A Two-Gyre Ocean Model Based on Similarity Solutions

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ABSTRACT

An ocean model based on similarity solutions derived from the thermocline equations is defined, and its properties are studied in a stationary case. This model only applies to cases that show small departures from zonality. It is continuously stratified, has open boundaries (in particular it extends indefinitely westward), and does not represent boundary currents. The zonal gradient of density is prescribed constant at the northern and southern boundaries, the density along the eastern coast is prescribed at the surface and bottom, and a zonal (null) Ekman pumping is applied at the surface (bottom). The mean state obtained with this simple model shows a quite realistic thermocline pattern. In particular, two gyres are represented, which improves the results obtained from the previous models based on similarity solutions. Waters of nearly constant density are found in the first 600 m in the subpolar gyre, and the northward “tilt of the thermocline” is also reproduced. In this model, the role of the horizontal diffusion coefficient is crucial as it allows mass exchanges between the gyres and has a strong impact on the meridional and vertical velocities, which in turn affect the thickness of the thermocline. Eventually, the results suggest that, as soon as the classical integral constraints exerting on the potential vorticity fluxes are taken into account, a (detailed) representation of the boundary currents is not needed to model the gross features of the thermocline in the open ocean.

1. Introduction

The large-scale ocean circulation outside the frictional or inertial boundary layers may be studied using the planetary geostrophic equations (sometimes called thermocline equations). They assume geostrophy, hydrostatic equilibrium, and take into account the spherical geometry of the earth and the horizontal variations of the density field. These equations, though considerably simpler than the primitive equations used in ocean general circulation models, remain difficult to solve, and their mathematical properties poorly known. To get a better understanding of the latter, particular solutions have been searched in the stationary case. A famous one is that of Needler [1967; for the ideal fluid thermocline, see also Welander (1971)]. However, it presented a deficiency since, imposing an arbitrary relationship between the surface density and the Ekman pumping, it could not satisfy physical boundary conditions. More systematic studies, which laid emphasis on the stationary case, have since been undertaken; in particular, symmetry group methods have led to more general solutions (e.g., Salmon and Hollerbach 1991; Hood 1996; Hood and Williams 1996). Salmon and Hollerbach (1991) listed 16 similarity solutions; two of them led to a linear, two-dimensional, advection–diffusion equation for the temperature. In the limit where the vertical diffusion coefficient remained small (there was no horizontal diffusion), these solutions showed some features of the real thermocline. However, Salmon and Hollerbach failed to obtain a two-gyre solution and speculated that it might represent a general limitation of the thermocline equations without horizontal diffusion. Hence, they suggested studying solutions where its role would be taken into account.

Some of these solutions had been previously searched and found in a stationary context by Filippov (1968). He listed four families of solutions where the horizontal diffusion was taken into account. These solutions were characterized by a density that depended only on the ratio between the vertical and horizontal diffusion coefficients. The pattern density remained thus unchanged as soon as this ratio was not modified. Unfortunately,
Filippov did not describe the solution patterns to which his complicated expressions led. Only the mathematical expressions were given. He also listed three families of solutions for which the horizontal diffusion vanishes. Among the latter, only one was described in a more detailed way: a one-gyre circulation was obtained with reasonable patterns of surface density and vertical velocity. The nature of the solutions when both horizontal and vertical diffusion coefficients are taken into account has therefore never been analyzed.

The time-varying solutions of the planetary equations have been studied from symmetry group methods only by Edwards (1996), to our knowledge. In this theoretical study, Edwards neglected dissipative processes, but he performed a comparison with a numerical solution including these processes to evaluate the accuracy of his solutions. At annual frequency the agreement was good, but it deteriorated at decadal time scales because diffusive effects became important.

In this paper, we define a model based on thermocline equations, which differs from those suggested by Filippov (1968) since it is time dependent. Moreover, the degree of realism of the solutions and how they depend on the horizontal diffusivity is discussed, which was not done by Filippov. The model introduced here also differs from that of Edwards (1996) since it takes into account dissipative processes. Following the suggestion of Salmon and Hollerbach (1991), solutions of the thermocline equations with vertical and horizontal diffusion are built from a combination of their two “linear” similarity solutions to preserve simplicity. These solutions verify a system of two quasi-linear partial differential equations, which leads to a mathematically well-posed problem under simple boundary conditions. The particular solution described here consists in a continuously stratified two-gyre ocean and succeeds in reproducing a quite realistic thermocline.

The model developed here is intended to study the decadal to centennial variability of the interior ocean in response to atmospheric forcings and, consequently, has time-dependent solutions. However, only stationary solutions are presented here because they are both simpler than those varying in time and are very suitable for estimating how realistic the model is. They also allow one to analyze the respective role of horizontal and vertical diffusions, which would be more difficult from time-dependent solutions. The paper is organized as follows. The model is presented in section 2. Section 3 describes how the zonal gradient of density is affected by the horizontal diffusion. The density pattern is presented in section 4, and the discussion of the main results is developed in section 5.

2. The model

The planetary geostrophic equations are in dimensional form:

$$\rho_0 2\Omega \sin \theta \nu = \frac{1}{R \cos \theta} \frac{\partial \rho}{\partial \lambda},$$

(1)

$$\rho_0 2\Omega \sin \theta \mu = \frac{1}{R} \frac{\partial \rho}{\partial \theta},$$

(2)

$$g p = \frac{\partial \rho}{\partial z},$$

(3)

$$\frac{1}{R \cos \theta} \frac{\partial \mu}{\partial \lambda} + \frac{1}{R \cos \theta} \frac{\partial}{\partial \theta} \left( \cos \theta \nu \right) + \frac{\partial w}{\partial z} = 0,$$

(4)

and

$$\frac{\partial \rho}{\partial t} + \frac{u}{R \cos \theta} \frac{\partial \rho}{\partial \phi} + \frac{v}{R} \frac{\partial \rho}{\partial \theta} + \frac{w}{z} \frac{\partial \rho}{\partial z} = \kappa \frac{\partial^2 \rho}{\partial z^2} + \frac{\kappa h}{R^2 \cos^2 \theta} \left[ \frac{\partial^2 \rho}{\partial \lambda^2} + \cos \theta \frac{\partial}{\partial \theta} \left( \cos \theta \frac{\partial \rho}{\partial \theta} \right) \right]$$

(5)

where \( t_\nu, \lambda, \theta, \) and \( z_\nu \) denote the time, longitude, latitude, and depth; \( u_\nu, v_\nu, w_\nu, p_\nu, \) and \( \rho_\nu \) denote the zonal velocity, meridional velocity, vertical velocity, pressure, and density; \( \Omega \) denotes the earth’s rotation speed and \( R \) the earth’s radius; \( \rho_0 \) denotes the reference density; \( g \) the gravity; and \( \kappa h / (\kappa v) \) the horizontal (vertical) diffusion coefficient.

To a good approximation, the large-scale motions of the ocean obey Eqs. (1)–(5) far from the coasts. These motions have a horizontal scale equal to the earth’s radius \( R \) and a vertical scale equal to the mean thermocline depth \( D \); the horizontal velocity scale \( U \) allows us to define the vertical velocity scale \( W = U D R \) and the time scale \( T = R / U \). For the computations, the following values will be used: \( R = 6400 \) km, \( D = 1 \) km, \( U = 1 \) cm s\(^{-1}\), \( W = U D R = 1.56 \times 10^{-6} \) m s\(^{-1}\) = 49.3 m yr\(^{-1}\), and \( T = R / U = 20 \) yr.

The change of coordinates \( T t = t_\nu, x = \lambda, y = \sin \theta, \) and \( D z = z_\nu \) associated with the change of variables \( U u = u_\nu / \cos \theta, U v = \cos \theta v_\nu, W w = w_\nu, p_\nu = -\rho_0 g z_\nu + \rho_0 2\Omega U R p \) and

$$\rho_\nu = \rho_0 \left( 1 + \frac{2\Omega U R}{g D^2 \rho_0} \right)$$

leads to the nondimensional system:

$$y v = \frac{\partial \rho}{\partial x},$$

(6)
A thorough discussion of this equation in a stationary case and for $\kappa_H = 0$ may be found in Samelson (1999) or Vallis (2006). In particular, it is shown that taking into account the horizontal variations of the density field through the advective terms modifies the scaling of the thermocline characteristics: its thickness is shown to be proportional to $k_V^{1/3}$, whereas the traditional one-dimensional advective–diffusive scaling leads to a thermocline thickness varying as $k_V^{1/3}$.

Salmon and Hollerbach (1991) studied similarity properties of this equation in the stationary case, neglecting the horizontal diffusion. They found a large family of similarity solutions; two of them (solutions $M_{12}$ and $M_{13}$ in their Table 3) led to linear equations, which greatly simplified the study. Here, we combine their solutions $M_{12}$ and $M_{13}$ and generalize them to the time-dependent problem, setting

$$M = xA(y, t) + xzB(y, t) + xz^2C(y, t) + G(y, z, t).$$

With this choice, Eqs. (6), (7), (8), and (12) become

$$y u = \frac{\partial p}{\partial y},$$

$$\rho = -\frac{\partial p}{\partial z},$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

and

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} = k_V \frac{\partial^2 \rho}{\partial z^2} + \frac{\kappa_H}{1 - y^2} \frac{\partial^2 \rho}{\partial x^2} + \kappa_H \frac{\partial}{\partial y} \left(1 - y^2 \right) \frac{\partial \rho}{\partial y}.$$  

This system is formally analogous to the planetary geostrophic equations in the $\beta$ plane except for the horizontal diffusion term where the effects of sphericity are still apparent [through the $(1 - y^2)$ term]. Combining Eqs. (6), (7), and (9) leads to the vorticity equation

$$y \frac{\partial w}{\partial z} = \frac{1}{y} \frac{\partial p}{\partial x}.$$  

With $\kappa_V = \kappa_V T/D^2$ and $\kappa_H = \kappa_H T/R^2$. Equations (6)–(8) thus become

$$y u = \partial_{xx}M, y v = -\partial_{yy}M,$$

and $\rho = -\partial_{zz}M$. Replacing in (10) then leads to a unique equation for $M$: the $M$ equation,

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} = k_V \left(1 - y^2 \right) \partial_{zzz}M + \kappa_H \left[1 - y^2 \right] \partial_{zzz}M + \partial_y \left[1 - y^2 \right] \partial_{zz}M.$$

(hence $u$ and $w$ no longer depend on the longitude). Equation (10) splits in two linear equations. The first links the zonal gradient of pressure $B + 2Cz$ (or by geostrophy the meridional velocity) and the zonal gradient of density $-2C$:

$$\frac{\partial z}{\partial \rho} + \frac{B \partial z}{y} = \kappa_H \partial_y \left[1 - y^2 \right] \partial_{zz}M.$$

The second one allows us to compute the pressure at the eastern side [$p(0, y, z, t) = \partial_z G = F$]:

$$\partial z F - \frac{2C}{y} \partial_{zz}F + \frac{B + 2zC}{y} \partial_{zzz}F + \frac{A + zB + z^2C}{y^2} \partial_{zz}F = k_V \partial_{zzz}F + \kappa_H \partial_y \left[1 - y^2 \right] \partial_{zz}F.$$  

An arbitrary function of $y$ and $t$ can be added to $G(y, z, t)$ without the expressions of $u, v, w, \rho$, and $\rho$ being modified.

A flat bottom at $z = 0$ and a rigid lid at $z = H$ are assumed. As the Ekman layers are not represented in this model, the vertical velocity is equal to the Ekman pumping $w_b$ at the top (bottom) of the ocean. For compatibility with the restrictive form of $M$, $w_b$ and $w_d$ are assumed to be independent of $x$. For simplicity, $w_d$ is set to 0. These conditions imply that

$$A = 0; \quad B = W_0(y, t) - HC(y, t).$$  

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where \( W_0(y, t) = y^2 w(y, t)/H \), and thus enable us to eliminate the unknown functions \( A \) and \( B \) from (16) and (17). Since the dissipative terms in the momentum equations (1) and (2) are neglected, the boundary layers cannot be represented. To avoid sources or sinks of water across the model boundaries, we assume that the volume flux through the southern \((y = y_s)\), northern \((y = y_n)\), and eastern \((x = 0)\) boundaries vanishes. Equivalently, this means that the barotropic velocity is assumed to vanish at these boundaries, while the baroclinic velocity can freely evolve. These hypotheses induce the relations \( \int_0^H u(y_n, z, t)\,dz = y_n y_c W(y_n, t) \), which in turn lead us to impose that the Ekman pumping vanishes along the northern and southern boundaries. They also imply the relation \( \int_0^H u(0, y, z, t)\,dz = -y^{-1} \partial_y \left[ G(y, H, t) - G(y, 0, t) \right] \) or equivalently

\[
G(y, H, t) - G(y, 0, t) = s(t),
\]

where \( s(t) \) is an arbitrary function that will be set to 0 for simplicity. Note that, because of these hypotheses, the volume flux at the surface (owing to the Ekman pumping \( w_e \)) is balanced by a volume flux through the western boundary.

With these boundary conditions, (16) becomes

\[
\partial_t C + \frac{W_0 \partial_y C - C \partial_y W_0}{y} = \kappa_H \partial_y \left[ (1 - y^2) \partial_y C \right],
\]

which can be easily solved since the only unknown, \( C \), appears. In this model, the Ekman pumping (via \( W_0 \)) thus determines both the zonal gradients of density (via \( C \)) and pressure (because of the hydrostatic hypothesis). More precisely, Eq. (20) shows that, in an area of upwelling \((W_0 > 0)\), the zonal gradient of density is transported northward; a northward increase of the upwelling \((\partial_y W_0 > 0)\) acts as a damping and therefore tends to make the distribution of the density more zonal. Another interpretation of this equation is possible, which focuses on the wavelike character of the advective term, \( y^{-1} W_0 \partial_y C \). Indeed, this term accounts for a propagation along characteristics determined by the differential equation \( dy/dt = W_0/y \). During a phase of adjustment, the anomalies thus move with a velocity that depends on the prescribed Ekman pumping, the propagation being cancelled when the latter vanishes (between the gyres). If the horizontal diffusion was missing, the characteristics in the subtropical gyre and those in the subpolar gyre would never cross. Here they can cross, owing to the horizontal diffusion, which therefore connects the two gyres.

On the other hand, Eq. (17) becomes

\[
\frac{\partial_t F - 2C}{y} \partial_y F + \frac{B + 2zC}{y} \partial_y z F + \frac{zB + z^2 C}{y^2} \partial_{zz} F = \kappa_H \partial_{zz} F + \kappa_H \partial_y \left[ (1 - y^2) \partial_y F \right]
\]

with \( B = W_0 - HC \), as previously. It characterizes the changes of density by advection and diffusion in a meridional plane. Eventually, initial and boundary conditions on \( C \) and \( F \) must be added to Eqs. (20) and (21) to complete the problem.

By deriving (21) with respect to \( z \), one obtains an advective–diffusive equation for the potential vorticity,

\[
\partial_z q + y \partial_y q + wz \partial_z q = \kappa_H \partialzz q + \kappa_H y \partial_y \left[ (1 - y^2) \partial_y \left( \frac{q}{y} \right) \right].
\]

Obviously, because the distribution of potential vorticity does not depend on the longitude in this model, it is not affected by advection along the parallels.

The zonal propagation of the long Rossby waves is mathematically described by terms that contain the partial derivative \( \partial_x \), where \( x \) is the zonal coordinate; since only small departures from zonality are allowed in this model, these terms are negligible and the zonal propagation of long Rossby waves is not represented. Consequently, the model will suit only to represent a low frequency process (with a time scale longer than a few decades) once the adjustment for the westward propagation of long Rossby waves has occurred.

3. Setup of the zonal gradient of density

Equation (20) determines the evolution of the zonal gradient of density \(-2C\), provided that the surface Ekman pumping \( W_0 \) is prescribed. As (20) is parabolic, it admits a unique solution when an initial condition, \( C(y) \), and two boundary conditions, \( C(y) \) and \( C(y) \), at the southern and northern boundaries are prescribed.

If stationary solutions are looked for, Eq. (20) simplifies, yielding

\[
W_0 \partial_y C - C \partial_y W_0 = \kappa_H \partial_y \left[ (1 - y^2) \partial_y C \right].
\]

First, note that an increase of the diffusion coefficient by a factor \( k \) has the same effect as a decrease of the Ekman pumping by a factor \( k^{-1} \), which was not the case in the time-varying case. This stationary equation shows that the meridional gradient of density is set up by a balance between horizontal diffusion and the effects of Ekman pumping. The time scales associated with these processes—which also correspond to the adjustment time...
of the system—must therefore match. In particular, the time scale associated with the diffusive term $\kappa_H \partial_y[(1 - y^2)\partial_y]$ cannot be too long in comparison with that defined by $W_0$, which is about $H/w_1$ (a time scale around 500 yr seems reasonable). This remark sets limits to the possible values for $\kappa_H$. They must range between about 0.015 (700 yr) and 0.03 (300 yr), which gives dimensional values between 970 and 2000 m$^2$ s$^{-1}$. They also tally with the values currently used in numerical models of low horizontal resolution (around 1°). Finally, note that the vertical diffusion coefficient used in conjunction with the horizontal one to solve Eq. (21) is also reasonable: it remains close to the observed values (see section 4).

When the density at the northern and southern boundaries depends only on the depth ($C_n = C_s = 0$), $C$ is everywhere null and the density is zonal. We focus here on the much more interesting case where $C$ differs from 0. The choice of $C$ at the northern and southern boundaries is tricky. For a basin extending from 12°N ($y = 0.20$) to 64°N ($y = 0.90$), we set $C_s = -0.02$ and $C_n = 0.05$ so as to match the slight westward decrease (increase) of surface density currently observed at these latitudes. It will be shown that this choice leads to a quite realistic solution (see section 4). The zonal Ekman pumping is shown in Fig. 1 (left) with the function $W_0$ and its derivative. It peaks at $-26$ m yr$^{-1}$ at 28°N and 15 m yr$^{-1}$ at 46°N and allows for the subtropical and subpolar gyre to be represented. The solution $C(y)$ corresponding to this setup is plotted for various horizontal diffusion coefficients in Fig. 1 (right).

When the coefficient $\kappa_H$ is small, about 0.005 (the corresponding time scale is about 2000 yr), the solution of (23) can be split into an inner solution, where $C$ is equal to 0, and two “boundary layers,” where $C$ exponentially varies, to match the prescribed boundary values. The width of the boundary layers is given by

$$l_b \simeq \frac{\kappa_H y_1 (1 - y_1^2)}{(W_{\|} + 2\kappa_H y_1^2)} = \frac{(1 - y_1^2)}{[y_1(2 + w_{h})/(H\kappa_H)]},$$

considering that the terms containing the derivatives of $C$ dominate there (the index $l$ indicates typical values of $W_0$ and $y$ in the boundary layer). Along the northern boundary $y_1 \simeq y_n = 0.9$; hence $l_b \simeq (1 - y_1)(1 + w_{h}/(2H\kappa_H))$. Since $|w_{h}| \ll (2H\kappa_H)$ close to the northern boundary, $l_b \simeq (1 - y_1) \simeq 0.1$ and the extension of the boundary layer is about 10° and, to a large extent, independent of the coefficient $\kappa_H$ (curve 2, Fig. 1). Along the southern boundary, $y_1 \simeq y_s = 0.2$; hence, $l_b \simeq y_s^{-1}[2 + w_{h}]/(H\kappa_H)]^{-1}$. The size of the boundary layer thus crucially depends on the nondimensional coefficient, $w_{h}/H\kappa_H$: it extends up to the latitude where $w_{h}$
becomes much larger than $H\kappa_H$. With $w_d \approx 0.2$ (the order of magnitude of $w_d$ around 20°), $H = 4$, and $\kappa_H = 0.005$ this leads to $l_0 \approx 0.25$, hence an extension of about 20° (curve 2, Fig. 1).

For large $\kappa_H$ (e.g., around or larger than 0.1, which corresponds to a time scale of 100 yr), diffusive processes dominate, and the solution is given by solving $\frac{\partial}{\partial t}[(1 - y^2)\partial_y C] = 0$. This leads to $\tanh[(C - b)/a] = y$ in which the constants $a$ and $b$ are determined by the boundary conditions (curve 6, Fig. 1).

Between these two extreme cases, the pattern of $C$ is influenced by both diffusive processes and Ekman pumping with a comparable impact. An interesting case corresponds to $\kappa_H \approx 0.03$ (associated time scale: 330 yr). Equation (23) looks like the equation of a harmonic oscillator everywhere there is a balance between the diffusion and $-C\partial_y W_0$ with $\partial_y W_0 > 0$ (i.e., for $y$ ranging from 0.43 to 0.72). Consequently, it could admit “resonant solutions”: curve 4 in Fig. 1 gives an example of such a behavior. A coarse analysis of (23) enables one to estimate the order of magnitude of $\kappa_H$ associated with this solution. The spatial period associated to Eq. (23) may be estimated as the mean value of

$$2\pi \sqrt{\frac{\partial_y W_0}{(1 - y^2)\kappa_H}}.
$$

The characteristic value we obtain is approximately equal to $5 \times \sqrt{\kappa_H}$, which matches the length of the domain for $\kappa_H = 0.03$.

This “resonant value” defines two domains where the behavior of the solutions is dramatically changed: in the first one, the zonal gradient of density $-2C$ is positive in the southern part of the domain (corresponding to the subtropical gyre) whereas, in the second one, it becomes negative everywhere, leading to a decrease of the density eastward. We choose a value for $\kappa_H$ corresponding to the first domain, which is more realistic. Because of this resonant value, the meridional structure of the currents is also sensitive to changes in the coefficient $\kappa_H$. For $\kappa_H$ varying in the range $0.025 \pm 0.003$, the coefficient $C$ doubles at 25°N. This has a strong impact on the meridional and vertical velocities, hence on the thermocline pattern (see section 5).

The density gradient $C$ being known, the meridional and vertical velocities can be easily computed using (15) and (18). They are plotted in Fig. 2 for $\kappa_H = 0.025$ [this value will be used to integrate (21) in section 4]. The meridional velocity is southward south of 45°N in the first 2000 m. The vertical velocity is downward in the subtropical gyre to about 1600 m and then upward. It is everywhere upward in the subpolar gyre. As the coefficient $C$ is proportional to the vertical gradient of the meridional velocity, it is maximal along the boundaries, the maxima of the meridional velocity are found near the northern and southern boundaries. However, the effect of these unrealistic values remains limited to a narrow area near the boundaries and does not affect the inner solution.

For brevity, we do not show the patterns of velocities obtained for coefficients $\kappa_H$ that significantly differ from $\kappa_H = 0.025$. Let us just point out that the solution is close to that obtained in the zonal case (no volume exchange between the gyres) when $\kappa_H$ is smaller than 0.01. Indeed, $C$ vanishes everywhere except in two narrow areas near the southern and northern boundaries. On the contrary, for large values of $\kappa_H$ ($\approx 0.04$), a positive meridional cell is obtained in nearly all the basin, except close to the southern boundary.

4. The meridional density pattern

The resolution of (21) is more difficult than that of a linear advective–diffusive equation because of the extra term $(-2C/y)\partial_y C$. The density is prescribed at the south, north, upper, and lower limits of the eastern boundary (see below and Fig. 3) and an iterative algorithm is applied. It gives a unique $F$ and a unique $G$ apart an additive function of $y$ and $t$ without physical meaning (see the appendix). The algorithm uses (19), which had
not yet been used. More complex boundary conditions could be assumed (for example the surface density could be prescribed in the subtropical gyre and the surface potential vorticity in the subpolar gyre): preserving simplicity is the main reason of our choice.

The numerical integration of (21) is made in a basin that has the same limits in latitude as that used for $C \left[12^\circ (y = 0.20) \text{ and } 64^\circ N (y = 0.90)\right]$ and extends westward in longitude from $0^\circ$ (the results are given from $0^\circ$ to $60^\circ W$).

The mesh size is regular in $y$ with 101 points; consequently, the resolution varies from 0.4° at the southern boundary to 0.9° at the northern boundary. There are 101 equally spaced levels on the 4000-m depth so as to resolve the thermocline with reasonable accuracy. The horizontal diffusivity $k_H$ is set to 0.025 (curve 3, Fig. 1). This value agrees with the discussion of the previous section (associated time scale 400 yr). As previously indicated, it is in the range of values currently used in numerical models, considering the moderate horizontal resolution of the model near the northern boundary [they range between 500 and 2000 m$^2$ s$^{-1}$ depending on the resolution (non-dimensional values between 0.0078 and 0.031)]. Finally, measurements (e.g., Toole et al. 1994) suggest that the vertical diffusivity is about 0.1 – 0.5 ($\times 10^{-4}$ m$^2$ s$^{-1}$), which leads to a non-dimensional vertical diffusivity ranging between 0.0064 and 0.032. To define the reference experiment, we choose $k_V = 0.015$. Some indications are given about the sensitivity of the results to this value.

Figure 3 (left) shows the prescribed profiles of density at the northern and southern limits of the eastern boundary. Northward, the thermocline is around 1500 m and separates two weakly stratified water masses, whereas southward it is close to the surface. These profiles crudely mimic the stratification of the Atlantic Ocean at 10° and 65°N. The density at the surface and bottom of the ocean must also be specified along the eastern side. For simplicity, it is assumed that the density anomalies at the bottom are null ($\rho = 0$ at $x = z = 0$) and that they increase linearly with $y$ from $-3.3$ to $-0.5$ at the surface. Thus, the density anomalies are continuous on the domain limits.

The zonal gradient of density $-2C$ being known, this leads to the surface density field shown in Fig. 3 (right). It displays small departure from zonality, as expected from the assumptions made to obtain the model equations. North of 50°N the densest waters are found westward, whereas the reverse occurs south of this latitude, according to the pattern of $C$ (curve 3 in Fig. 1, right). This surface pattern is in qualitative agreement with the observations (see Peixoto and Oort 1992, p. 193).

The density field at the eastern side (Fig. 4) is computed using these boundary conditions. In the subtropical gyre the main thermocline does not penetrate below about 800 m (isopycnal $-0.5$, Fig. 4). The northward tilt of the thermocline—indicated by the + line with each + sign corresponding to the maximal depth of an isopycnal—is represented; this tilt becomes slightly stronger in the western part of the basin (not shown). Around 25°N a domain with less stratified waters is found between a thin surface thermocline and the main thermocline. This two-thermocline structure is apparent in Fig. 5, which shows the vertical gradient of density at 24°N for three different values of the vertical diffusion coefficient. The two-thermocline structure has been
thoroughly discussed in Samelson and Vallis (1997). We give here only a brief account of the relation observed in our model between the stratification and $k V$ to show that this relation agrees with their conclusions. The surface thermocline allows the density to satisfy the surface boundary condition (it is recalled that the Ekman boundary layer is not represented here). The main thermocline has a depth of about 250 m for $k V = 0.030$ and 300 m for $k V = 0.0064$ (depth of the local extremum of the density gradient), suggesting that the depth slightly increases when $k V$ decreases. This is confirmed by other experiments not presented here. The orders of magnitude are compatible with the scaling proposed by Samelson and Vallis (1997). The thickness of the main thermocline increases with $k V$. We have not tried to verify whether this increase is in $k V^{1/2}$ because of uncertainties in estimating the thickness. However, the main thermocline is thicker here than in the Samelson and Vallis (1997) model. This is due to our choice of horizontal diffusion; a stronger horizontal diffusion (say about 0.027) would lead to a thinner internal thermocline (not shown). Such a behavior agrees with the Filippov solutions (see section 5). When the internal thermocline becomes thinner, the estimation of its thickness becomes more precise and the agreement with the scaling of Samelson (1999) or Samelson and Vallis (1997) improves (let us recall that these scalings are established for a vertical diffusion tending to 0; i.e., for a thermocline becoming very thin).

To end the description of the meridian pattern of density, let us remark that, north of 50°N, the stratification becomes unstable close to the surface. This is caused by the advection of light waters coming from the intermediate layers of the subtropical gyre (see Fig. 2); they are less dense than the water at the surface of the subpolar gyre, whose density is prescribed. Similarly, the stratification is unstable near the bottom because of water of density ranging between 0 and +0.03, which come from the northern boundary. If a convective adjustment scheme had been introduced, these unstable profiles of density would be replaced by neutral profiles.

Figure 6 shows the isopycnals in the subtropical gyre at 400-m depth. The westward deepening of the thermocline, which is characteristic of these latitudes, is represented but remains moderate (the order of magnitude corresponds to the southern Pacific): indeed, only solutions showing a small departure from zonality can be represented by this model.

![Figure 4](th动工图.png)

**FIG. 4.** (Top) Meridional profile of the density (nondimensional units) when a zonal gradient of density is applied along the southern and northern boundaries; (bottom) a zoom of the first 1000 m.

![Figure 5](th动工图.png)

**FIG. 5.** Vertical gradient of density at 24°N (nondimensional units) for three vertical diffusion coefficients. The vertical velocity is also shown (longer arrow: $0.5 \times 10^{-7}$ cm s$^{-1}$).
The barotropic velocity (Fig. 7, left), which is in Sverdrup balance, illustrates that the model succeeds in representing two gyres. It contrasts with all previous models based on similarity equations, which were able to represent only a single gyre. The zonal velocity has a strong baroclinic structure (Fig. 7, right), the highest velocities being found in the upper 500 m.

In this open model, the advective and diffusive mass anomaly fluxes through the boundaries do not vanish and therefore must be considered. Because their order of magnitude is more familiar, we give values for the heat fluxes, rather than for the mass fluxes, using the linear relation

\[
\Phi_T = -\rho_0 C_p \Phi_m/\alpha
\]

(one sets \(\alpha = 0.2 \text{ kg m}^{-3} \text{ C}^{-1}\), \(C_p = 4.2 \times 10^3 \text{ J kg}^{-1} \text{ C}^{-1}\), and \(\rho_0 = 10^3 \text{ kg m}^{-3}\)). Heat fluxes are very weak at the bottom of the ocean; they are largely dominated by the advective flux at the surface (recall that it corresponds to the base of the Ekman layer), and their sum is equal to \(0.31 \times 10^{15} \text{ W}\), corresponding to a mean heat flux density of \(10 \text{ W m}^{-2}\). Positive values indicate a warming of the ocean (or a mass loss). The heat flux associated with baroclinic currents across the southern and western limits of the basin is about \(0.44 \times 10^{15} \text{ W}\). Since the whole sum is null—a thermal equilibrium being assumed—this warming is balanced by a cooling across the eastern boundary at \(0^\circ\) longitude and across the northern boundary with heat fluxes equal to \(-0.60 \times 10^{15}\) and \(-0.15 \times 10^{15} \text{ W}\).
respectively. These heat transports have orders of magnitude compatible with the observed or modeled values, showing that the solution that we consider does not introduce a spurious source or sink of heat. Note also that, at the eastern and western boundaries, the flux is dominated by the advective component by two orders of magnitude. The strong eastward current at 35°N, which dominates the oceanic circulation, ensures the main part of this heat transport. At the northern and southern boundaries, the advective component is only about three times larger than the diffusive one.

5. Summary and discussion

a. The horizontal diffusion and the thermocline

The role of the horizontal diffusion has been already emphasized in section 2, when Eq. (20) was discussed. The coefficient $\kappa_H$ allows one to connect the meridional and vertical velocities of the subtropical gyre to those of the subpolar gyre; otherwise, these velocities would evolve independently in each gyre. Moreover, when $\kappa_H$ takes a value that is close to the resonant value, the solution may change quickly and lead to very different patterns of velocities. The thickness of the thermocline is thus modified, with the latter decreasing when $\kappa_H$ increases to the resonant value.

The importance of the horizontal diffusion coefficient is coherent with the results of Filippov (1968), who looked for similarity solutions of the thermocline equations in the $\beta$ plane. He found that, when the horizontal diffusion was kept in the thermocline equations, the density of the similarity solutions he studied could be written as

$$\rho = \left(\frac{d}{k}\right) \arctan\left(\frac{\phi}{k}\right) + c \quad \text{or}$$

$$\rho = f \exp\left[\left(\frac{d}{k}\right) \arctan\left(\frac{\phi}{k}\right)\right] + c,$$

where $c$, $d$, and $f$ are arbitrary constants and $\phi = \beta(ux + uy)/(f_0 - \beta y)$ ($x$ and $y$ denote the zonal and vertical coordinates). The constant $k$ depends on two constants, $a$ and $b$, and the ratio between the vertical and horizontal diffusion coefficients, both playing a comparable role. More precisely, multiplying the horizontal diffusion by $\kappa$ produces the same changes on the density as dividing the vertical diffusion by $\kappa$.

The solutions presented here depend on both coefficients in a slightly more complex way than in Filippov’s. On the one hand, owing to Eq. (20), the horizontal diffusion modifies the zonal gradient of density, then the meridional and vertical velocity as explained in section 2. On the other hand, the coefficient $\kappa_H$ appears in Eq. (21) or (22) in a place similar to that of $\kappa_V$: indeed, the diffusion term is written as $\kappa_V \partial_z q + \kappa_H [y\partial_y (1 - y^2) \partial_y (q(y))]$. In the previous expression, the first term is preponderant, but this does not mean that the horizontal diffusion now has a negligible role: indeed, it keeps on acting, through $\nu$ and $\omega$, which ensure the transport of potential vorticity. Despite this complex behavior, the decrease of the thermocline thickness when $\kappa_H$ is increased—the decrease that could be inferred from the Filippov solutions—is still observed here (see section 4).

These results obviously do not contradict the scalings of Samelson and Vallis (1997), Samelson (1999), or Vallis (2006). Besides that the thermocline thickness decreased in $k_V^{1/2}$ (see section 4 for a discussion of this result), they showed that the thermocline depth or thickness depended on the vertical velocity: in particular, when the latter was intensified, the thermocline became thinner. In our model, in which the vertical velocity depends on the horizontal diffusion, we again find this result.

b. The thermocline and the existence of boundary currents

Marshall (2000), and Polton and Marshall (2003) suggested that the internal thermocline balance emerges as a constraint on potential vorticity fluxes: the integral of these fluxes over any area enclosed by a Bernoulli potential or density contour must vanish. Such a constraint is present here. It is easy, from Eqs. (6)–(10), to rewrite the potential vorticity equation in flux form, $\partial_z q + \text{div} \mathbf{J} = 0$, where the potential vorticity flux vector $\mathbf{J} = q\mathbf{U} + y\mathbf{Dk}$ consists of two components: the first one accounts for the advection of the potential vorticity, $q = -y\partial_z \rho$, by the flow, whereas the second one involves the forcing by the dissipative processes [here, $\mathbf{D}$ denotes the rhs of Eq. (10) and $\mathbf{k}$ a unitary vector along the vertical]. The relation $\mathbf{J} = \nabla \times \mathbf{V} \rho = \mathbf{V} \times (\mathbf{BV} \rho)$, which gives an expression of the vorticity flux independent of the dissipative operators in a stationary case, becomes for the particular model considered here $\mathbf{J} = \mathbf{V} \times (\mathbf{BV} \rho)$, where $\mathbf{V} \rho = (\partial_z P, \partial_y P, 0)$ (the gradient of the Bernoulli function is thus replaced by the horizontal pressure gradient). Using the hydrostatic balance, this relation may be written $\mathbf{J} = \mathbf{V} \times (P \mathbf{V} \rho - \rho g^2/2k)$. Consequently, integrating $\mathbf{J}$ over any surface $A$ enclosed by a contour $C$, Stokes theorem leads to

$$\oint_C \mathbf{J} \cdot d\mathbf{l} = \iint_A \left(P \mathbf{V} \rho - \rho g^2/2k\right) d\mathbf{A}. \quad (24)$$

Over an area closed by a contour with $\rho$ constant, $\mathbf{V} \rho$ is normal to $d\mathbf{A}$ so that $\int_C P \mathbf{V} \rho \cdot d\mathbf{l} = 0$. Moreover, since $\rho$ is
constant, it can be extracted from the integral on the right-hand side and \( \oint d\mathbf{A} = 0 \), the contour being closed. In conclusion, \( \int \mathbf{J} \, d\mathbf{A} = 0 \).

This constraint is the same as that studied by Polton and Marshall (2003). However, because the domain is open, the isopycnal paths can never be closed in a horizontal plane; obviously, isopycnal closed paths are possible as soon as the paths are in three-dimensional space and are allowed to go up or down. The particular similarity solution that we study thus can satisfy local balances, which are the same as those obeying the Polton and Marshall constraint. This condition seems sufficient to have a realistic thermocline.

Polton and Marshall added that closing a gyre with boundary currents was necessary to obtain a thermocline. The problem of closing a gyre or equivalently the existence of closed isopycnal paths in a horizontal plane is however largely independent of the existence or nonexistence of integral constraints. Indeed, it mainly depends on the diffusion operator chosen (we have noticed that the stationary expression of \( \mathbf{J} \) as its integral over a given area were independent of the expression of dissipation or diffusion). In the model used by, for example, Samelson and Vallis (1997) or Polton and Marshall (2003), an ad hoc bi–Laplacian operator is added so as to satisfy no-slip boundary conditions. In Colin de Verdière (1989), a simple Laplacian heat diffusion is used, but a Laplacian momentum diffusion is added in the momentum equation. The existence of a closed circulation in the subtropical gyre thus depends on these operators in the models. Here, because we chose a Laplacian operator for the diffusion in (10), no solution of (6)–(10) can satisfy realistic boundary conditions. Consequently, the isopycnal cannot be closed in a horizontal plane. Our results thus seem to suggest that closing a gyre with boundary currents is not necessary to obtain a thermocline.

6. Conclusions

Salmon and Hollerbach (1991) searched similarity solutions of the thermocline equations, taking into account the role of the vertical diffusion. They analyzed the physics of two of them, the simplest ones, but failed to obtain a two-gyre solution. Hence, they suggested studying simple solutions of the thermocline equations that would take into account the role of horizontal diffusion. In this paper, we follow their suggestion and develop a simplified model of the thermocline from a combination of the two simple similarity solutions that they discovered and studied. A drawback sometimes encountered with similarity solutions is avoided here since, though simple, the model satisfies essential boundary conditions. Indeed, no barotropic flux of volume is allowed across the northern, eastern, and southern boundaries of the ocean and one can prescribe 1) a zonal Ekman pumping at the top and bottom of the ocean, 2) a vertical profile of buoyancy along the northern and southern boundaries that can vary linearly following the longitude, and 3) the buoyancy at the top and bottom of the ocean along the eastern coast. With these boundary conditions a two-gyre circulation is obtained, and the internal thermocline, including its tilt in the subtropical gyres, is represented.

The solutions found in the stationary case for realistic forcing fields seem in satisfactory agreement with the observations and the simulations from planetary geostrophic models (e.g., Colin de Verdière 1989; Samelson and Vallis 1997), considering the simplicity of the model. The two-thermocline structure is a common stratification feature for all subtropical basins. In most diffusive models (e.g., Samelson and Vallis 1997), this is quite well represented: close to the surface, where the isopycnals outcrops, there is a thin adiabatic thermocline, whereas a diffusive thermocline appears below, whose dynamics is described by the internal boundary layer theory (Stommel and Webster 1962; Salmon 1990). This vertical structure is also present in our experiments. Less stratified waters are found around 25°N at 100-m depth, south of the eastern intergyre current, and separate two areas of more stratified waters. The thickness of these less stratified waters decreases when the vertical diffusivity coefficient increases, as expected.

In addition to these results already proved in a more realistic context by, for example, Samelson and Vallis (1997), this simple model has allowed enlightenment on the role of horizontal diffusion. It connects, in an efficient way, the subtropical and subpolar gyres and has a direct impact on the zonal density gradient and, consequently, on the meridional and vertical velocities. Hence, it has a direct impact on the thermocline thickness, a larger coefficient leading to a thinner thermocline. This result agrees with the similarity solutions found by Filippov (1968). It also suggests that horizontal and vertical diffusivities both act on the structure of the thermocline—the latter in a more direct way by smoothing the potential vorticity.

Marshall (2000) and Polton and Marshall (2003) suggested that the internal thermocline balance emerges as a constraint on potential vorticity fluxes. They added that boundary currents were necessary to obtain a realistic thermocline. The latter are not represented here, but the constraint on potential vorticity fluxes is verified. It seems sufficient to ensure the existence of a thermocline, our solutions being allowed to satisfy local balances.
or constraints similar with those described by Polton and Marshall (2003).

This model, in spite of its simplicity, has allowed us to represent the thermocline far from coasts in a quite realistic way. It shows that previous studies about similarity solutions do not present only an academic interest; they can lead to quite realistic solutions, even though numerous simplifications have been done. They can also be of theoretical interest, for example, by illustrating the role of horizontal diffusion or of the integral constraints in ocean dynamics. Finally, as it occupies an intermediate position between layer models, zonally averaged models, and planetary geostrophic models with diffusion and dissipation, it constitutes an interesting tool for both theoretical and numerical studies.

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APPENDIX

Numerical Resolution of Eq. (21)

The method we follow to solve Eq. (21) in the stationary case is iterative. Let $F^n$ be the value of $F$ obtained at the $n$th iteration; to compute $F^{n+1}$ we replace Eq. (21) by the following relation:

$$-\left(\frac{2C}{y}\right)\partial_y F^n = -\nu \partial_y F^{n+1}_y - w \partial_z F^{n+1}_z + \kappa_H \partial_y (1-y^2) F^{n+1}_y + \kappa_H \partial^2_y (1-y^2) F^{n+1}_y = \mathcal{L} F^{n+1}_y,$$

where the unknown is $F^{n+1}_z = \partial_z F^{n+1}_z$. To compute $F^{n+1}_z$, the operator $\mathcal{L}$ is discretized by a finite difference method with second-order precision. Since $\partial_z F = F_z$ is prescribed on the boundaries, this leads us to invert the squared matrix associated with $\mathcal{L}$. By integrating over the vertical coordinate, $F^{n+1}$ is then obtained:

$$F^{n+1} = \int_0^H F^{n+1}_z - H^{-1} \left( \int_0^H F^{n+1}_z \right).$$

$[F^{n+1}$ thus satisfies the condition (19), $G(y, H, t) - G(y, 0, t) = \int_0^H F(y, z, t) dz = 0$, with $s(t) = 0$. The iterations are initialized with $F^0 = 0$. They are stopped when a quasi-stationary series is obtained. The computational cost is minimal because inversion of the operator $\mathcal{L}$ is made only once.

REFERENCES


