On the Asymmetry between Cyclonic and Anticyclonic Flow in Basins with Sloping Boundaries

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ABSTRACT

The authors present results from laboratory experiments and numerical simulations of the barotropic circulation in a basin with sloping boundaries forced by a surface stress. Focus is placed on flows with large-scale Rossby numbers that are significantly smaller than unity. The results of the laboratory experiments and simulations show that cyclonic circulation follows the isobaths, the flow pattern being independent of the strength of the forcing. For anticyclonic circulation, the flow pattern changes with forcing strength. It is similar to the cyclonic topographically steered pattern for weak forcing, and it develops strong cross-slope flows for strong forcing.

Linear dynamics are symmetric between cyclonic and anticyclonic circulations and give a good description of the cyclonic and weakly forced anticyclonic circulations. The analysis of the nonlinear dynamics shows that topographically steered cyclonic flows are all stable and steady energy-minimum solutions to the inviscid nonlinear equations. This implies that the nonlinear terms (advection of relative vorticity) are always small for the topographically steered cyclonic flow.

For anticyclonic flow, the situation is very different. It is possible that no anticyclonic topographically steered flow is ever a solution to the steady inviscid equations. And if such a steady anticyclonic flow does exist, it is likely to be unstable, since it must correspond to a saddle point in energy rather than to a minimum or a maximum. The nonlinear terms are important when the Rossby number is larger than the Ekman number, which is the case for the anticyclonic experiments with strongest forcing. For these experiments, the advection of relative vorticity prevents the flow from following topography, creating locations with strong relative vorticity and cross-slope flow. The development of cross-slope flow can be understood from the conservation of potential vorticity in basins with irregular topography.

The separation of anticyclonic flow from steep topography shown in the laboratory experiments and the theoretical analysis herein are in agreement with features like the Gulf Stream separation from the continental slope at Cape Hatteras, North Carolina.

1. Introduction

In high latitudes, where the density stratification is weak, the large-scale ocean currents tend to follow contours of constant depth. In the Nordic seas, for example, Helland-Hansen and Nansen (1909) noted early on that the surface circulation was strongly guided by the topography. The underlying physics is that the large-scale circulation tends to conserve potential vorticity. In other locations the constraint to follow topography seems less strict. A classical example is the Gulf Stream that separates from the coast and a steep continental slope near Cape Hatteras on the East Coast (Fofonoff 1981; Veronis 1981; Hurlburt and Hogan 2000; Matsumoto and Lynch-Stieglitz 2003). A number of mechanisms have been proposed to explain the Gulf Stream separation, and the effect of topography has also received much attention (Smith and Fandry 1976; Özgökmen et al. 1997; Stern 1998; Tansley and Marshall 2000; Marshall and Tansley 2001; for a review, see also Dengg et al. 1996).

In this paper we will analyze the steady barotropic flow in basins with steep sloping boundaries. Our analysis assumes an $f$ plane. This can be done without a loss of generality because the results also apply to cases with...
a varying Coriolis parameter, $f$, by considering $f/H$ instead of the topography, $H$, when $f$ is varying. We demonstrate that as long as the Rossby number is larger than the Ekman number, steady anticyclonic flow in a basin will be characterized by regions of strong cross-slope flow. Thus, our results may have relevance for the understanding of features like the Gulf Stream separation.

In regions with closed depth contours, the topographic steering of the time-mean flow can be striking (Walin 1972; Nøst and Isachsen 2003; Isachsen et al. 2003). The underlying dynamics are that the geostrophic circulation cannot accomplish a net transport across a closed-depth contour on an $f$ plane. Thus, if the wind forces a net cross-isobath transport in the surface Ekman layer, continuity demands the presence of a compensating transport in the bottom Ekman layer. Furthermore, if the bottom friction is small, the leading-order bottom velocity will be directed along the isobaths, implying that the bottom pressure is a function of the depth alone. Accordingly, an along-isobath circulation is established such that its associated net cross-isobath transport in the bottom Ekman layer balances that of the wind-forced surface Ekman layer (Greenspan 1968; Nøst and Isachsen 2003; Nilsson et al. 2005). As a consequence, wind forcing over a localized region on a closed isobath induces a nonlocal isobath-following circulation. Nøst and Isachsen (2003) showed that these rather basic dynamical considerations, in combination with observations of wind and hydrography, can be used to calculate the time-mean circulation in the Arctic Ocean and the Nordic Seas.

The model of Nøst and Isachsen (2003) is linear, and the validity of the model is dependent on a strict topographic steering. A strong cross-isobath flow at one point may change the total flow pattern. However, the circulation in the Nordic Seas and Arctic Ocean, where the model is successful, is mainly cyclonic, and when advection of relative vorticity is considered, there is a difference between cyclonic and anticyclonic circulations.\(^1\) This difference can be shown with an idealized example illustrated in Fig. 1. Assume that the solid black lines in Fig. 1 are isobaths and that the flow is parallel to the isobaths. The figure shows two identical examples, except that one has shallow water to the right and the other has shallow water to the left. In the limit of geostrophic flow, the transport between two isobaths must be constant, implying that the flow will speed up and slow down as illustrated by the arrows. Assume also that the flow conserves potential vorticity given by $q = (f + \zeta)/H$, where $f$ is the Coriolis parameter, $\zeta$ is the relative vorticity, and $H$ is the depth. Since relative vorticity is changing along the flow, so must $H$ to keep $q$

\(^1\) Note that the important dynamical distinction is whether the flow has shallow water on the right or on the left. Since we are dealing with a basin, “cyclonic” and “anticyclonic” are shorthand for “with shallow water on the right” and “with shallow water on the left,” respectively (on the Northern Hemisphere). For flow around a seamount the situation is opposite.
constant. Therefore, the flow cannot follow the isobaths exactly, but it will follow contours of \( q \). The dashed lines in the figure are lines of constant \( q \), with the relative vorticity estimated from the isobath-following flow.

For the cyclonic flow (shallow water to the right) the narrowing of the new streamlines (\( q \) contours) is weaker, and this will cause weaker along-flow changes in relative vorticity than what is used to calculate the \( q \) contours. This means that the real streamlines conserving potential vorticity will be located between the dashed lines and the isobaths. Thus, if the relative vorticity calculated from isobath-following flow is small, the real streamlines will stay close to the isobaths.

For anticyclonic flow (with shallow water to the left), the narrowing of the \( q \) contours is stronger than the narrowing of the isobaths. Streamlines following contours of \( q \) will therefore cause larger changes in the relative vorticity than will the isobath-following flow. Contours of potential vorticity will then be even farther away from the isobaths, causing even larger along-stream changes in relative vorticity. We see that even if relative vorticity calculated from the isobath-following flow is small, it might be amplified at certain locations. If the relative vorticity becomes of the same order of magnitude as the Coriolis parameter, flow may no longer be restricted to follow isobaths. Although this example is almost naive in its simplicity, we will show through laboratory experiments, simulations, and theory that the process illustrated in Fig. 1 is important for large-scale currents in the ocean.

In addition to the simple example above, several other existing theories predict an asymmetry between cyclonic and anticyclonic flows in a basin. Bretherton and Haidvogel (1976) showed that an initial 2D turbulent field above topography will approach a state of minimum enstrophy for a given kinetic energy. This state is characterized by streamlines parallel to the isobaths: cyclonic around basins and anticyclonic around seamounts. The same result is also obtained from statistical mechanics, which predict that random 2D flow over topography has a nonzero mean given by the state of maximum entropy (Salmon et al. 1976). Such flows, driven by eddy-induced rectifications, will also be cyclonic around ocean basins and anticyclonic around seamounts. Parameterizations of this effect (Holloway 1987, 1992) have been widely used in models of the Arctic Ocean to improve the representation of the topographically trapped currents (Nazarenko et al. 1998; Nazarenko and Tausnev 2001; Polyakov 2001).

The asymmetry between eastward and westward planetary jets has been studied in the laboratory using a setup with a flow of homogeneous water over a bottom slope (Sommeria et al. 1989; Flexas et al. 2005). Eastward jets tend to be narrow, strong, and wavy, while westward jets are broad and weak. This difference can be explained by instabilities on both sides of the jet and the associated mixing of potential vorticity (Marshall 1981; Ivchenko et al. 1997; Marcus and Lee 1998). However, time-independent stable flow also shows an asymmetry between cyclonic and anticyclonic circulations over topography. This has been shown by Nycander and LaCasce (2004) for flow around an isolated topographic feature on an infinite plane.

The simple example illustrated by Fig. 1 indicates that along-isobath flow with shallow water on the left does not conserve potential vorticity. If this is correct, such flow will not be a solution to \( \mathcal{J}(\psi, q) = \mathbf{v} \cdot \nabla q = 0 \), where \( \psi \) is the transport streamfunction associated with the velocity \( \mathbf{v} \). The relationship \( \mathcal{J}(\psi, q) = 0 \) has a linear subset given by \( q = \lambda \psi \), where \( \lambda \) is a constant. This subset was analyzed by LaCasce et al. (2008) for flow over topography, motivated by the simple example above and the results of the laboratory experiments presented in this paper. They analyzed the solutions in an idealized elliptical basin and for the topography of the Nordic seas. They found that the linear subset has no topography-following anticyclonic solutions. Only the cyclonic circulation resembled the topography, while the anticyclonic flow was dominated by small-scale structures and cross-slope flow.

We will continue to explore the possibility that anticyclonic isobath-following flow is not a steady solution in the inviscid case. To do this, we will here extend the analysis by Nycander and LaCasce (2004) to the large-scale circulation in a closed basin with topography. The analysis is based on the nonlinear stability theory of Arnol’d (1965, 1966) and shows that there is a fundamental difference between cyclonic and anticyclonic circulations in basins with sloping boundaries. While cyclonic circulation is a stable energy-minimum flow, anticyclonic circulation cannot be an extreme value in energy. Steady anticyclonic circulation may therefore not exist; if it does exist, it is probably unstable.

The motivation of the present paper stems from the result of a rotating tank experiment, designed to study the nonlocal nature of a homogeneous wind-forced flow on closed isobaths. The experimental setup, illustrated in Fig. 2, consists of two bowl-shaped basins that communicate over a sill. The “wind forcing” is confined to one of the subbasins and is provided by the stress from a rotating disc in contact with the water surface. According to linear theory (Greenspan 1968; Nøst and Isachsen 2003; Nilsson et al. 2005), the flow field should simply be reversed if the direction of the disc rotation were to change. However, the flows in the laboratory experiments responded dramatically differently to cy-
clonic and anticyclonic rotations, once the rotation rate exceeded a certain threshold. For cyclonic disc rotation, the resulting circulations were broadly consistent with the linear theory. For anticyclonic disc rotation, on the other hand, the circulation in the unforced basin tended to be cyclonic rather than anticyclonic as predicted by the linear theory, and the flow was not everywhere directed along isobaths.

The remainder of the paper is organized as follows: the outcome of the laboratory experiments and a set of numerical experiments are presented in section 2. In section 3, theoretical considerations are presented, which are discussed in section 4 in relation to the observed flow in the laboratory and the numerical simulations. Oceanographic implications and basic mechanisms responsible for the asymmetry are discussed in section 5, before we summarize in section 6.

2. Laboratory and numerical experiments

a. Setup

The laboratory experiments were done at the Coriolis tank at SINTEF in Trondheim, Norway. The tank has a diameter of 5 m and a depth of 50 cm. Our to-
pography was set up within this tank and has the dimensions shown in Fig. 2. The topography is flat in the middle of the two subbasins and on the sill. Everywhere else the bottom has a slope of 30° to the horizontal. The rotation period of the tank was 20 s, equivalent to a Coriolis parameter $f = 4\pi/20 = 0.6$ s$^{-1}$.

The circulation was forced by a rotating disc in contact with the surface. The forcing was confined to one of the subbasins as shown in Fig. 2, and the tank was filled with homogeneous water; the circulation was therefore barotropic.

For the numerical simulations we used the Regional Ocean Model System (ROMS) (Shchepetkin and McWilliams 2005). The model has a free surface and was used in 2D mode, which means that it solves only the depth-integrated equations. We used a linear bottom friction defined by

$$\tau_b = u_w v,$$  \hspace{1cm} (1)

where $u_w = 10^{-3}$ m s$^{-1}$. The surface stress was represented as proportional to the difference between the disc velocity and the velocity in the water:

$$\tau_w = u_w (\mathbf{\omega} \times \mathbf{r} - \mathbf{v}); \hspace{1cm} |\mathbf{r}| \leq 1$$

$$\tau_w = 0; \hspace{1cm} |\mathbf{r}| > 1.$$  \hspace{1cm} (2)

Here, $\mathbf{\omega}$ is the angular velocity of the disc and $\mathbf{r}$ is the radius vector originating in the center of the disc.

b. Results

The laboratory results and the numerical simulations gave nearly identical outcomes. We first present the main structure of cyclonic and anticyclonic flows based on the results of the numerical simulations. Figure 3 shows the results from cyclonic forcing with four different disc rotation periods. We see that although the strength of the flow varies with the strength of the forcing, the structure of the flow remains the same. A topographically steered current is driven around the unforced basin with the anticyclonic forcing with the same four-disc rotation strength of the forcing. Figure 4 shows the results from cyclonic forcing with four different disc rotation periods. We see that although the flow pattern is similar to the cyclonic case, but with the opposite sign (see Fig. 3). However, when the forcing strength is increased, the flow pattern gradually changes to a flow regime with cyclonic flow in the unforced basin and anticyclonic flow in the forced basin. For the strongest forcing ($\omega = 2\pi/60$ s$^{-1}$), the circulation below the disc is anticyclonic while the unforced basin is dominated by a weaker cyclonic flow. This flow pattern is completely different from the cyclonic flow pattern shown in Fig. 3. Note that for strong forcing, the anticyclonic flow completely separates from the slope (in the region where it leaves the forced basin). A part of this flow crosses the sill and flows into the unforced basin where it forces a cyclonic circulation, but most of it takes place in a closed anticyclonic circulation in the forced basin. The region of separation does not coincide with the edge of the disc (cf. Figs. 2, 4).

The patterns for cyclonic and anticyclonic flow observed in the laboratory experiments were highly similar to the results from the numerical simulations described above. Snapshots of the ink distributions for strong anticyclonic forcing ($\omega = 2\pi/63$ s$^{-1}$) are shown in Figs. 5 and 6. Figure 5 clearly shows that the flow in the unforced basin is cyclonic. It also shows a weak anticyclonic flow along the rim of the basin, which becomes gradually stronger closer to the inflow from the forced to the unforced basin (see figure caption). This flow structure is similar to the structure of the simulated flow for $\omega = 2\pi/60$ s$^{-1}$ (cf. Fig. 4, bottom panel). Figure 6 shows snapshots of the ink distribution from the area where the anticyclonic flow separates from the slope and the streamlines instead close within the forced basin. The figure clearly shows how water over the slope in the forced basin separates from the slope and recirculates within the forced basin instead of following the topography into the unforced basin. Comparison with Fig. 4 shows the agreement between numerical simulations and laboratory experiments. In the next section we summarize linear and nonlinear theories in search of an understanding of the observed asymmetry between cyclonic and anticyclonic flows.

3. Theory

Starting with the shallow-water equations and assuming a rigid lid renders the following equations:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + f \mathbf{k} \times \mathbf{v} = -\frac{\nabla p}{\rho} + \frac{\tau_w - \tau_b}{H},$$  \hspace{1cm} (3)

$$\nabla \cdot (\mathbf{v} H) = 0.$$  \hspace{1cm} (4)

Here, $\mathbf{v}$ is the horizontal velocity vector, $f$ is the Coriolis parameter, $\mathbf{k}$ is the vertical unit vector, $p$ is pressure, $\rho$ is the constant density, and $H$ is the depth. The $\tau_b$ and $\tau_w$ are given by Eqs. (1) and (2), respectively. Because
we are mainly interested in steady-state solutions and because the length scales of the problem are considerably smaller than the Rossby radius of deformation, a rigid lid is a reasonable assumption.

The curl of Eq. (3) together with Eq. (4) gives

$$\frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q = \frac{1}{H} \nabla \times \left( \frac{\tau_w - \tau_b}{H} \right),$$

where $q$ is the potential vorticity, given by

$$q = \frac{\zeta + f}{H},$$

and $\zeta$ is the relative vorticity.

### a. Nondimensional parameters and scaling

Assuming a steady state, multiplying Eq. (5) by $H$, integrating over the area enclosed by a potential vorticity contour ($C_q$), and replacing $\tau_b$ by the expression in Eq. (1) gives

$$\oint_{C_q} \left( \frac{\tau_w - H_s V}{H} \right) \cdot d\mathbf{l} = 0.$$  \hspace{1cm} (7)

Thus in a steady state the velocity scale is determined by a balance between the surface and bottom stresses. For potential vorticity contours closing under the disc, this balance is achieved with a velocity in the water equal to half the disc speed. Therefore, we choose
$U \sim \omega R/2$ to represent the velocity scale. The magnitude of the nonlinear advection terms and the friction terms relative to the terms in the geostrophic balance is given by the Rossby number (Ro) and the Ekman number (Ek), respectively. These nondimensional parameters are given by

$$\text{Ro} = \frac{\omega}{2f}, \quad \text{Ek} = \frac{u^*}{fH_0}.$$  \hspace{1cm} (8)

Ro varies with the disc rotation speed $\omega$, while Ek is constant and independent of the forcing. Values of Ro and Ek for the different forcing strengths and values of depth $H_0$ are given in Table 1. For all the disc rotation speeds in the laboratory experiments and simulations, we have $\text{Ro} \ll 1$ and $\text{Ek} \ll 1$. From Eq. (7) we see that friction has a clear influence on the steady-state solution no matter how small it is; however, at first sight it seems reasonable to ignore the nonlinear terms. This leads to the simplified linear solutions, which we look at in the next section. However, Ro is larger than Ek for most of the experiments, depending on the value of $H_0$, and we shall see in the section treating inviscid dynamics that the nonlinear terms play a vital role in determining anticyclonic circulation patterns.

**b. Linear theory**

In this section we derive the simplified linear solution that was used successfully by Nøst and Isachsen (2003) to model the bottom currents in the Nordic Seas and the Arctic Ocean. Nøst and Isachsen (2003) included a variable Coriolis parameter and effects of hydrography,
but the main physics are included in the model presented here.

As \( \text{Ro} \ll 1 \) and \( \text{Ek} \ll 1 \), \( \mathbf{v} \) can be represented to first order by the geostrophic relation, and it is also clear that \( \mathbf{v} \) must closely follow isobaths for the lhs and rhs of the linear version of Eq. (5) to be of the same order of magnitude. This means that \( p \) can be treated as a function of \( H \), which leads to

\[
\mathbf{v} = \frac{1}{\rho \text{f}} \frac{dp}{dH} \mathbf{k} \times \nabla H. \tag{9}
\]

In the absence of forcing and dissipation, Eq. (9) is a solution to the linearized version of Eq. (5) for any value of \( dp/dH \). Whether these are also solutions to the nonlinear inviscid equations will be addressed in the next section. Now we will continue by finding the values of \( dp/dH \) that agree with the conditions set by the balance between forcing and dissipation.

Ignoring the relative vorticity in Eq. (6), assuming a steady state, multiplying Eq. (5) by \( H \), and integrating over the area enclosed by an isobath \( C_H \) leads to

\[
\oint_{C_H} (\tau_w - u_\theta \mathbf{v}) \cdot d\mathbf{l} = 0. \tag{10}
\]

An expression for \( dp/dH \) can now be found by using \( \mathbf{v} \) from Eq. (9) in Eq. (10). This expression can then be substituted back into Eq. (9), which leads to the following expression for \( \mathbf{v} \):

\[
\mathbf{v} = \frac{1}{u_\theta} \frac{\int_{C_H} \tau_w \cdot d\mathbf{l}}{\int_{C_H} (\mathbf{k} \times \nabla H) \cdot d\mathbf{l}} \mathbf{k} \times \nabla H. \tag{11}
\]

In our topography (Fig. 2), the slope is constant along the isobaths and the linear solutions are therefore especially simple. For contours closing under the disc, \( \mathbf{v} = \mathbf{v_d}/2 \), where \( \mathbf{v_d} \) is the speed of the disc. The surface stress will act on about half the length of the contours that enclose both the forced and the unforced subbasins. On these contours, \( \mathbf{v} = \mathbf{v_d}/3 \). The simplified solution is shown in Fig. 7. We see that this solution reasonably well represents the cyclonic flow (Fig. 3) as well as the anticyclonic flow with weak forcing (Fig. 4). The magnitude of the velocity is not shown in Fig. 7, but it is also in excellent agreement with laboratory experiments and simulations for cyclonic and weakly forced anticyclonic flows.
The simplified linear solution depends heavily on the assumption that $dp/dH$ is a function of $H$, and it will be valid as long as the relative variations of $dp/dH$ along the isobaths are small. This requires $Ek \ll R/L$, where $R$ and $L$ are the length scales across and along the isobaths, respectively (Nøst and Isachsen 2003). Nøst and Isachsen (2003) also found that the simplified solution gave good results and was valid for most of the Nordic seas and the Arctic Ocean. However, nonlinear terms were not considered. In the next section, we look at nonlinear inviscid theory to see how this affects the solutions.

c. Nonlinear inviscid theory

We will here consider the nonlinear stability properties of steady inviscid flows in the absence of forcing. The rationale is that in the limit when $Ro/Ek \gg 1$, the frictional spindown time scale becomes much greater than the advective time scale. Thus, although the flows in the laboratory experiments and the numerical simulations are subjected to both forcing and dissipation, we anticipate that a stability analysis of inviscid, unforced flows can illuminate the observed asymmetry between cyclonic and anticyclonic circulations. It should be noted, however, that the forcing–dissipation constraint given by Eq. (7) still applies and determines which nearly inviscid steady-state flows are realized in our experiments and simulations.

Ignoring the forcing and dissipation terms, Eq. (5) describes the conservation of potential vorticity as

$$\frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q = 0.$$  \hspace{1cm} (12)

We represent the transport by the streamfunction $\psi$,

$$\mathbf{v}H = k \times \nabla \psi,$$  \hspace{1cm} (13)

so that the relative vorticity is given by

$$\zeta = \nabla \cdot \frac{\nabla \psi}{H}.$$  \hspace{1cm} (14)

The potential vorticity $q$ is still given by Eq. (6). The flow is in a closed basin with the coastline $\gamma$, so that the boundary condition is

$$\psi = 0, \quad (x, y) \in \gamma.$$  \hspace{1cm} (15)

Since the rigid-lid approximation is employed, the kinetic energy constitutes the total energy
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Equation (12) implies that \( q \), and therefore any function \( F(q) \), is conserved in every fluid element. Every element also conserves its volume \( dV = H dA \), where \( dA \) is its surface area. Therefore, the volume of water with a certain value of \( F(q) \) will always remain the same. Consequently, the values of the integral

\[
C_F = \int_A F(q)H dA
\]

will remain constant for flow controlled by Eqs. (12) and (13). Here, \( F \) is an arbitrary function. The parameter \( C_F \) is called a Casimir integral, and flows that have the same value as all the Casimir integrals are called isovortical flows. Perturbations of the potential vorticity field that keep the flow in the same set of isovortical flows are said to be isovortical perturbations. Thus, a flow controlled by Eq. (12) will always be within the same set of isovortical flows.

According to a general variational principle, a flow field that minimizes or maximizes the energy \( E_T \) within the same set of isovortical flows must be a steady and stable solution of Eq. (12): the only way to conserve both \( E_T \) and \( C_F \) when the flow is in an energy minimum/maximum state is to keep the flow field constant in time. This principle was used by Nycander and LaCasce (2004) and Nycander and Emamizadeh (2003) to prove the existence of steady and stable vortex flows at an isolated seamount on an infinite plane.

Although no ocean basins are circular symmetric, the case of a symmetric basin is of interest to understanding the mechanisms causing asymmetries between cyclonic and anticyclonic flows. In a circular symmetric basin, \( H \) is only a function of \( r \), where \( r \) is the distance to the center of the basin. The dynamics in such a basin conserve the angular momentum \( M \), defined by

\[
M = \int_A \mathbf{r} \cdot (\mathbf{r} \times \mathbf{v}) H dA = \int_A \mathbf{r} \cdot \nabla \psi dA.
\]

(18)

This invariant can be used similarly as the energy to prove stability in a circular basin. In other words, a flow field that minimizes or maximizes the angular momentum \( M \) within the same set of isovortical flows must be stable.

1) **FLOW IN A CIRCULAR SYMMETRIC BASIN**

In a circular symmetric basin, any streamfunction that is only a function of \( r \) [i.e., \( \psi = \psi(r) \)] is a steady solution of Eq. (12). If it maximizes or minimizes the angular momentum within a set of isovortical flows, it is also a stable solution. Maximizing/minimizing \( M \) while keeping \( C_F \) constant is equivalent to maximizing/minimizing \( J_M = M + C_F \) without constraints. With an appropriate choice of \( F(q) \) (see appendix A), the first variation of \( J_M \) vanishes, while the second variation is given by

\[
\delta^2 J_M = \int_A H \frac{dG}{dq} (\delta q)^2 dA,
\]

(19)

where \( G(r) \) is defined by

\[
\frac{dG}{dr} = rH.
\]

(20)

See appendix A for detailed derivations of the first variation of \( J_M \) and Eqs. (19) and (20). Hence, we have nonlinear stability if \( dG/dq \) is sign definite. Because

\[
\frac{dG}{dq} = rH \frac{dq}{dr},
\]

(21)

the flow is nonlinearly stable if it has a monotonic radial profile of the potential vorticity. For weak flow, the
potential vorticity is essentially given by $f/H$. Therefore, in a basin with a monotonic radial profile of $H$, a weak circular flow is stable regardless of the direction of the flow.

2) Flow in an Irregular Basin

We will now analyze the nonlinear stability of flows in an irregular basin. By an irregular basin we mean any basin that is not circular symmetric. In such a basin, the total energy $E_T$ is conserved, but the angular momentum $M$ is not. Stationary flows are solutions of the equation $J(\psi, q) = 0$, the steady version of Eq. (12). In this case the streamfunction is only a function of $\psi$, $\psi = \psi(q)$. Minimizing/maximizing $E_T$ with $C_F$ kept constant is equivalent to minimizing/maximizing the function $J_E = E_T + C_F$. For the choice $dF/dq = \psi(q)$, the first variation of $J_E$ is zero (see appendix B) and the second variation can be written as

$$\delta^2 J_E = \int_A \left[ \frac{(\nabla \psi)^2}{H} + \frac{d\psi}{dq} (\delta q)^2 H \right] dA. \quad (22)$$

If $d\psi/dq > 0$, then $\delta^2 J_E$ is positive definite, and the flow minimizes the energy. Such a flow is therefore stable. For quasigeostrophic theory, this result has been shown by many authors (Arnol’d 1965, 1966; Carnevale and Frederiksen 1987; Nycander and LaCasce 2004). It has been extended here to the case of finite topography, described by Eqs. (12) and (13).

For weak flows, $q$ is essentially given by $f/H$, so that $q$ decreases toward the center of the basin. For cyclonic flow following topography, $\psi$ also decreases toward the center of the basin, and therefore $d\psi/dq > 0$. Thus, cyclonic flows in which the relative vorticity is sufficiently weak such that the potential vorticity is essentially given by topography are stable minimum-energy flows.

The same cannot be concluded for weak anticyclonic flows following topography. For these flows, $d\psi/dq < 0$, which means that the two terms in the integrand of Eq. (22) have different signs. The relative magnitude of these two terms depends on the spatial scale of the isovortical perturbation, as $(\delta q)^2/(\nabla \psi)^2$ can be made arbitrarily large by choosing a perturbation with a sufficiently small spatial scale. Thus, for a perturbation with a sufficiently small length scale, the second term will dominate and $\delta^2 J_E$ is negative. This shows that the anticyclonic flow cannot be an energy-minimum state.

A perturbation with a sufficiently large length scale will give a positive $\delta^2 J_E$, as the first term in the integrand of Eq. (22) will dominate. The question is how large these length scales need to be: Are they larger than the size of the basin? That large-scale geostrophic flows in the ocean are not maximum-energy flows is illustrated by the following idealized example: first consider a vanishing flow, with zero energy (clearly, a minimum-energy flow). In this case, the potential vorticity everywhere equals $f/H$. We then prescribe an isovortical perturbation by moving water columns from the shallow regions near the coast to the deep regions in the middle, and vice versa, while conserving their potential vorticity. This will lead to a very strong flow where the contribution from relative vorticity to $q$ is dominant. This type of rearrangement can be done to any weak flow, which shows that within the same set of isovortical flows there exist flows with very large energy and large Rossby numbers. The length scales of the perturbations are in this case similar to the spatial scale of the topography, and the idealized example shows that maximum energy flows are unrealistic in the real ocean.

As the energy might be decreased or increased depending on the length scale of the perturbation, there also exist perturbations in which both terms in the integrand of Eq. (22) are of equal magnitude and cancel each other, giving $\delta^2 J_E = 0$. If the assumption that $J(\psi, q) = 0$ (giving $\delta^2 J_E = 0$) is true, then the anticyclonic flow must be a saddle point in energy.

We conclude that if a weak steady anticyclonic circulation exists, it is neither a minimum nor a maximum energy flow. Instead, it must correspond to a saddle point in energy and is therefore likely to be unstable. Saddle points of conserved integrals usually correspond to steady but unstable flows; some examples of this are given for baroclinic instability in Pierrehumbert and Swanson (1995) and for barotropic instability in Benilov et al. (2004).]

However, it is also possible that no steady isobath-following anticyclonic circulation satisfying $J(\psi, q) = 0$ exists. This possibility is supported by the results of LaCasce et al. (2008), who analyzed a linear subset of $J(\psi, q) = 0$ and found that it had no weak anticyclonic solutions following topography.

We finally mention that there is a possibility for stable minimum-energy flows with weak cyclonic circulation inside a narrow anticyclonic coastal jet. Let $\gamma^*$ denote the streamline that separates the cyclonic and anticyclonic regions. The area integral of relative vorticity inside any streamline in the cyclonic region must be positive, since the circulation around the streamline is cyclonic, while the corresponding integral inside $\gamma^*$ vanishes. Thus, the relative vorticity must be negative (anticyclonic) near $\gamma^*$. As for all minimum-energy flows, we must have $d\psi/dq > 0$ everywhere. Since $\psi$ decreases outward in the anticyclonic jet, $q$ must also
decrease outward there. This means that the relative vorticity $\zeta = Hq - f$ decreases rapidly from $\gamma^*$, where it is already negative, toward the coast. Nevertheless, if the coastal jet is narrow enough, it may not be unrealistically strong.

In some of the experiments a flow with an anticyclonic coastal jet and a cyclonic interior circulation is in fact observed (cf. Figs. 4, 5). In the next section we argue that this is probably explained by the locally large Ekman number near the coast, but the nonlinear effect described here may also contribute to making this kind of circulation possible.

4. The observed flow in relation to theory

In the present theoretical considerations, we have separately analyzed linear weakly viscous dynamics and nonlinear dynamics in the entirely inviscid limit. Obviously, the flows observed in the laboratory and simulated numerically are influenced by both dissipation and nonlinear effects. From Eq. (7) we see that dissipation will always be important for the forced steady-state solutions, no matter how small the friction. Furthermore, even for small basin-scale Rossby numbers, the nonlinear terms can be important locally, which may affect the shape of the streamlines globally. From our analysis of the inviscid equations, we expect the importance of the nonlinear terms to depend on the sense of rotation of the flow as well as on the Rossby number.

Nonlinear effects can be expected to become important when $Ro \gg Ek$, and the linear solution is dominant when $Ro \ll Ek$. In the case of $Ro \sim Ek$, the nonlinear effects will be damped significantly by friction. Figures 3, 4 show the cyclonic and anticyclonic flow fields for different strengths of the forcing. When varying the forcing, we are varying $Ro$ while keeping $Ek$ constant. The strongest forcing ($\omega = 2\pi/60$ s$^{-1}$) has $Ro \gg Ek$ for most of the basin, while in the case with the weakest forcing ($\omega = 2\pi/600$ s$^{-1}$), $Ro \sim Ek$ (see Table 1).

The observed flow in the laboratory and simulations agrees remarkably well with the theory. In the cyclonic circulation (Fig. 3), the flow field agrees with the linear solution for all forcing strengths (Fig. 7). This means that $q$ is essentially given by $fH$ and that the relative vorticity is small everywhere. This is a stable minimum-energy solution in the inviscid case, satisfying $J(\psi, q) = H\nu \cdot \nabla q = 0$. The nonlinear terms in Eq. (5) are therefore zero, explaining why the linear solution (Fig. 7) is a good description of the cyclonic flow.

The anticyclonic flow the situation is very different. The analysis in section 3c(2) shows that if an anticyclonic topographically steered flow is a solution to

$$J(\psi, q) = 0,$$

it is a saddle point in energy. In that case the flow is most likely unstable. It is also possible that the flow is not a solution to $J(\psi, q) = 0$ at all, which is supported by the results of LaCasce et al. (2008), who analyzed a linear subset of $J(\psi, q) = 0$ and found no anticyclonic topography-following solutions. In either case, the nonlinear terms will drive the flow away from the topographically steered linear flow pattern, provided that $Ro \gg Ek$, so that the nonlinear terms dominate the friction terms. This agrees with the observations for strong forcing (Figs. 4–6), which show a flow field that is very different from the linear solution (Fig. 7).

Thus, when the forcing is anticyclonic, strong cross-slope flows will appear. This means that the relative vorticity must change strongly along the flow to balance the depth changes so that potential vorticity is conserved. This effect is illustrated in Fig. 8, where $\zeta f$ is plotted for cyclonic and anticyclonic circulations when $\omega = 2\pi/180$ s$^{-1}$. In the anticyclonic case, we clearly see the locations where the relative vorticity is comparable to $f$. We also see that these features of strong cross-slope flow coincide with the locations where the iso-
baths change curvature. These locations are illustrated by contouring the potential vorticity of the linear solution in Fig. 8. The strong cross-slope flow seems to have been initiated by the changing curvature of the isobaths.

For weaker forcing, Ro approaches the value of Ek, and the nonlinear effects are then more and more likely to be balanced by friction. The observed flow also approaches the linear solution for weaker forcing, as the anticyclonic circulation near the rim of the unforced basin becomes increasingly dominant as the forcing decreases. That linear dynamics will start to dominate near the coast is to be expected as the frictional effects scale as $1/H$ [see rhs of Eq. (5)]. The depth dependence of the friction terms is incorporated into the Ekman number given in Table 1. For the two experiments with the strongest forcing, Ro is larger than Ek for all depths listed in Table 1. For these experiments, the anticyclonic flow completely separates from the topography as the isobaths turn into the unforced basin (see Fig. 4). For the forcing with $\omega = 2\pi/300$ s$^{-1}$, the anticyclonic flow near the coast follows the isobaths into the unforced basin. For this experiment, $Ro \sim Ek$ ($H_0 = 0.1$ m) (see Table 1), and linear dynamics can be expected to be valid near the rim of the basin, explaining the anticyclonic flow there. For the experiment with the weakest forcing, Ro is smaller than Ek everywhere but in the deepest regions, which explains why this experiment is reasonably well described by linear theory (cf. Figs. 4, 7).

5. Discussion

a. Potential vorticity conservation over a sloping topography

From the above theoretical analysis and laboratory experiments we conclude that as long as $Ro \gg Ek$, an anticyclonic flow will not follow topography. Or alternatively, for an anticyclonic flow to be topographically steered, friction is needed to damp out the nonlinear effects. This result is valid even when large-scale estimates suggest that $Ro \ll 1$, which is normally an argument for linearization of the equations. To understand this we refer back to our simplified drawing in Fig. 1, which illustrates how a narrowing of the isobaths leads to an along-flow gradient of relative vorticity. This relative vorticity gradient must be followed by a cross-slope flow to conserve potential vorticity. For a flow with shallow water on the left, the cross-slope flow will act to enhance the along-flow relative vorticity gradient, which again will enhance the cross-slope flow, and so on. There is a positive feedback between relative vorticity and cross-slope flow, which for $Ro \gg Ek$ will lead to strong relative vorticity and cross-slope flow. The example in Fig. 1 is not general but rather constructed to illustrate the effect. However, the theoretical analysis of the nonlinear inviscid case in section 3c indicates that this process is general. It is clear from Fig. 1 that if the isobaths are straight and parallel, then anticyclonic solutions following topography can exist, as there will be no change in relative vorticity along the isobaths. The same is true for a circular symmetric basin, which we know has stable anticyclonic solutions [section 3c(1)]. Therefore, both the theory and the idealized example show that irregularities in the topography are needed for the asymmetry to occur.

Figure 1 provides a simple explanation to the observed asymmetries in the case of a changing steepness of the slope. Our numerical simulations and laboratory experiments clearly show the asymmetries of a changing curvature of the isobaths with a constant steepness of the slope. In this case, standing topographic waves may play an important role. Such standing topographic waves can exist if the basin flow is anticyclonic, but not if it is cyclonic. In Fig. 4 such waves are clearly visible near the region of flow from the unforced to the forced basin, and they are associated with locations of strong relative vorticity (see Fig. 8) and cross-slope flow. The flow in this region does not separate from the slope, and the waves have amplitudes smaller than the width of the slope. The separation of the flow from the slope in the region of flow from the forced to the unforced basin (Figs. 4, 6) may be caused by a standing topographic wave with an amplitude larger than the width of the slope. In the laboratory, this brings the flow onto the flat bottom of the sill area. Here, it is no longer under the influence of a sloping topography and may move freely away from the slope.

In the real ocean there will always be irregularities on the topography, and a flow over a sloping topography with shallow water on the left (in the Northern Hemisphere) should then develop locations of strong cross-slope flow as long as $Ro \gg Ek$. In the next section we examine whether this new insight may help to improve our understanding of the ocean.

b. Application to the ocean

We expect cyclonic flow in ocean basins to be strongly guided by the topography. Nøst and Isachsen (2003) have demonstrated that a linear model, similar to the one described in our linear theory section, gives a good description of the circulation in the Nordic Seas and the Arctic Ocean. In this region the circulation is mainly cyclonic, and the conclusions of Nøst and Isachsen (2003) are therefore in agreement with the results of this work. Also in the subpolar basin of the
North Atlantic, the cyclonic circulation is strictly guided by the topography (Fratantoni 2001). An obvious example is the East Greenland Current that sticks to the topography as it rounds the south tip of Greenland.

The Gulf Stream, which is part of the anticyclonic subtropical gyre of the North Atlantic, does not follow the topography as it separates from the continental slope near Cape Hatteras. The separation of the Gulf Stream shows many similarities to the separation of the anticyclonic flow from the topography in our laboratory experiments and simulations; we therefore believe that the theory presented herein contributes to the understanding of this phenomenon.

At the shelf break region near Cape Hatteras, rough estimates of the bottom velocity and the width of the Gulf Stream give 0.1 m s\(^{-1}\) and 100 km, respectively (Richardson and Knauss 1971; Fratantoni 2001). The Rossby number is then on the order of 10\(^{-2}\). Using \(u_0\) \approx 10^{-2} \text{m s}^{-1}, which is a reasonable value for the ocean (Haidvogel and Beckmann 1999), we estimate the Ekman number to be 10\(^{-2}\), with \(H = 1000 \text{m}\). Thus, \(Ro \gg\) Ek, and asymmetry between cyclonic and anticyclonic flow should be present. The continental slope at Cape Hatteras is an area where, from the theory presented above, strong cross-slope flow is expected. At this point the slope steepens dramatically and at the same time turns left. In our laboratory experiments the slope remains constant, but when the isobaths turn left the anticyclonic flow separates from the topography. We suggest that a similar mechanism operates at the separation of the Gulf Stream at Cape Hatteras. A separation mechanism caused by the topography, as we propose, is also supported by the fact that the Gulf Stream seems to have separated at the same point during the last glacial period (Matsumoto and Lynch-Stieglitz 2003).

In this context, it is relevant to note that Stern (1998) analyzed a Gulf Stream separation mechanism related to a downstream increase of the bottom slope. In essence, he studied the linearized potential vorticity equation for a quasigeostrophic one-layer system: the model describes a near-shore barotropic slope current and a seaward baroclinic current resting on a deep passive layer. The upstream condition is that the barotropic slope current has a constant cyclonic vorticity. Stern found that when the bottom slope increases downstream, the relative vorticity is generally augmented. The conservation of potential vorticity results in a seaward shift of the slope current that separates by flowing into the seaward baroclinic region. There are obvious similarities between the separation mechanism studied by Stern (1998) and the ideas of the present paper. One difference, however, is that the separation in Stern’s model is somewhat sensitive to the upstream values of the slope and the cyclonic vorticity, parameters that determine the ratio between the flow speed and the wave speed of the topographical waves. Our results suggest that the separation of the Gulf Stream may not be dependent on the upstream conditions, as long as the Ekman number is small compared to the Rossby number. The acceleration of the current due to the increase in bottom slope will depend on the nonlinear feedback mechanism between alongstream changes in relative vorticity and cross-slope flow (see Fig. 1). This will probably lead to high flow speeds at the separation point regardless of upstream conditions. Then the leftward turn of the isobaths will lead to a sharp increase in relative vorticity and send the flow toward deeper water, resulting in a separation much like that of the anticyclonic flow in our simulations and laboratory experiments (Figs. 4, 6).

6. Summary and conclusions

We have performed laboratory experiments and simulations of the barotropic circulation in a basin with topography as shown in Fig. 2. The circulation is forced by the drag from a rotating disc touching the water surface, as illustrated in Fig. 2. The circulation in the basin is weak, in the sense that the large-scale Rossby number (Ro) is significantly smaller than unity. The results show that there is a clear asymmetry between cyclonic and anticyclonic flow. Cyclonic flow is strictly topographically steered, and the flow pattern remains unchanged for varying forcing strength (Fig. 3). In contrast, the anticyclonic flow pattern changes with the forcing strength. For weak forcing, the pattern is similar to the cyclonic one, but with the opposite sign. For stronger forcing, locations of strong cross-slope flow develops, and the flow in the unforced basin (see Fig. 2) changes from anticyclonic to cyclonic (Fig. 4).

In section 3, we look for steady solutions by separately analyzing linear frictional theory and nonlinear inviscid theory. The linear theory is symmetric between cyclonic and anticyclonic flows, and it gives a good description of the cyclonic flow and the weakly forced anticyclonic flow. The inviscid nonlinear theory exhibits its strong asymmetry between cyclonic and anticyclonic flows. According to this theory, a topographically steered cyclonic flow with small relative vorticity everywhere is a stable minimum-energy state. For such a flow, \(J(b, q) = \mathbf{v} \cdot \nabla q = 0\), and the circulation is therefore well described by the linear theory. Weak anticyclonic flow is, according to the inviscid theory, neither a minimum-energy nor a maximum-energy state. If an anticyclonic topographically steered flow is a steady solution, it must be a saddle point in energy; it is therefore
most likely unstable. It is also possible that no steady anticyclonic flows following topography exist, as suggested by the results of LaCasce et al. (2008). In this case \( J(\psi, q) = \mathbf{v} \cdot \nabla q \neq 0 \), and the nonlinear terms will therefore contribute to the vorticity balance given by Eq. (5).

In either case—that is, whether anticyclonic steady flows are unstable or do not exist—the nonlinear terms will drive the flow away from topography, leading to the development of locations with strong relative vorticity and cross-slope flow. Without friction, this is the case even in the limit where the flow approaches zero. The physical mechanism is that conservation of potential vorticity involves a feedback mechanism between relative vorticity and cross-slope flow. For an irregular topography, the relative vorticity and cross-slope flow are amplified.

The nonlinear asymmetry between cyclonic and anticyclonic flows will come into play once the Rossby number is larger than the Ekman number. For cyclonic currents in the ocean commonly have Rossby numbers that are larger than the Ekman number. For cyclonic currents, like in the Nordic seas and the Arctic Ocean (Nøst and Isachsen 2003), this is not important because cyclonic currents are well described by linear theory. The last integral is a special case of Eq. (17), and its choice made in Eq. (A2) gives

\[
\frac{dF}{dq} = G(r), \quad \frac{dG}{dr} = Hr. \tag{A2}
\]

This choice of \( F \) requires both \( \psi \) and \( q \) to only be functions of \( r \), and it leads to \( \delta J_M = 0 \). This can be seen by using \( H\delta q = \nabla \cdot (\nabla (\delta \psi)/H) \), which leads to the following expression for \( \delta J_M \):

\[
\delta J_M = \int_A \nabla \cdot \left( \frac{G(r)}{H} \nabla (\delta \psi) \right) dA. \tag{A3}
\]

Since \( G \) is only a function of \( r \), \( \delta J_M \) can be written as

\[
\delta J_M = G(r) \int_A \nabla q \delta H \cdot dA = G(r)\delta \int_A qH dA = 0. \tag{A4}
\]

The last integral is a special case of Eq. (17), and its variation must therefore be zero. This shows that the choice made in Eq. (A2) gives \( \delta J_M = 0 \). The second variation of \( J_M \) can be found from Eq. (A1) with \( dF/dq = G \). This gives the expression for \( \delta^2 J_M \) given by Eq. (19).

**APPENDIX B**

**Variation of Energy in an Irregular Basin**

Minimizing \( E_T \) [given by Eq. (16)] subject to the constraint that \( C_F \) [given by Eq. (17)] is constant is equivalent to minimizing the function \( J_E = E_T + C_F \) without constraints. The first variation of \( J_E \) is given by

\[
\delta J_E = \int_A \left[ \nabla \psi \cdot \left( \frac{\delta (\nabla \psi)}{H} + \frac{dF}{dq} \delta qH \right) \right] dA. \tag{B1}
\]

From \( \delta q = \delta \zeta/H \) and expressing \( \zeta \) by Eq. (14), \( \delta J_E \) can be expressed as

\[
\delta J_E = \int_A \left[ \nabla \psi \cdot \left( \frac{\delta (\nabla \psi)}{H} + \frac{dF}{dq} \nabla \cdot \left( \frac{\delta (\nabla \psi)}{H} \right) \right) \right] dA. \tag{B2}
\]

We now choose \( dF/dq \) so that \( \delta J_E \) vanishes:

\[
\frac{dF}{dq} = \psi. \tag{B3}
\]
Since \( F \) is only a function of \( q \), this choice is possible only if \( \psi \) is a function of \( q \) alone; that is, if it represents a stationary solution. We then obtain

\[
\delta I_E = \int_A \left[ \nabla \psi \cdot \frac{\delta (\nabla \phi)}{H} + \psi \nabla \cdot \frac{\delta (\nabla \phi)}{H} \right] dA
\]

\[
= \int_A \nabla \cdot \left[ \psi \frac{\delta (\nabla \phi)}{H} \right] dA = \int_{\Gamma} \psi \frac{\delta (\nabla \phi)}{H} \cdot n \, dl = 0.
\]

(B4)

To see that this expression equals zero we have used the boundary condition of Eq. (15).

Thus, for stationary solutions, and with the choice of \( F \) defined by Eq. (B3), we have \( \delta I_E = 0 \). The second variation with the same choice of \( F \) is then given by Eq. (22).

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