-dipole line ($z = -0.54$). The horizontal axis is time in $T_{ip}$. 
=5T_{ip} and seem to be related to the vortex phase oscillations mentioned earlier. The wave generation occurs in the first 5T_{ip} after initialization, starting at short r and propagating forward. It can be also appreciated that, in the quasi-steady state, the horizontal (along dipole) and vertical wavenumbers, |\k_1| and |\m_1|, respectively, increase with r and |z|; that is, the wavelength decreases as the distance from the dipole’s center increases.

Numerical experiments carried out with the ABC model initialized with the balanced fields \( \phi_0(x, t_0) \) at times \( t_0 > 5T_{ip} \) show the generation of the wave packet irrespective of the initial time \( t_0 \). The wave front is therefore independent of the idealized initial conditions (ellipsoidal PV surfaces) of the dipole.

Apart from the time variability associated to the dipole’s heading, it seems that, after the wave packet generation, the dipole’s flow, the extreme Rossby numbers, and the wave packet remain stationary, so there is no time-dependent geostrophic adjustment due to the wave packet.

2) INERTIA–GRAVITY WAVE SOLUTIONS IN BACKGROUND FLOW

The relations between the unbalanced quantities \( u_{it} \), \( u_{in} \), \( w_{in} \), and \( D \), are locally similar to those of IGW solutions, close to the inertial frequency, in a background flow. Consider the momentum, mass, and volume conservation equations for the disturbances \( \tilde{u} \) and \( \tilde{D} \) in the total flow \( u \),

\[
\frac{d\tilde{u}}{dt} + f\mathbf{k} \times \tilde{u} = -\alpha_0 \nabla \rho - N^2 \mathbf{\tilde{D}}, \quad \frac{d\tilde{D}}{dt} = \tilde{w}, \quad \nabla \cdot \tilde{u} = 0. \tag{2}
\]

where \( dX/dt = \partial X/\partial t + u \cdot \nabla X \) is the material derivative.

We seek plane wave complex solutions for \( \tilde{u} = (\tilde{u}, \tilde{v}, \tilde{w}) \) and \( \tilde{D} \) in the \((r, z)\) plane, with no \( n \) dependence, of the form

\[
\tilde{X} = \tilde{X}_0 \exp(i\theta), \tag{3}
\]

where the phase \( \theta(x, z, t) \). The ambiguity in the usual notation, where the same symbols are used to denote functions both in the spatial \((x, y, z, t)\) and material description, is extended now since the same symbols are used also to denote a function of variables \((r, n, z, t)\), including the spatial Cartesian coordinates in the orthogonal reference frame \((i, n, k)\) moving with the dipole. To avoid an excess of notation and since distinction in only needed when time operations, like time differentiation or averaging, are involved, the usual convention is followed here and the symbol \( \partial X/\partial t \) is
used for the time derivative of $\dot{x}(x, y, z, t)$, $\dot{x}(x, y, z, t)$ is used for the time derivative of $X$ in the material description, and symbol $D_x \dot{x} = \partial \dot{x}/\partial t + u \partial \dot{x}/\partial r$ is the more general rate of change of $X$ relative to an observer moving with velocity $v$ along the $r$ axis.

For wavenumbers $k(x, z, t), l = 0$, and $m(x, z, t)$,

$$k = (k, l, m) = \nabla \theta,$$

and for the local (or absolute) frequency $\omega(x, z, t),$

$$\omega_l = -\frac{\partial \theta}{\partial t}.$$

The isochoric (volume preserving) condition in the along-dipole plane,

$$\frac{\partial u_t}{\partial r} + \frac{\partial w_z}{\partial z} = 0,$$  \hspace{1cm} (4)

is very well satisfied in the frontal wave packet as deduced from the time series of the slope $b(t)$ in the linear fitting $(\partial u_t/\partial r)(t) = a(t) + b(t) (\partial w_z/\partial z)(t)$ (Fig. 8). Thus, gradients of the wave quantities in the $n$ direction are small $(l = 0)$. The vortex phase oscillations, with a period $-5 T_{cp}$, are also noticeable in the time evolution of $b(t)$.

Equations (2) admit plane wave solutions where the frequency relative to the fluid particle $\omega_p$ (or intrinsic) satisfies the dispersion relation:

$$\omega_p = -\frac{d \theta}{dt} = \frac{j^2 m^2 + N^2 k^2}{k^2 + m^2}.$$

Neglecting the advection by the total vertical velocity the total rates of change of $\dot{u}$ and $\dot{v}$ in (2) can be expressed as

$$\frac{d \dot{x}}{dt} = D_x \dot{x} + (u_t - \mathbf{U}) k \dot{x}.$$

Consistency of the plane wave solutions (which must have constant coefficients) requires that $\omega_p$, and therefore the ratio $M = m/k$, be constant. Defining the frequency of the waves relative to the moving dipole, $\omega_d = -\left( \frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial r} \right)$, we obtain the kinematical relations:

$$\omega_p = -\frac{d \theta}{dt} = -\omega_l - u_t k = \omega_d - (u_t - \mathbf{U}) k.$$

The above expression is a particular case of the invariance of the wavenumber $k$ to arbitrarily moving rigid frames of reference.
The mean dipole’s speed \( \overline{U} \equiv 0.22 \) (Fig. 4) while \( O(u) = 1 \) (Fig. 11). Thus \( u_i - \overline{U} > 0 \) implies \( k < 0 \); that is, at a fixed time \( u_{hi} \) rotates anticyclonically with increasing \( r \), as can be inferred from Fig. 6.

Since \( |k|, |m| \to \infty \) as \( u_i \to \overline{U} \), the increase of the wavenumbers is always limited by the numerical grid size. However, the larger wavelengths of the frontal wave are well resolved with the current resolution (128\(^3\) grid points), and are observed also with 256\(^3\) and 512\(^3\) grid point simulations (Fig. 12). The imbalanced fields were not extracted in these cases because the OPVB is very costly with these high spatial resolutions. The increase of \( |k| \) and \( m \) happens because the horizontal distance completed in a period by a fixed fluid particle, which is undergoing quasi-inertial oscillations, decreases as the background flow decreases with increasing \( r \) and \( |z| \).

The numerical results approximately satisfy relations (9). The time average on the along-dipole line in the dipole’s reference frame of the wave quantities (Fig. 13) show that \( \overline{\pi}(r, z_0) \equiv -i\sqrt{2} \overline{w}(r, z_0) \) and \( \overline{\pi}(r, z_0) \equiv -i\sqrt{2} \overline{\pi}(r, z_0) \), especially the rotation, anticyclonic for the vector \( [\overline{\pi}_r(r), \overline{\pi}_r(r)] \) and cyclonic for \( [\overline{\pi}_r(r), \overline{\pi}_r(r)] \), and the ratios \( |\overline{\pi}_r|/|\overline{\pi}_w| \sim \sqrt{2} > 1, |\overline{\pi}_r|/|\overline{\pi}_w| \sim \sqrt{2}f = 2\pi\sqrt{2}/10 \equiv 0.89 < 1 \). Relations (9) also imply that the ratio between horizontal and vertical wave velocity components, which is obtained from the volume conservation on the along-dipole line neglecting the \( n \) dependence, is \( u_i/\overline{w} \equiv c = 10 \), which can be inferred in the numerical results shown in Figs. 6 and 7.

The results above also imply \( \omega_j < 0 \); that is, at a fixed location the wave horizontal velocity \( u_{hi} \) rotates cyclonically with time. Thus, an anchored current meter located at some point on the along-dipole line will measure cyclonic horizontal velocity perturbations as the dipole passes along, which are in fact related to the anticyclonic quasi-inertial oscillations experienced by the fluid particle.

The fields shown in the horizontal and vertical distributions are only those in the lower half of the domain...
(\(z \in [0, -\pi]\)) because, owing to the initial PV symmetry \(\varpi(x_0, z) = \varpi(x_0, -z)\), it results that \(u(x_0, z) = u(x_0, -z)\), \(v(x_0, z) = v(x_0, -z)\), \(w(x_0, z) = -w(x_0, -z)\), \(\partial(x_0, z) = -\partial(x_0, -z)\), and the same relations hold for the balanced and unbalanced components so that there is one frontal wave in each half of the domain. The frontal wave in the lower half corresponds to \(m > 0\), implying that, at a fixed time, \(\mathbf{u}_{hi}\) rotates cyclonically (anticlockwise) with increasing \(z\) (as observed in Fig. 7). In the upper half \(m < 0\), so that at a fixed time \(\mathbf{u}_{hi}\)

\[\Delta w_x = 5 \times 10^{-3}\] Contours of \(u_x(r, z, t) - U(t)\) are \([1, 1/2, 1/3, 1/4]\) and \(u_x(r, z, t) - U(t) = 0\) (circular dotted line). The dotted vertical line marks the location of the dipole’s center. The solid vertical line crossing the along-dipole line at the black square location (Fig. 1) is included for reference. Horizontal and vertical extents are 2.2 and \(z \in [0, -2.2]\), respectively: time in \(T_{ip}\).

\(\text{FIG. 11. Vertical distributions of } w(r, z, t)\) (wave packet) and \(u_x(r, z, t) - U(t)\) (circular solid lines) on the (moving) plane of the along-dipole line (thin horizontal line at a depth \(z = -0.54\)) as defined in Fig. 1. \(\Delta w_x = 5 \times 10^{-3}\). Contours of \(u_x(r, z, t) - U(t)\) are \([1, 1/2, 1/3, 1/4]\) and \(u_x(r, z, t) - U(t) = 0\) (circular dotted line). The dotted vertical line marks the location of the dipole’s center. The solid vertical line crossing the along-dipole line at the black square location (Fig. 1) is included for reference. Horizontal and vertical extents are 2.2 and \(z \in [0, -2.2]\), respectively: time in \(T_{ip}\).

![Fig. 11. Vertical distributions of w(r, z, t) and u_x(r, z, t) - U(t) on the along-dipole line.](image1)

\(\text{FIG. 12. Horizontal distribution of } w\) at \(z = -1.18\) \((i_z = 161)\) and \(t = 7T_{ip}\) in a numerical simulation with 512³ grid points of a dipole with \(\varpi^+ = 1.8, \varpi^- = -0.8, A^+_x = 1.4, A^+_z = 1.2, \) and \(c_y = 200; x, y \in 2\pi[-1, 1]\) and \(w \in [-4, 4] \times 10^{-3}\).

\(\text{FIG. 13. (a) The hodograph of } [\pi_x(r), \pi_y(r)] \times 10^6\) on the along-dipole line at \(z = -0.54\). The time average is from \(t = 5\) to \(39T_{ip}\), and the bars mean one standard deviation. (b) As in (a) but for \([\pi_y(r), \partial_z(r)] \times 2 \times 10^4\).}
rotates anticyclonically (clockwise) with increasing \( z \) (not shown).

4) THE WAVE PACKET IN THE MATERIAL DESCRIPTION

A mean value of the wave vertical velocity \( \bar{w}_i(R, Z, t) \) as a function of the fluid particle \( (R, Z) \) and time \( t \), that is, \( \bar{w}_i \) in the material description, can be obtained from \( u_i(r, z, t) \), and \( w_i(r, z, t) \) in the following way. First, the time averages \( \bar{u}_i(r, z, t) = \langle u_i(r, z, t) \rangle \) and \( \bar{w}_i(r, z, t) = \langle w_i(r, z, t) \rangle \) are computed on the along-dipole line. Then, the time \( \bar{t} \) taken for the fluid particle \( (R, Z) = (r_0, z_0) \), located at \( r = r_0 = 0 \) and \( z = z_0 \) at \( \bar{t} = 0 \), to move a distance \( r \) on the along-dipole line with a velocity \( \bar{v}_i(r, z) \) is computed as

\[
\bar{t}(r_0, r, z) = \int_{r_0}^{r} \frac{dr'}{\bar{v}_i(r', z)} = \frac{T_{tr}}{U}.
\]  

(11)

Last, the unbalanced vertical velocity \( \hat{w}_i \) experienced by the fluid particle moving on the along-dipole line, as a function of \( \bar{t} \) and at a depth \( z = z_0 \), can be visualized in a scatterplot of \( \bar{w}_i(r, z_0) \) versus \( \bar{t}(r_0, r, z_0) \) and joining values at consecutive locations \( r \). This is equivalent to invert \( \bar{t}(r_0, r, z_0) \), to get the \( r \) location of the fluid particle \( (r_0, z_0) \) at time \( \bar{t} \), \( r(r_0, r, z_0, \bar{t}) \), and composite it with \( \bar{w}_i(r, z_0) \), that is, \( \hat{w}_i = \bar{w}_i \circ r \). The results (Fig. 14) show the amplification and decay of the amplitude of \( \hat{w}_i \) as the fluid particle moves on the along-dipole line, and that the oscillatory motion has an intrinsic frequency \( \omega_p = \sqrt{2f} \).

More information can be obtained from the two-dimensional \((Z, t)\) distribution \( \hat{w}_i(r_0, Z, \bar{t}) \), that is, as a...
The distribution of phase is the intrinsic period of the IGWs located below. The time interval between points of wave packet propagates afterward to the fluid particles in the region of the maximum vertical shear gravity oscillations are those in the upper layer;  

\[ \text{FIG. 17. Time series of the spatial average of the total energy of the total flow } E_T, \text{ the balanced flow } E_{b}, \text{ the unbalanced flow } E_{i} = 5 \times 10^4, \text{ and the interaction } E_{b}. \text{ The plots represent } E_x = \langle E_x \rangle = \langle E_{b} \rangle, \text{ where the time averages and standard deviations are } \langle E_{b} \rangle = (1557.4 \pm 0.9) \times 10^{-3}, \langle E_{i} \rangle = (0.005 \pm 0.001) \times 10^{-1}, \text{ and } \langle E_{b} \rangle = (1.1 \pm 0.2) \times 10^{-4}. \text{ The vertical axis is in units of } 10^{-3}, \text{ horizontal axis is time in } T_{ip}. \]

The budget of the total energy may be expressed as

\[ \text{(12)} \]

\[ \tilde{\theta}(r, \theta, z) = \frac{1}{\omega_p} \int_0^t k \, dr' = \frac{1}{\omega_p} \int_0^t \frac{\partial \theta}{\partial r} \, dr' \]

The above result follows also from the integration of \( d\theta/dt = -\omega_p \), with \( \omega_p \) constant. Thus, if the initial phase \( \theta(r_0, z) = \theta_0 \) (const), then \( \omega_p \, \nabla \tilde{\theta} = -\nabla \theta \), that is, \( \nabla \theta \) is in the direction of \( \nabla \tilde{\theta} \) (in the direction of the \( x \) axis in Fig. 15).

One might think of Fig. 15 also as the representation reciprocal to plotting \( \tilde{\eta}(r, \theta, z) \) together with \( \tilde{\eta}(r, \theta, z) \) (Fig. 16). Both representations contain the same information though the downward propagation (relative to the fluid particles) and the IGW phase synchronization is observed better in Fig. 15. The distribution \( \tilde{\eta}(r, \theta, z) \) may be interpreted as a space contraction of \( \tilde{\eta}(r, z) \) (depending on \( z \), with larger contraction in the upper layers) so that contours of \( \tilde{\eta}(r, \theta, z) \) in Fig. 16 transform into vertical parallel lines. Reciprocally, the distribution \( \tilde{\eta}(r, \theta, z) \) may be interpreted as a time dilation of \( \tilde{\eta}(r, \theta, z) \) (depending on \( z \), with larger dilation in the upper layers) so that contours of \( r(r_0, \theta, z) \) in Fig. 15 transform into vertical parallel lines.

5) Energy budget

The budget of the total energy may be expressed as

\[ \langle E_T \rangle = \langle E_{b} \rangle + \langle E_{i} \rangle + 2 \langle u_b \cdot u_i \rangle + 2 N^4 \langle d_b \cdot d_b \rangle, \]

where \( \langle E_{b} \rangle = \langle \omega_b \rangle + N^2 \langle \omega_b \rangle \) and \( \langle E_{i} \rangle = \langle \omega_i \rangle + N^2 \langle \omega_i \rangle \)

are the domain averages of the total energy of the bal-

\[ \text{FIG. 18. As in Fig. 11 but with } \Delta \eta = 10^{-5} \text{ and } z \in [\pi, -\pi + 2], \text{ that is, the four vertical sections correspond to the rear part below the dipole, in the same location relative to the moving along-dipole line: time in } T_{ip}. \]
anced and unbalanced flow, respectively, and $2(u_p \cdot u) + 2N^2(\partial u_i/\partial z)$ are the interaction terms. The energy of the balanced flow $\langle E_T \rangle$ (Fig. 17) has a behavior similar to that of the total flow $\langle E \rangle$ (displaying several extrema but with an overall decrease due to numerical diffusion) except between $t = 5T_{ip}$ and $t = 7T_{ip}$, where $\langle E_T \rangle$ decreases when $\langle E_T \rangle$ increases. The $\langle E_T \rangle$ increases as a result of the development of the wave packet. Since, in order of magnitude, $(\langle E_T \rangle - \langle E_T \rangle)/\langle E_T \rangle \sim 200$, the largest part of the energy deficit $(\langle E_T \rangle - \langle E_T \rangle)$ is transferred to the interaction term $(\partial \tau/\partial z)$. The $\langle E_T \rangle$ and $\langle E_T \rangle$ experience small-amplitude oscillations with the same period of $5T_{ip}$ as the dipole’s heading.

d. Transient wave packets

The $t-z$ distributions of the unbalanced variables (Fig. 7) show the amplification of the frontal wave packet during the first inertial periods as well as the downward propagation of a wave packet, different from the stationary frontal wave packet, at lower levels ($z < -1$). The vertical group velocity of this transient wave packet decreases as it approaches the critical level $u_i = U$. The wave becomes progressively horizontal, and therefore it is observed better in $(u_i, u_{ip})$ than in $w$, until it finally no longer propagates vertically and its amplitude, relative to the moving dipole, slowly decays. These transient, downward propagating waves do, however, leave the dipole’s rear (or, equivalently, are left behind as the dipole propagates forward) and, before arriving to the bottom layer, form a small-amplitude wave tail left behind by the dipole (Fig. 18). The origin of these transient waves seems to be related to the local variability due to the initial development of the frontal wave packet. Once the frontal wave packet becomes stationary, the transient waves are no longer generated.

e. Robustness of the frontal wave packet

A series of different numerical experiments were carried out to verify that the generation of the frontal wave packet is a robust phenomenon in dipoles where (7) is physically possible. Figure 19 shows $w$ in five different cases. The reference case (Fig. 19a) is included for comparison. The case, Fig. 19b, has vortices with a larger number of PV contours ($n_c = 60$, proving that the frontal wave is independent of $n_c$), and an anticyclone with larger vertical extent ($A_{z} = 0.7$), resulting in a flow with $\zeta_{min} = -0.75, \zeta_{max} = 0.48, f_{max} = 0.44$, and $\sigma_{max} = 0.39$. The anticyclone is stronger, relative to the case, Fig. 19a, and this causes an increase in the horizontal (cross dipole) extent of the frontal wave packet in the anticyclonic side of the dipole. The case in Fig. 19c is as in Fig. 19b, but with vortices of equal vertical extent ($A_{z} = 0.7$) and larger $\sigma_{max} = 0.87, f_{max} = 0.60, \sigma_{max} = 0.34$. The increase of cyclonic vorticity causes an increase in both $U$ and in the trajectory curvature of the dipole. In this case the cyclone experiences phase oscillations (as the anticyclone does in the previous cases) so that waves also develop in the cyclone as stationary spiral waves (better noticed in the next cases). The case in Fig. 19d is as in Fig. 19b but with larger $\sigma_{max} = -0.95, 2.1$, which causes a flow at the margin of the static and inertial stability ($\zeta_{min} = -1.0, \zeta_{max} = 0.87, f_{max} = 0.98, \sigma_{max} = 0.93$). In this case $U$ is larger and the

Fig. 19. Vertical velocity $w$ on plane $t_z = 54$ and $t = 10T_{ip}$ in five different numerical simulations explained in the text: $w \in \{[-0.21, 0.22], [-0.19, 0.22], [-1.3, 1.0], [-1.6, 1.9], [-1.5, 2.0]\} \times 10^{-3}$.

Fig. 20. Vertical distributions of $w(x, y, z)$ on plane $y_0 = 0$ for the isolated frontal wave packet simulation with initial time $t_i = 11T_{ip}$; $x \in [-\pi, \pi], z \in [-\pi, 0], w \in \{[-2.5, 2.5], [-1.9, 1.7], [-1.6, 1.6], [-1.7, 1.5], [-1.2, 1.3]\} \times 10^{-3}$; time in $T_{ip}$.
amplitude of the frontal wave increases in relation to the background mesoscale $w$.

The case in Fig. 19e has the same dipole configuration as in Fig. 19b but was simulated using the $ABc$ model (with $c_f = 100$), initialized with the potential $\phi(x, t_0)$ provided by case (d) at $t_0 = 5 T_{wp}$. The results are very consistent with the case in Fig. 19d, suggesting that the stationary frontal wave is a robust phenomenon, independent of the explicit conservation of PV (e.g., PV contour advection on isopycnals and PV inversion). The $ABc$ model, and the PV initialization approach, was however used to obtain the balanced initial conditions at $t_0 = 5 T_{wp}$.

f. Evolution of an isolated frontal wave packet

It is plausible that the small-amplitude waves spontaneously generated by the otherwise balanced dipole may amplify and cause the large-amplitude frontal wave packet. Thus, the frontal wave continually depends on the dipole’s flow. Intuitively, in absence of the dipole, the frontal wave packet would move backward, in a direction opposite to the dipole’s velocity. To verify this behavior the evolution of the isolated frontal wave packet is simulated by initializing the $ABc$ model only with the unbalanced potential $\phi(x, t_0)$ and $\sigma = 0$ everywhere, that is, only the frontal wave packet is present initially. In the initial conditions ($t_0 = 11 T_{wp}$) the frontal wave packet and the deeper tail wave are clearly observed (Fig. 20).

As expected, as time evolves the wave packet rapidly moves backward and downward, with group velocity perpendicular to the wavenumber vector, and intersects the tail wave, which is more stationary in the fixed reference frame than the wave packet above, causing interference patterns. The wave packet continues its downward propagation and spreading, rapidly arriving at the domain’s lower layers and interfering with the upward propagating wave packet of the upper half ($z \in [0, \pi]$, not shown). Thus, the unbalanced flow behaves as an isolated internal wave packet, which supports the fact that the OPVB approach is correctly extracting most of the wave packet from the balanced flow.

g. Frontal wave packets in a mesoscale tripole and quadrupole

The dipole is not the only vortical structure able to support a frontal wave packet. Other examples of coherent vortical structures generating frontal wave packets are the stable mesoscale tripole and quadrupole. These structures may be thought of as consisting of a set of partial dipoles.

An anticyclonically rotating tripole is simulated here with three ellipsoidal vortices: a central anticyclone with $\sigma^- = -0.85$ and two outer cyclones with $\sigma^+ = 1.5$, and semiaxes $A_x^+/c = 1, A_y^+/c = 1$, and $A_z^+/c = A_x^+ = 0.8$ in the initial reference configuration. The tripole remains coherent (Fig. 21a) and is static and inertially stable with $\kappa_{\text{min}} = -0.89, \kappa_{\text{max}} = 0.73, \gamma_{\text{max}} = 0.56$, and $D_{\text{max}} = 0.38$. Soon after the initialization time ($t = 5 T_{wp}$) two stationary (relative to the rotating tripole reference frame) wave packets are generated at the frontal part of the rotating vortices (Fig. 21b). A stable cyclonic tripole of similar characteristics generates the frontal wave as well (Fig. 21c).

An anticyclonic quadrupole, simulated with three outer cyclones (Fig. 21d) remains coherent and stable with $\kappa_{\text{min}} = -0.96, \kappa_{\text{max}} = 0.75, \gamma_{\text{max}} = 0.49$, and $D_{\text{max}} = 0.30$. In this case three stationary wave packets are generated at the frontal sides of the rotating vortices (Fig. 21c). These frontal waves are very similar to the one studied in detail in the single dipole, and are in fact the same phenomenon, being always located where the flow decelerates, in the frontal part of the dipolelike vortical structures.

4. Concluding remarks

We have seen that wave packets of large-amplitude vertical velocity, relative to the background vertical velocity, can be generated in the frontal part of translating vortical structures in rotating, statically and inertially stable, mesoscale geophysical flows. The frontal wave packets remain stationary relative to the translating vortical structures and are due to the inertia–gravity
The interplay between balanced and unbalanced dynamics is evident in the evolution of the large speed regions. Consistent with these quasi-inertial oscillations, the magnitude of the horizontal and vertical wavenumbers decreases as the fluid speed decreases, the ratio between the horizontal and vertical wavenumbers remains however constant and equal to the ratio between the horizontal and vertical shears of the background horizontal velocity. It is suggested, though not proved, that, among the wavenumber components of the small-amplitude waves spontaneously generated by vortex phase oscillations (dipole’s heading), only those components with wavenumbers \( k \) satisfying the relation of stationarity relative to the moving vortical structure \( (\omega_0 - Uk = 0) \) are amplified and cause the frontal wave packet in the region where \( -u_x/u_z \sim -k/m \sim f/N \). Thus, the dipole interior behaves like a cavity resonator for IGWs in the background flow.

Further theoretical work is therefore needed to understand the physical mechanism that excites and maintains with finite amplitude these inertia–gravity components. Whatever the cause, it seems that the persistence of this frontal wave, spontaneously generated and continually supported by the vortical flow, is a phenomenon beyond the range of application of the strict separation between balanced and unbalanced dynamics.

Acknowledgments. I thank two anonymous reviewers for their positive comments, which significantly improved the original manuscript. Partial support for this research has come from the Spanish Ministerio de Educación y Ciencia (Grant CGL2005-01450/CLI), and the U.K. Engineering and Physical Sciences Research Council (Grant XEP294). Computer resources and technical assistance provided by the Barcelona Supercomputing Center-Centro Nacional de Supercomputación is also acknowledged.

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