Near-Inertial Oscillations and the Damping of Midlatitude Gyres: A Modeling Study

AARON GERTZ AND DAVID N. STRAUB

Department of Atmospheric and Oceanic Sciences, McGill University, Montréal, Quebec, Canada

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ABSTRACT

The classic wind-driven double-gyre problem for a homogeneous (unstratified) thin aspect ratio fluid is considered, but allowing for the flow to be depth dependent. Linear free modes for which the vertical wavenumber $k_z \neq 0$ are inertial oscillations, and they are excited with a large-scale stochastic forcing. This produces a background sea of near-inertial oscillations and their interaction with the vertically averaged flow is the focus of this study. In the absence of 3D forcing, the near-inertial motion vanishes and the barotropic quasigeostrophic system is recovered. With 3D forcing, 2D-to-3D energy transfers—coupled with a forward cascade of 3D energy and scale-selective dissipation—provide an energy dissipation mechanism for the gyres. The relative strength of this mechanism and a Rayleigh drag applied to the 2D flow depends on both the 3D forcing strength and the Rayleigh drag coefficient.

1. Introduction

Mechanical energy is input into the wind-driven circulation primarily at large scales. While geostrophic processes such as baroclinic instability and interactions with western boundaries can effectively transfer energy from basin scales to the ocean mesoscale, a further transfer to the centimeter scales where the dissipation occurs is more problematic. In this manuscript, we argue that a transfer of energy from geostrophic to inertial modes, coupled with a forward cascade of inertial mode energy, may play a role in this process.

More generally, there are several energy sinks for the large-scale circulation. Bottom friction provides an obvious dissipation mechanism in which the transfer to small scales occurs in association with a no-slip condition at the lower boundary. The net effect is often modeled by Ekman pumping, which damps energy in a horizontal-scale-independent fashion (e.g., Pedlosky 1987). A similar mechanism acting at the sea surface has also recently received attention (Duhaut and Straub 2006; Dawe and Thompson 2006; Hughes and Wilson 2008; Xu and Scott 2008; Zhai and Greatbatch 2007). This is related to a dependence of the air–sea momentum flux on the surface ocean velocity. However, this can be thought of as causing a reduction in the mechanical energy input rather than as an energy sink per se.

A third dissipation mechanism relates to a modification of geostrophic energy cascades by surface effects. Typically, quasigeostrophy (QG) is implemented assuming the surface to correspond to an isopycnal (e.g., $\theta_z = 0$ at $z = 0$, where $\theta$ is the quasigeostrophic streamfunction and $z$ is the vertical coordinate). A more careful implementation allows for $\theta_z|_{z=0}$ to evolve, and the total solution includes a ‘‘surface QG’’ contribution. Examination of these dynamics shows that a forward cascade of surface density variance leads to a change in the energy spectrum from $-3$ at large scales to $-5/3$ at smaller scales (e.g., Tulloch and Smith 2009). The transition scale is typically smaller than the baroclinic Rossby radius. Associated with this is a quasigeostrophic forward cascade of energy at small scales and in the near-surface layers. Note also that this does not imply a forward cascade of surface kinetic energy (Capet et al. 2008).

Loss of balance has also been suggested as a plausible candidate for dissipation (Molemaker and McWilliams 2005; Straub 2003; Wunsch and Ferrari 2004). Perhaps the most obvious mechanism is the generation of gravity waves by geostrophic flow over rough topography. Molemaker and McWilliams (2005) argued that this should be an effective sink only if the interior dynamics systematically redistribute energy toward the lower
boundary. While this may indeed occur, we focus instead on the loss of balance in the fluid interior. It has been conjectured (e.g., Molemaker and McWilliams 2005; Ngan et al. 2008) that the loss of balance in the fluid interior is related to instability mechanisms whereby balanced flow is unstable to unbalanced perturbations. In a statistical equilibrium, there would be a transfer to unbalanced modes and a related increase in dissipation associated with a forward cascade of unbalanced energy. McWilliams et al. (1998) examined solubility conditions for the Charney balance equations and related these to circumstances in which one might anticipate a loss of balance. Of particular interest is an ageostrophic anticyclonic instability that occurs when the magnitude of the strain field is comparable to or larger than the absolute vorticity. Similar criteria have been derived from parcel arguments or for the limit where horizontal pressure gradient forces are weak (e.g., Weiss 1991; Leblanc and Cambon 1997; Straub 2003; Ngan et al. 2008). Ngan et al. (2008) emphasize a random strain mechanism that occurs at high vertical and low horizontal wavenumbers—that is, in the part of the wavenumber space normally associated with near-inertial motion.

A series of rotating tank experiments lends further weight to the idea that geostrophic-to-gravity wave transfers can significantly enhance dissipation (Lovegrove et al. 1999, 2000; Williams et al. 2003, 2005). An evolving baroclinic flow was considered. For baroclinically stable flows, Kelvin–Helmholtz instability appeared to generate the waves, whereas baroclinically unstable flows showed “spontaneous wave generation.” A net effect was an increase in the overall dissipation.

In this manuscript we consider a thin aspect ratio homogeneous fluid on a $\beta$ plane. Specifically, we consider the hydrostatic equations in the absence of stratification, but allowing for the velocity field to have a vertical structure. Loosely speaking, we take the 2D (vertically averaged) modes as rough analogs to balanced motion and the 3D modes as unbalanced. Because of the thin aspect ratio, one might anticipate the 3D modes saturating at relatively low energy levels (Ngan et al. 2005)—and therefore producing only weak 2D-to-3D transfers. Our hypothesis is that external forcing of the 3D modes will allow for higher saturation levels of 3D energy and thus a stronger effective drag on the 2D flow. We therefore apply a large-scale stochastic forcing to the 3D modes. No attempt is made to tailor this forcing to represent any specific physical process; rather, we think of it as crudely representing sources of near-inertial motion other than balanced-to-unbalanced transfers occurring in the fluid interior (e.g., interactions with topography, high-frequency winds, tides, etc.). The idea is to maintain a background field of 3D motion with which the 2D flow can interact. We are particularly interested in interactions where the 2D flow is energetic, since this is where we anticipate the interaction being strongest. For this reason, our large-scale 3D forcing is concentrated in the western portion of the basin, near the boundary layer confluence. The 2D modes are forced as in the classic double-gyre problem, (e.g., Pedlosky 1987).

Our main result is that—even for zero Rayleigh drag—oceanographically relevant double-gyre circulations can be obtained, provided there is sufficient external forcing of the 3D modes. By “oceanographically relevant,” we mean those solutions whose time means exhibit fast western boundary currents and inertial recirculation, as well as a time mean Sverdrup regime far from the boundaries.

The paper is organized as follows. The next section describes the numerical model and introduces the diagnostics. Section 3 gives the results. We first describe the energetics for the reference cases with and without 3D forcing and then focus on the relative importance of the Rayleigh drag and the 2D-to-3D transfers as dissipation mechanisms over a range of Rayleigh drag coefficients and 3D forcing levels. A few robustness tests relating to vertical resolution and the horizontal structure of the 3D forcing are presented. Our conclusions are given in section 4.

2. Numerical model and experimental design

a. Model equations and discretization

We consider the incompressible Boussinesq equations for a homogeneous fluid in the thin aspect ratio, or hydrostatic, limit:

$$\frac{\partial \mathbf{u}_h}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{u}_h + \mathbf{f} \times \mathbf{u}_h = -\nabla_h P$$

$$\mathbf{v} = \mathbf{u}_h + \mathbf{w} \quad \nabla \cdot \mathbf{v} = 0 \quad \frac{\partial}{\partial z} P = 0,$$

where the subscript $h$ is for horizontal, $f$ is the Coriolis parameter, and the density has been absorbed into $P$. We further make the $\beta$-plane approximation: $f = f_0 + \beta y$. Because we assume the fluid to be both homogeneous and hydrostatic, the horizontal pressure gradient term applies only to the vertically averaged equations. It is then convenient to split the velocity into depth-averaged (denoted by an overbar) and depth-varying (denoted by a

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1 It is straightforward to show that nonhydrostatic pressure terms in the horizontal momentum equation are $O(\delta^3)$ with respect to the advective terms, where $\delta$ is the vertical to horizontal aspect ratio of the flow (e.g., Pedlosky 1987).
prime) components. Further, taking the vertical velocity, \( w \), to vanish at upper and lower (flat) boundaries, it follows that \( \mathbf{u}_n \) is nondivergent and may be represented by a streamfunction, \( \psi \). The equations can thus be rewritten as

\[
\frac{\partial}{\partial t} \mathbf{V}^2 \psi = -J(\psi, \mathbf{V}^2 \psi + \beta y) - \mathbf{z} \cdot \left[ \mathbf{V} \times (\mathbf{V}' \cdot \mathbf{V}) \mathbf{u}_h^2 \right] + \mathbf{F} - \rho \mathbf{V} \frac{\partial \psi}{\partial t} + A \mathbf{V}^2 \psi,
\]

where forcing and dissipation terms have been added.

Terms involving primed quantities in (2) allow for the 3D modes to feedback onto the 2D flow. In the absence of such feedback, (2) reduces to the barotropic quasigeostrophic equation. In this limit, a Rayleigh drag provides the bulk of the energy dissipation. When 3D effects are included, both the terms involving primed quantities and the hyperviscous dissipation also play active roles in the 2D energetics.

Note that the linear free modes of (3) are inertial oscillations. That is, ignoring the forcing dissipation and nonlinearity, (3) reduces to

\[
\frac{\partial}{\partial t} \mathbf{u}_h' + \mathbf{z} \times \mathbf{u}_h' = 0.
\]

We emphasize that no pressure gradient force term appears. As mentioned, this results from our assumptions of a homogeneous (unstratified) fluid and a thin aspect ratio. The last two terms on the rhs of (3) allow for transfers between different depth-dependent modes. Since most of our runs use only two evenly spaced vertical levels (see below) and hence have only a single depth-dependent mode, these terms vanish identically. In section 3c, modestly higher vertical resolution is considered and found not to alter our main conclusions. Dissipation takes the form of a hyperviscous term only; that is, no Rayleigh drag is applied to the 3D modes. Our reason for this is that we think of the 3D modes as being roughly analogous to near-inertial motions occurring in the interior of the ocean water column and of the Rayleigh drag as a crude model for bottom friction. Rayleigh drag is thus kept in (2) since the 2D problem does not yield realistic solutions without it. The Rayleigh drag is dropped from (3) because we think of the 3D modes as representative of motion that is largely unaffected by bottom drag in the actual ocean.

The use of hyperviscosity requires additional boundary conditions. Slip conditions were used; that is, \( \mathbf{V} \mathbf{\cdot} \mathbf{u}_h, \mathbf{V}^4 \mathbf{\cdot} \mathbf{u}_h \), \( \mathbf{V}^6 \psi, \partial(\mathbf{u}_n', \mathbf{n})/\partial t, \partial^3(\mathbf{u}_n', \mathbf{n})/\partial t^3 \), and \( \partial^5(\mathbf{u}_n', \mathbf{n})/\partial t^5 \), where \( n \) is the direction perpendicular to the boundary, are all zero at the boundaries.

Before discretization, it is convenient to rewrite (3) in vorticity-Bernoulli form:

\[
\frac{\partial}{\partial t} \mathbf{u}_h' = -\mathbf{z}(f + \mathbf{V}^2 \psi) \times \mathbf{u}' - \mathbf{z} \mathbf{\cdot} \mathbf{u}_h' - \mathbf{V}_h B' + \mathbf{F}' + A \mathbf{V}^2 \mathbf{u}_h' - (\mathbf{V}' \mathbf{\cdot} \mathbf{u}_h') + \mathbf{z} \mathbf{\times} \mathbf{u}_h',
\]

where

\[
B' = \mathbf{u}_h' \mathbf{\cdot} \mathbf{u}_h + \frac{1}{2} \mathbf{u}_h' \mathbf{\cdot} \mathbf{u}_h' - \frac{1}{2} \mathbf{u}_h' \mathbf{\cdot} \mathbf{u}_h'.
\]

As discussed above, for the special case of two evenly spaced levels, this simplifies: the last four terms of (5) and the last two terms on the rhs of (6) vanish identically.

Equations (2) and (5) were solved numerically on a 512 \times 512 \times N grid. For most of our simulations, \( N = 2 \). The severe truncation in the vertical allows us to concentrate limited computing resources on horizontal resolution and diagnostic calculations. It also has the advantage of simplicity; that is, our system with this corresponds to a horizontal resolution of 8 km. The domain was a 4000 km2 box. On a 512 \times 512 grid, arranged for the elliptical inversion and a third-order Adams–Bashforth discretization was used for time stepping. For our \( N = 8 \) simulations, the \( w' \) terms in (5) are nonzero and \( w' \) was calculated by imposing a rigid-lid condition at the surface and integrating the horizontal divergence vertically downward.

b. Standard parameters and forcing

The domain was a 4000 km2 box. On a 512 \times 512 grid, this corresponds to a horizontal resolution of 8 km. The value of \( \beta \) was chosen to be \( 2 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1} \) and the center latitude value of \( f \) was \( 7.5 \times 10^{-5} \text{ s}^{-1} \). In (2), \( \mathbf{F} \) was taken to correspond to the classic double-gyre forcing:

\[
\mathbf{F} = -F_0 \sin \left( \frac{2\pi y}{L} \right),
\]

where \( L = 4000 \text{ km} \). In addition, \( F_0 \) was chosen so that the inertial scale, \( F_0/\beta \) (e.g., Pedlosky 1987), was 44 km—or 5.5 grid points.
A large-scale stochastic forcing was applied to the $x$ component of (5). Specifically, $F' = \hat{x}F'_x$, with

$$F'_x = \alpha(t) \cos\left(\frac{\pi z}{H}\right) \exp^{-b^2(\text{cubic})} \sin\left(\frac{\pi y}{L}\right),$$

where $b = L/2$, the origin is at the southwest corner of the basin, and $H$ is the ocean depth. The horizontal structure of $F'_x$ has a maximum at the center latitude on the western side of the basin and tapers to zero like a sinusoid in $y$ and like a Gaussian in $x$. This structure was chosen so as to concentrate the forcing in that part of the domain where we expect the double-gyre circulation to be most energetic and the local Rossby number to be sufficiently large so as to allow for significant 2D-to-3D transfers. The reason for this is that we expect the “loss of balance” mechanism to be most active where the local Rossby number is large. In section 4c, we also briefly consider a spatially uniform horizontal structure. The coefficient, $\alpha(t)$, was given by an Ornstein–Uhlenbeck—or, equivalently, an autoregressive AR(1)—process. That is,

$$\alpha(t) = F_{3D} \hat{\alpha}(t),$$

where

$$\hat{\alpha}(t + \Delta t) = \hat{\alpha}(t)(1 - \epsilon) + \alpha_0 \Delta t^{1/2},$$

in which $r$ is randomly selected from a normal distribution, $\epsilon = \Delta t/1$ day and $\alpha_0 = (\sqrt{2} \text{ day})^{1/2}$. The RMS value of $\hat{\alpha}$ is 1 and its frequency spectrum is approximately white for periods greater than 1 day and falls off like $\omega^{-2}$ for higher frequencies, $\omega$. The amplitude is determined by $F_{3D}$. We used three different values of $F_{3D}$, corresponding to $2L F_0 / \pi$, $4L F_0 / \pi$ and $6L F_0 / \pi$ [cf. Eq. (7)]. These will be referred to as our low, medium, and high levels of 3D forcing. The time step $\Delta t$ [for both $\alpha(t)$ and the discretized versions of Eqs. (2) and (5)] was taken to be less than or equal to $\sqrt{2}$ of a day. This ensured that inertial frequencies were well resolved. Numerically, the value of the hyperviscous coefficient limited the size of the time step.

**c. Diagnostics**

We are interested in the spectral energy transfers. The 2D energy equation is obtained by multiplying (2) by $\psi_p$ and integrating over the domain. To get a Fourier space version, we first define $\psi_p$ to be the periodic extension of $\psi$ into a checkerboard pattern. That is, for $x$ and $y$ between 0 and $L$, $\psi_p(-x, y) = -\psi(x, y)$, $\psi_p(x, -y) = -\psi(x, y)$, and $\psi_p(-x, -y) = \psi(x, y)$. Similar periodic extensions are made for the various terms on the rhs of (2). For example, let $a_J$ denote the periodic extension of the Jacobian term on the first line of (2). Similarly, $a_{3D}$, $a_{F}$, $a_{\text{drag}}$, and $a_{\text{hyper}}$ are the extensions of the primed, forcing, drag, and hyperviscosity terms, respectively. An equation for the domain-integrated 2D energy, $E_{2D}$, is then given by

$$\frac{\partial E_{2D}}{\partial t} = A_J + A_{3D} + A_F + A_{\text{drag}} + A_{\text{hyper}},$$

where

$$A_J = \frac{1}{2} \left( \hat{\psi}_p \hat{a}_J^* + \hat{\psi}_p^* \hat{a}_J \right),$$

$$A_{3D} = \frac{1}{2} \left( \hat{\psi}_p \hat{a}_{3D}^* + \hat{\psi}_p^* \hat{a}_{3D} \right),$$

in which hats denote Fourier transforms and asterisks denote complex conjugation. In addition, $A_F$, $A_{\text{drag}}$, and $A_{\text{diss}}$ are defined similarly. When negative, $A_{3D}$ represents 2D-to-3D energy transfers and will be of particular interest in the next section.

The 3D energy equation is formed by taking the scalar product of $\mathbf{u}_h$ with (5):

$$\frac{\partial E_{3D}}{\partial t} = B_F + B_{\text{hyper}} + B_{\text{Cor}} + B_{2D} + B_{3D-3D},$$

where the $B$s are defined by analogy with the $A$s. For example, $B_{2D}$ collects terms involving the 2D flow field. Specifically, let

$$b_{2D}^x = \left\{ \dot{x} \cdot \left[ -\mathbf{V}_h \cdot \mathbf{u}_h^* - \mathbf{z} \times \mathbf{u}_h^* - \tilde{z} \mathbf{V}_h^* \psi \times \mathbf{u}_h^* \right] \right\}_p,$$

$$b_{2D}^y = \left\{ \dot{y} \cdot \left[ -\mathbf{V}_h \cdot \mathbf{u}_h^* - \mathbf{z} \times \mathbf{u}_h^* - \tilde{z} \mathbf{V}_h^* \psi \times \mathbf{u}_h^* \right] \right\}_p,$$

where, as before, the subscript $p$ denotes a periodic extension. All terms on the rhs of (14) are quadratic, involving products of 2D and 3D fields. The periodic extensions are made such that terms in the $\dot{x}$ equation are antisymmetric about $x = 0$ and symmetric about $y = 0$ and terms in the $\dot{y}$ equation are symmetric about $x = 0$ and antisymmetric about $y = 0$. Then,

$$b_{2D} = \frac{1}{2} \left( \mathbf{u} \cdot \mathbf{b}_{2D}^x + \mathbf{u}^* \cdot \mathbf{b}_{2D}^x + \mathbf{u} \cdot \mathbf{b}_{2D}^y + \mathbf{u}^* \cdot \mathbf{b}_{2D}^y \right).$$

Physically, $B_{2D}$ includes both transfers from 2D to 3D (or vice versa) as well as energy transfers between different 3D modes.
The terms \( B_{\text{Cor}} \), \( B_F \), \( B_{\text{hyper}} \), and \( B_{3D-3D} \) are defined similarly to \( B_{2D} \). One might anticipate that \( B_{\text{Cor}} \) vanishes since the Coriolis term can do no work. Indeed, on an \( f \) plane it is straightforward to show that this is the case. On a \( \beta \) plane, however, the \( y \) dependence can allow for transfers. A physical interpretation of this is given in section 3a. Note that \( B_{3D-3D} \) involves the last four terms of (5). These terms are identically zero for our \( N = 2 \) simulations and \( B_{3D-3D} \) will not be considered further.

The \( \mathcal{A}s \) are functions of the horizontal wavenumbers, \( k_x \) and \( k_y \), and of time. We will consider only time averages and will also integrate over horizontal wave-numbers in (binned) rings of constant \( k_h = \sqrt{k_x^2 + k_y^2} \). The \( B \)s are also functions of the vertical wavenumber. Since most of our runs have \( N = 2 \), however, there is only one nonzero vertical wavenumber. As such, both the time averaged \( \mathcal{A}s \) and \( \mathcal{B}s \) can be considered to be functions of \( k_h \) alone. Note, however, that the \( \mathcal{A}s \) are functions of the horizontal wavenumber of the relevant 2D mode, that is, the mode to which energy is either being added (\( A > 0 \)) or removed (\( A < 0 \)). Similarly, the \( \mathcal{B}s \) are functions of the \( k_h \) associated with the 3D mode in question. Thus, for example, if \( A_{3D} \) is positive at a given \( k_h \), 3D modes are putting energy into a 2D mode with that \( k_h \). Since \( a_{3D} \) involves only 3D modes, this energy must be coming from the 3D flow; however, which \( k_h \) are losing 3D energy are not identified.

Conversely, if \( B_{2D} \) is positive at some \( k_h \), energy is being added to 3D modes at this wavenumber. As with \( A_{3D} \), the wavenumber of the modes losing energy is not known. Moreover, since \( b_{2D} \) involves both 2D and 3D modes, it is also not known whether the energy is coming from 2D modes or from other 3D modes. For example, the transfer could be from 3D modes with low \( k_h \) to 3D modes with higher \( k_h \). Similar triadic interactions have been described in other systems: for example, Lelong and Riley (1990) for rotating-stratified flows and Waleffe (1993) for homogeneous rotating turbulence.

At low Rossby number, the 2D (or vortical) member of a triad acts as a “catalyzer.” That is, it allows 3D (or gravity wave) energy to be transferred between wave vectors while the catalytic (2D or vortical) member of the triad does not lose or gain energy itself. Note that this is a low Rossby number result; at higher Rossby numbers, 2D–3D exchanges are also possible.

3. Results

We wish to consider how the addition of 3D forcing affects the 2D double-gyre problem. We first describe the energetics for a reference simulation. Following this, we consider aspects of the energetics and how they vary with the size of the Rayleigh drag coefficient and the 3D forcing level. Various robustness tests are then briefly considered in section 3c.

### a. Reference simulation

As a reference case, we consider \( r = 5 \times 10^{-8} \text{ s}^{-1} \). A purely 2D (\( F_{3D} = 0 \)) simulation is compared to a 3D simulation using our medium value for \( F_{3D} \). For both the 2D and 3D simulations, time mean streamfunctions show familiar features of the double-gyre problem: energetic western boundary currents, inertial recirculations, and a large-scale Sverdrup flow in the interior (not shown). The 3D forcing causes \( E_{3D} \) to saturate at a nonzero level. Although this remains small compared to the \( E_{2D} \) in all cases considered, there is an appreciable effect on the 2D energy. This is evident in Fig. 1, which shows energy time series for both simulations. In the simulation with 3D forcing, \( E_{2D} \) equilibrates at about 78% of its mean value in the \( F_{3D} = 0 \) case. Moreover, the two \( E_{2D} \) time series show distinctly different characteristics, with more low-frequency variability in the case with 3D forcing.

We emphasize that the 3D motion has a strong near-inertial component. This is evident in Fig. 2, which shows the frequency spectra, \( E(\omega) \), computed for four time series of \( u_x \) taken at regularly spaced intervals in \( y \) and positioned about \( 1/4 \) of the basin width eastward of the western boundary. Peaks are seen at the local inertial frequency, and energy-weighted frequencies have values within a few percent of \( f \) in all cases (not shown). Note that the 3D energy level is somewhat weaker in the northern portion of the domain. This is consistent with the dominant period of the stochastic forcing (1 day) being longer than the inertial period in the northern portion of the domain.

For both simulations, the dominant 2D energy balance has the Rayleigh drag providing the main sink for
the power input. In both cases, the power source (which is dependent on the large-scale flow field) is about the same. With $F_{3D} = 0$, the Rayleigh drag accounts for about 97% of the energy dissipation, with the hyperviscous term making up the remaining 3%. With 3D forcing, the Rayleigh drag accounts for 74%, with the remaining 26% being split between a net 2D-to-3D transfer (10%) and the (2D) hyperviscosity term (16%). As seen below (Fig. 3), the drastic increase in dissipation by the 2D hyperviscosity term is associated with a 3D-to-2D energy transfer occurring at small scales.

Consider further the various terms in the 2D energy budget as functions of the horizontal wavenumber. We first examine the case where $F_{3D} = 0$, for which the standard barotropic double-gyre problem is recovered. Figure 3a shows $A_j$, $A_{3D}$, $A_{\text{hyper}}$, and $A_p$. When multiplied by $k_h$, the area “under the curve” corresponds roughly to the amount of energy added (positive $A$) or removed. This correspondence is imperfect due to finite bin size effects. Recall also that the various $A$s have been integrated over rings of constant $k_h$ in the $k_x$, $k_y$ plane. The overall balance has that forcing injects energy into low horizontal wavenumbers and $A_j$ transfers the energy to the moderately higher energy-containing wavenumbers where it is removed by Rayleigh drag. The term $A_j$ is broken up into its $\beta$ and nonlinear components in Fig. 4b. As might be anticipated, the $\beta$ term is associated with a forward energy transfer and the nonlinear term is associated with an inverse energy transfer. That is, $A_{j-\text{(beta)}}$ removes energy from small wavenumbers and adds energy at moderate wavenumbers. The opposite is true for $A_{j-\text{(nonlinear)}}$. This is to be expected given the inverse cascade of 2D turbulence (Kraichnan 1967) and the well-known phenomenon of westward intensification (e.g., Pedlosky 1996). Note that the inertial range of an inverse energy cascade would appear in the figure as near-zero values over a range of $k_h$ between positive values on the left and negative values on the right. What is seen instead is that the source and sink regions are adjacent—that is, not separated by an inertial range. This is not too surprising, given that the inverse cascade begins at relatively small $k_h$.

Figure 3c is similar, but for our standard run with 3D forcing. The main balance is essentially as in Fig. 3a; however, there are differences. The term $A_{3D}$ is negative (indicating 2D-to-3D transfers) between about $k_h = 4$ and 30. Positive transfers are seen at higher $k_h$ as well as at very low $k_h$. Note that the high-wavenumber peak is similar in amplitude and position to the peak in $A_{\text{hyper}}$. Thus, there is a 2D-to-3D energy transfer at moderate $k_h$ and an injection back to 2D modes at low $k_h$ and near the dissipation range. As mentioned above, these changes result in a net energy sink for the 2D flow, although the Rayleigh drag nonetheless remains dominant in this simulation.

Figure 3d is similar but shows the terms in the 3D, instead of the 2D, energy balance. As above, the error term is small; the various terms sum approximately to zero. Figure 3d shows energy being input by the forcing at large scales and transferred to intermediate $k_h$ by the Coriolis term. The main effect of $B_{2D}$ is to remove

3 Apparent errors are particularly large when low wavenumbers correspond to large values. Note that some terms (e.g., $A_j$, $A_{j-\text{(beta)}}$, $A_{j-\text{(nonlinear)}}$) are transfers that should identically integrate to zero. We verified that they do to a good approximation: average values are less than 1%–2% of the RMS value. Some error is to be expected since we are using a spectral diagnostic on data obtained from a gridpoint model. Because of finite bin size effects and large values at low $k_h$, the apparent errors in Fig. 3 appear somewhat larger.

4 We normalize such that $k_h = 1$ corresponds to the gravest Fourier component in our extended domain (see section 2c). That is, the dimensional wavenumber, $k_h = 2m/L$, becomes $n$ after normalization. Note that our smallest physical wavenumber corresponds to a structure, $\psi \sim \sin(\pi x / L) \sin(\pi y / L)$, or to a normalized $k_h = 1/\sqrt{2}$. Contributions from wavenumbers $k_h < 1$ are included in the $k_h = 1$ bin in Fig. 4 and similar figures. Note also that although the forcing is applied as a single sinusoid in $y$ (see section 2c), an $x$ dependence is introduced upon reflection. As such, $A_p$ is not a delta function as one might imagine but does work out to be concentrated in a few small wavenumbers.
energy from small-to-intermediate $k_h$, transferring it to high wavenumbers, where dissipation becomes active.

Note from Figs. 3c and 3d that both $B_{2D}$ and $A_{3D}$ are positive at high wavenumbers. This may seem counterintuitive: after all, there cannot simultaneously be a net positive energy transfer in both directions. The apparent paradox is resolved when it is recalled that the horizontal axis in Figs. 3c is the $k_h$ of the 2D modes whereas in Fig. 3d it is the $k_h$ of the 3D modes. Recall also that $B_{2D}$ (labeled 2D–3D transfers in Fig. 3d) may also include 3D-to-3D transfers in which the 2D flow acts as a catalyst.

The forward energy transfer at low $k_h$ associated with the Coriolis force (Fig. 3d) merits further attention. It can be understood in terms of the $y$ dependence of the Coriolis parameter. Consider the following simple example. Ignoring boundary effects, we take the initial conditions to be given by $u_h = (1, 0)$ everywhere and further suppose linear dynamics, so that Eq. (4) applies. Then, $u_h(y, t)$ is given by $u_h = (u', v') = [\cos(ft), \sin(ft)]$. Thus, for example, while $u'$ remains bounded, $\partial_t u' = -\beta t \sin(ft)$ increases linearly with time. This can only be the case if the meridional length scale collapses. In physical space, a zonally banded structure of inertial oscillations results (cf. Fig. 10).

The transfers described above are broadly consistent with a 2D-to-3D transfer at relatively large scales, coupled with a forward cascade of 3D energy. At the high-wavenumber end of this cascade, scale-selective dissipation removes the energy. Below, we argue that this scenario becomes a more dominant part of the overall energy balance when either the 3D forcing level is increased or the Rayleigh drag coefficient is decreased.

b. Parameter dependence

We consider (i) how our results depend on the level of 3D forcing, $F_{3D}$, and (ii) how they depend on the magnitude of the Rayleigh drag coefficient.

Figure 4 shows the 2D and 3D spectra for three levels of $F_{3D}$. The 3D spectra are shallow and for sufficiently high levels of $F_{3D}$ can cross the 2D spectra to the left of the dissipation range. The 2D spectra increase from $k_h = 1$
to a maximum at around $k_h = 5$, followed by an approximately $-3$ slope extending either to the dissipation range (for low $F_{3D}$) or to the vicinity around which the 2D and 3D spectra cross (for larger $F_{3D}$). Stronger 3D forcing is associated with a reduction of energy at large horizontal scales. High-wavenumber 2D energy levels, by contrast, increase with $F_{3D}$. This shoaling of the 2D spectra at high $k_h$ is consistent with a reinjection of energy back into 2D modes by $A_{3D}$ at the tail end of the forward 3D energy cascade.

The important point is that the large-scale 2D energy is effectively damped by the presence of 3D motion. This is further evident from consideration of $A_{3D}$ and $A_{hyper}$ (Fig. 5). Both terms essentially maintain their forms, but become larger in amplitude with increased levels of 3D forcing. Since in the purely 2D problem $A_{hyper}$ is negligibly small (3% of total), we can consider the summed integrals of $A_{hyper}$ and $A_{3D}$ as a reasonable measure of the net damping of 2D energy via this loss of balance mechanism. In other words, additional sinks of 2D energy appear in the form of a net 2D-to-3D transfer and an increased dissipation by the 2D scale-dependent dissipation. From the weakest to the strongest level of $F_{3D}$, this increases from 13% to 48% of the total energy dissipation.

Figure 6 shows $A_{3D}$ and $A_{hyper}$ for different values of the drag coefficient, $r$. As might be anticipated, weaker drags lead to stronger transfers. Additionally, the positive $A_{3D}$ values seen at $k_h < 4$ for high values of $r$ disappear as $r$ is reduced. Comparing with Fig. 3, we see that for low values of $r$, $A_{3D}$ has a shape and a size that are comparable to those of $A_{drag}$ in the reference simulation. In other words, it appears that the loss of balance mechanism in some sense takes the place of the Rayleigh drag energy damping mechanism as $r$ is reduced. A notable difference is that $A_{3D}$ tends to damp 2D energy at slightly smaller scales than does $A_{drag}$, which directly damps the energy-containing scales.

Energy spectra for the different values of $r$ are shown in Fig. 7. It is clear that the 3D spectra have little dependence on $r$, whereas the 2D spectra do: smaller $r$ corresponds to increased levels of large-scale 2D energy. This is not unexpected and the point we emphasize is that this increase is considerably reduced from the purely 2D case. That this is the case is clearly seen from Table 1, which gives time-averaged 2D energy levels for the different values of $r$ for cases with and without 3D forcing.
The case where $r = 0$ is of special interest. We compare two 5000-day simulations: one with our medium value of $F_{3D}$ and one with $F_{3D} = 0$. A statistical equilibrium appeared to be reached after about 3500 days of integration when 3D forcing was included, but the equilibrium was less clear with $F_{3D} = 0$. (For the run without 3D forcing, energy appeared to be leveling off at the end of the simulation.) Time-averaged streamfunctions for the latter half of the runs are shown in Fig. 8. The case with 3D forcing remains in an oceanographically relevant regime whereas the purely 2D case is dominated by a pronounced four-gyre structure. The additional counterrotating gyres are also evident, albeit with much weaker amplitudes, when 3D forcing is included. This four-gyre structure has been seen elsewhere (Greatbatch and Nadiga 2000; Scott and Straub 1998). The strong counterrotating gyres suggest an equilibrium in which the large-scale velocity field projects only minimally onto the forcing, a possibility considered by Scott and Straub (1998). That is, with a four-gyre response and a two-gyre forcing, the wind power source, $\int \psi F \, dx \, dy$, tends to zero since the integral of $\sin(2\pi y) \sin(4\pi y) \, dy$ vanishes.

c. Robustness

We performed a few additional simulations to test the robustness of our results. In particular, we repeated our reference simulation with modestly higher vertical resolution and carried out a few simulations for which the horizontal structure of the 3D forcing, $F'$, was uniform (cf. Eq. (8)).

We repeated our reference simulation with eight vertical levels. As before, $F'$ had a $\cos(\pi z/H)$ structure so that only the first vertical mode was forced directly. One might anticipate that 3D-to-3D transfers would cascade energy to the higher vertical modes. In other words, it could be that disallowing such transfers (by

![Energy Spectra (different Rayleigh coeffs.)](image1)

![Time-averaged streamfunctions for runs with zero Rayleigh drag: (left) $F_{3D} = 0$ and (right) medium value of $F_{3D}$](image2)

![Table 1](table1)

<table>
<thead>
<tr>
<th>Rayleigh drag ($\times 10^{-8}$ s$^{-1}$)</th>
<th>10</th>
<th>5</th>
<th>2</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{2D}$ (with 3D forcing)</td>
<td>5.9</td>
<td>9.1</td>
<td>15</td>
<td>20</td>
<td>36</td>
</tr>
<tr>
<td>$E_{2D}$ (without 3D forcing)</td>
<td>6.0</td>
<td>12</td>
<td>28</td>
<td>49</td>
<td>88</td>
</tr>
</tbody>
</table>
setting \( N = 2 \) forced energy that might have otherwise cascaded to higher vertical wavenumbers to instead feedback onto the 2D flow. Figure 9a shows time averages of \( A_{3D} \) for both our standard run and its analog with \( N = 8 \). The difference between the two is minimal. An examination of the vertical energy spectra shows a steep drop-off, with roughly 90% of the 3D energy contained in the gravest vertical mode (not shown). Presumably, catalytic interactions associated with \( B_{2D} \) are simply more effective at cascading 3D energy forward in horizontal wavenumber space than interactions associated with \( B_{3D-3D} \) are at effecting a vertical wavenumber cascade.

We also performed several additional simulations using a more complex vertical structure for \( F' \). For these simulations, it was necessary to add a small amount of vertical (hyperviscous) dissipation to ensure stability. Because of this, some of the forcing was directly injected into the dissipation range and, although we increased \( F_{3D} \) by a factor of 3 relative to its value in our reference simulation, \( E_{3D} \) saturated at lower values than in our reference simulation. Instead, it was comparable to our low \( F_{3D} \) forcing case (cf. Fig. 4). Spectral transfers were very similar to those in that simulation (not shown).

Use of a uniform horizontal structure for \( F' \) led to more significant differences. We carried out a number of simulations with \( N = 2 \) and for which \( F' \) had the same vertical structure as in (8), but with a spatially uniform horizontal structure. Since the forcing was applied over the entire domain, \( F_{3D} \) was reduced by a factor of 2 relative to our reference simulation. Nonetheless, 3D energy levels were higher (by about a factor of 2) compared to those in our reference simulation. We also found it necessary to use a larger value of the hyperviscous coefficient (and correspondingly smaller time step) in these simulations.5 Figure 9b shows \( A_{3D} \) for a range of Rayleigh drag coefficients. Comparing with Fig. 6, the low-wavenumber 3D-to-2D transfers are considerably larger than were seen previously. Additionally, both the peak dissipation wavenumber and the high-wavenumber 3D-to-2D transfers were shifted to the left (e.g., the dissipation peak was approximately 70% of its previous value). For \( r = 0.5 \times 10^{-7} \) s\(^{-1}\), the low-wavenumber 3D-to-2D transfers dominated over the moderate wavenumber 2D-to-3D transfers, but as before the 2D-to-3D transfers became increasingly dominant as \( r \) was reduced.

We do not yet fully understand this difference in behavior; however, some insight comes from an inspection of the fields in physical space. Figure 10 compares typical snapshots of \( u' \) for our reference simulation and its horizontally uniform \( F' \) counterpart. With our previous forcing, 3D energy was concentrated near the center latitude and on the western half of the domain. With horizontally uniform forcing, 3D energy was weak in this region and stronger outside of the center latitudes (particularly near the southern boundary). We speculate that this may be due to the strong 2D flow in the confluence region resulting in a stronger horizontal cascade of 3D energy there (essentially, the 2D turbulence would stir the 3D flow fields). A look at time averages (not shown) of the physical space structure of \( A_{3D} \) showed a net 3D-to-2D transfer along the boundaries (particularly along the southern boundary) and negative net transfers in the center latitudes. This led us to

5 For comparison, purely 2D simulations were also carried out using the higher viscous coefficient. In these simulations, the hyperviscous dissipation accounted for roughly 15% of the total—considerably more than for the case of our reference simulation (i.e., with our middle value of \( F_{3D} \) and for which the hyperviscous dissipation was 3% of the total).
suspect that the sign of the transfer was dependent on a regional Rossby number; however, preliminary attempts to test this were not revealing. Clearly, more work must be done to understand both the spatial structure of $A_{3D}$ and how it depends on the horizontal structure of $F'$; however, we leave this for future consideration.

4. Discussion

We used an idealized model to illustrate the possibility that near-inertial motion in the ocean may interact with wind-driven ocean gyres in such a way as to provide an energy sink for the gyres. Our model has ignored many processes—stratification, topography, non-hydrostatic effects, mixed layer processes, and realistic winds—which are clearly important for a more complete description. Nonetheless, we believe that the results may apply more generally and suggest avenues of further research in ocean gyre energetics. In particular, we have found examples in which midlatitude gyre circulations are dissipated by interactions between vertically averaged (ostensibly “balanced”) and 3D (“unbalanced” or near inertial) modes. In our simulations, inertial modes have an appreciable effect on the 2D flow, but only when they are externally forced. We interpret this forcing as representative of unbalanced motion generated by effects not included in the model (interactions with topography, high-frequency winds, tides, etc.). Generation of unbalanced motion via a loss of balance mechanism alone does not appear to provide a sufficient source of 3D energy, although this may be different in a stratified setting.

The overall picture we suggest is that (i) factors other than the loss of balance may be essential to maintaining significant background levels of near-inertial motion and (ii) given the presence of such motion, a viable sink for the geostrophic circulation is that energy can be transferred to near-inertial modes. Energy shunted into the inertial modes then cascades forward in wavenumber space and is ultimately dissipated. This mechanism seems especially active in regions where the geostrophic
flow is strong and in situations where other energy sinks are not available. In particular, our runs with a weak or zero Rayleigh drag showed the 2D-to-3D transfer mechanism to be more effective than otherwise. Finally, when the 3D energy was large outside of the jet confluence region, large-scale 3D-to-2D transfers could occur. It thus remains possible that interactions with near-inertial modes could provide a net energy source, rather than a sink, for midlatitude gyres.

The 3D-to-2D energy transfers seen at low $k_b$ in some of our simulations are reminiscent of the results from studies of (nonhydrostatic) homogeneous rotating turbulence in more idealized studies than what was presented here (Bourouiba and Bartello 2007; Smith and Waleffe 1999). These studies show 3D-to-2D energy transfers occurring for $O(0.1)$ Rossby numbers, while 2D-to-3D transfers occur for larger Rossby numbers. In our simulations, local Rossby numbers (measured as $\nabla F_\beta / f$) attained $O(1)$ values while the global RMS Rossby number remained small. A preliminary look did not show a clear correlation between the local value of the Rossby number and the sign of the transfer. On the other hand, our results did show low-wavenumber 3D-to-2D transfers to be more prevalent when the Rayleigh drag was higher (and the RMS Rossby number was lower).

Our results are also relevant to the problem of inertial runaway in the ocean gyre circulation problem (e.g., Ierley and Sheremet 1995). Inertial runaway can be thought of as the tendency for ocean models to produce qualitatively unrealistic results as dissipation parameters are made small. The effect is at its worst when a net vorticity forcing is applied (e.g., Pedlosky 1996) and cross-gyre exchanges in potential vorticity (e.g., Marshall 1984) play a lesser role in the gyre vorticity balance. Scott and Straub (1998) have suggested that runaway is related to an inability of the geostrophic dynamics to dissipate energy. In our simulations, we tried to make the scale-dependent dissipation operator as small as possible. It therefore played only a negligible role in the energetics, except where there was a forward energy cascade. When the Rayleigh drag was also made small, 2D energy increased dramatically. That is, inertial runaway occurred. The effect, however, was markedly less when the 2D-to-3D energy transfer mechanism was active (cf. Table 1).

Finally, we note that while our stochastic 3D forcing resulted in fairly weak levels of 3D energy saturation, instantaneous values could be several times larger and increasing $F_{3D}$ required a further reduction of the time step. It remains to be seen whether the 2D-to-3D or, more generally, the balanced-to-unbalanced transfer and forward cascade mechanism described here will also play a key role when more realistic forcings are used. If so, it may be that this additional route to dissipation will help make models less sensitive to the precise tuning of poorly known parameters.

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