The Mixing Efficiency of Stratified Turbulent Boundary Layers

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ABSTRACT

The mixing efficiency observed in stratified turbulent boundary layers is considered within the framework of the Monin–Obukhov similarity theory. It is shown that the efficiency within the layer increases with distance from the boundary. Near the boundary, the efficiency is proportional to the distance from the boundary scaled with the Monin–Obukhov length. Far from the boundary, the efficiency relaxes to a value that depends on the overall thickness of the layer relative to the Monin–Obukhov layer. This value approaches 1/6 when the thickness is larger than 1/2 of the Monin–Obukhov length. The same analysis shows that the buoyancy Reynolds number cannot be used to unequivocally predict the efficiency. The $-1/2$ scaling between the efficiency and buoyancy Reynolds number that has been observed in field measurements and experiments is shown to depend on an extra dimensional scale and thus is not universal.

1. Introduction

Despite its significance for large-scale geophysical flows (Wunsch and Ferrari 2004), we still lack a precise understanding of how mixing in a stratified environment depends on the environmental conditions that drive it [see, e.g., the review by Ivey et al. (2008) and references therein].

If we view mixing as any mechanism that pumps buoyancy against a gravitational potential gradient, we can think of its efficiency as the fraction of the energy consumed by the system that irreversibly raises its center of mass. This intuitive concept can be given a precise mathematical formulation using the available potential energy (APE) framework (Winters et al. 1995; Winters and D’Asaro 1996; Tailleux 2013; Scotti and White 2014), but for the present discussion, it is sufficient to consider the flux Richardson number:

$$R_i = \frac{g}{\rho_0} \frac{\langle w' \rho' \rangle}{\phi_k},$$

where $\phi_k = -\langle u'w' \rangle \partial (u)/\partial z$ is the turbulent shear production term. Here and thereafter, $\langle \cdot \rangle$ indicates averaging, while the prime denotes fluctuating quantities, $u$ and $w$ denote horizontal and vertical velocity, $\rho$ is the density, $\rho_0$ is a reference density, and $g$ is the gravitational acceleration. Finally, $z$ is the geopotential height. If mixing is stationary and sustained by the production of turbulence via the shear production mechanism, then the efficiency coincides with $R_i$ (Osborn 1980; Scotti and White 2014); in this paper, we are solely concerned with such systems. Several nondimensional parameters can be used to characterize mixing in shear-driven turbulent flows (Stretch et al. 2010; Mater et al. 2013; Mater and Venayagamoorthy 2014b). For convenience, we can speak of “external” parameters, such as the bulk Froude number [as used, e.g., by Wells et al. (2010) in their study of entrainment in gravity currents] or a Richardson number (see, e.g., Pardyjak et al. 2002). These parameters describe the large-scale properties (relative to the scale of mixing) of the flow. An alternative description is based on parameters that describe the turbulent environment, such as the buoyancy Reynolds number (also known as isotropy index or activity number):

$$Re_b = \frac{\varepsilon}{\nu N^2},$$

where $\nu$ is the viscosity, $\varepsilon$ is the turbulent kinetic energy dissipation rate, and $N$ is the buoyancy frequency of the background stratification. It is related to the ratio of the Ozmidov scale $L_O = (\varepsilon/N^2)^{1/2}$ to the Kolmogorov scale $\eta = (\nu^3/\varepsilon)^{1/4}$ via

$$Re_b = \left( \frac{L_O}{\eta} \right)^{4/3}.$$
and heuristically measures the range of scales where turbulent eddies are both weakly affected by viscosity and by the large-scale stratification (Gibson 1980; Gargett 1988).

Many researchers have sought to collapse observational and numerical data with a Ri\_f(Re\_b) relationship. While different functional relationships have been proposed, in the large Re\_b limit, they asymptote to a power law

\[ \text{Ri}_f \simeq C \text{Re}_b^{-a}. \]  

The puzzling feature is that different experiments yield values close to 1/2 for the exponent, whereas the non-dimensional coefficient C varies over more than one order of magnitude (Shih et al. 2005; Lozovatsky and Fernando 2013; Bluteau et al. 2013). For a compilation of data, see Walter et al. (2014). The lack of universality is not surprising: Mater and Venayagamoorthy (2014a, b) argue convincingly that the efficiency cannot be expected to depend on a single non-dimensional parameter. Here, we show that, at least for a class of stratified boundary layers, which can be described by a similarity theory [Monin–Obukhov (M–O) layers], this is indeed the case, and the key to identifying the extra parameter is given by the exponent a in (4). Also, in M–O layers, a much better predictor for the efficiency is the local gradient Richardson number, a point that Linden (1979) already made several decades ago. We focus on M–O layers because the theory can be carried forward sufficiently to give a detailed description of mixing within the layer. We also have access to observational data-sets that can be used to test the theory. The drawback is that while M–O layers describe the atmospheric boundary layer well, they are rarely encountered in the ocean. At the bottom of the ocean, the geothermal heat flux is usually negligible. At the surface, the M–O length scale [for a formal definition, see (5)] based on a stress of 0.1 N m\(^{-2}\) and a 10\(^{-8}\) m\(^2\) s\(^{-3}\) buoyancy flux gives a scale O(100) m, which is comparable to a host of different scales associated with surface processes. Nonetheless, since the main goal is to provide a critical assessment of (4), the advantage of working with an M–O layer outweighs its limited range of applicability to the ocean.

Being able to quantify the efficiency is more than just an academic problem. It is well established that the predictive skill of general circulation models is sensitive to the characteristics of the turbulence parameterizations (Jayne 2009; Melet et al. 2013) of which the mixing efficiency is an essential element. For example, a recent analysis of water mass transformation shows that, if the efficiency is reduced at high Re\_b according to (4) with C \approx 4, the strength of the upwelling around Antarctica is reduced by a factor of 2 (de Lavergne et al. 2016).

To date, most parameterizations assume a constant Ri\_f = 1/6 (St. Laurent et al. 2002; Polzin 2009; Olbers and Eden 2013), at odds with (4), especially considering that, in the ocean, the buoyancy Reynolds number can reach values as high as O(10\(^5\)) in energetic regions, though values O(10\(^3\)) are more prevalent [see, e.g., Jackson and Rehmann (2014) for a compilation of values from different sources].

In this paper, we focus on stratified boundary layers in the “energetic” regime defined by Shih et al. (2005) as Re\_b > O(100). Many recent observational studies that have sought to collapse the efficiency in terms of the buoyancy Reynolds number (Lozovatsky and Fernando 2013; Bluteau et al. 2013; Walter et al. 2014) were conducted in flows that had in common the existence of a bottom boundary, but which differ on the mechanism that sustained stratification (surface heat flux in the atmospheric experiments versus a host of processes in the oceanic ones). Moreover, in these experiments, high values of Re\_b correlate with low values of the gradient Richardson number \( \text{Ri}_b = N^2/\bar{S}^2 \) [see, e.g., Fig. 9 in Lozovatsky and Fernando (2013)], where N is the Brunt–Väisälä frequency and \( \bar{S} \) is the mean shear. For this reason, we exclude strongly stratified layers. We emphasize that we are not implying that high values of the buoyancy Reynolds number cannot be achieved in strongly stratified turbulence. The reader ought to bear in mind that our theory applies to the regime \( \text{Ri}_b < \sim 0.2 \) and Re\_b > O(100), where a stratifying buoyancy forcing is applied at the boundary. The advantage of considering only this regime is that we can give a rather complete theoretical description, at the cost of forgoing direct applicability to realistic oceanic layers. The central element of the analysis is the existence of a length scale, the Monin–Obukhov length scale \( L_{MO} \), against which the other length scales of the problem are compared. The main results presented in this article are as follows:

1) The flux Richardson number \( \text{Ri}_f \) (i.e., the efficiency under our assumptions) within the boundary layer increases monotonically with \( |\bar{z}| \), the distance from the boundary. When \( |\bar{z}|/L_{MO} \) is small, \( \text{Ri}_f \sim |\bar{z}|/L_{MO} \). For values of \( |\bar{z}|/L_{MO} > O(1/2) \), it approaches an asymptotic value.

2) The overall (core) efficiency in the boundary layer is controlled by how thick the layer is in units of \( L_{MO} \). Layers that extend over several M–O lengths (thick layers) on average mix with an efficiency that approaches a constant value. Conversely, the average efficiency of thin layers is bounded from above by the thickness of the layer expressed in units of \( L_{MO} \).
3) Unlike the efficiency, the buoyancy Reynolds number varies very little across the layer and is proportional to the M–O length expressed in wall units.

4) Observing the scaling relationship (4) in a set of observations collected at a fixed location in an ensemble of M–O turbulent layers implies the existence of a “hidden” scale, a dimensional quantity that remains constant over the ensemble of observations.

Together, these results militate against the existence of a universal $Ri_j(Re_b)$ relationship.

2. Local mixing efficiency within a Monin–Obukhov layer

We consider how the local mixing efficiency changes with distance from the boundary in a layer described by the M–O similarity theory, which is based on two assumptions: (i) the layers are horizontally homogeneous and (ii) the total (turbulent plus molecular) ensemble-averaged momentum and buoyancy fluxes are constant along the vertical direction. The dimensional quantities of import within the M–O framework are the momentum flux divided by the mean density of the fluid, denoted by $\tau$, the buoyancy flux $B$ (negative in stably stratified layers), and $|z|$, the vertical distance from the boundary; the latter, without loss of generality, is assumed to be located at $z = 0$. To that, we add an external length scale $h$, which limits the vertical extent of the domain. The length could represent a physical boundary (e.g., a water column of finite depth) or could be due to a sharp increase in the stratification, for example, the depth of the permanent pycnocline in the ocean. Following standard praxis, we introduce the friction velocity $u_* = \sqrt{|\tau|}$. Combining the friction velocity with the buoyancy flux, we obtain the Monin–Obukhov length, defined as

$$L_\text{MO} = -\frac{u_*^3}{kB},$$

where $k$ is the von Kármán constant. Note that, following standard conventions, the Monin–Obukhov length is positive in stably stratified layers. In addition to the M–O length, we can define a viscous length scale

$$l = \frac{v}{u_*},$$

We denote with a $^+$ distances measured in units of $l$ and define $\zeta = |z|/L_\text{MO}$. In particular, let $\zeta_h = h/L_\text{MO}$. From the onset, we assume that $L_\text{MO}$ is large but not infinite. For a vertically bounded flow, we define the core region as the region where $|z|^+ \gg 1$ (i.e., outside the viscous sublayer). Since $\zeta = |z|^+L_\text{MO}^+$, in the core region $\zeta \gg 1/L_\text{MO}$. Thus, in the ideal limit $L_\text{MO} \to \infty$, the M–O layer covers the region $0 \leq \zeta$. In practical application, how small $\zeta$ can be before we approach the viscous sublayer depends on how big $L_\text{MO}^+$ is. While this is a technical point, it is of consequence when considering numerical simulations. To wit, in environmental application it is possible to have $\zeta = O(10^{-3})$ (see, e.g., the bottom panel of Fig. 1). In the DNS discussed below, $\zeta$ depends on the stratification, and in all cases considered is no smaller than $10^{-2}$.

Dimensional analysis requires that to order $(1/L_\text{MO}^+)$,

$$\frac{|\tau|}{u_*} \frac{|z|}{u_*} \frac{\hat{u}}{\partial z} = \phi_{ma} (\zeta, \zeta_h),$$

$$-\frac{|\tau|}{u_*} \frac{|z|}{u_*} \frac{\hat{b}}{\partial z} = \phi_{ba} (\zeta, \zeta_h),$$

$$\langle w' \tau' \rangle = -\tau \phi_{ma} (\zeta, \zeta_h),$$

and

$$\langle w' b' \rangle = B \phi_{ba} (\zeta, \zeta_h),$$

where $b = g(\rho_0 - \rho)/\rho_0$ is the buoyancy. We refer to the $\phi$ as the M–O structure functions. These equations apply without change whether the layer extends above the boundary or below$^1$ and regardless of the direction of the mean velocity. We are interested in turbulence acting within a stably stratified layer so that $B < 0$. In the following, we will use $\sim$ to denote equality up to $O(1/L_\text{MO}^+)$, while $\sim$ indicates that the quantity on the left of the symbol scales with the quantity on the right.

a. The $\zeta_h \gg 1$ case: Semibounded layers at high Reynolds numbers

The original M–O theory assumes that when $L_\text{MO}$ and $\zeta_h$ are very large, the structure functions depend on $\zeta$ alone [i.e., achieve complete similarity as defined by Barenblatt (1996)]. When these conditions are met, observations in the atmospheric boundary layer collapse to nearly identical linear structure functions for the mean buoyancy and momentum:

$$\phi_{ma} \sim \phi_{ba} = c_0 + \zeta/\gamma_\infty$$

(Foken 2008; Grachev et al. 2015), where $c_0$ is the intercept and $\gamma_\infty^{-1}$ is the slope. The mean profiles thus transition from a logarithmic to a linear profile as we move away from the boundary. Since the total momentum and

$^1$Here, above (below) denotes the direction of increasing (decreasing) geopotential height.
buoyancy fluxes are constant by assumption in the vertical direction, outside the viscous sublayer where $z^{+} \gg 1$, we have

$$\phi_{mt} = 1 - \frac{\nu}{\tau} \frac{\partial (u)}{\partial z} = 1 + O(1/|z|^{+}) \simeq 1,$$

(12)

and similarly $\phi_{ba} \simeq 1$. Outside the viscous sublayer, the turbulent flux structure functions are constant.

Under the assumption that TKE production is locally balanced by dissipation and buoyancy flux, we can write

$$Ri_f = \frac{\xi \phi_{ba}}{\phi_{ma} \phi_{mt}} \simeq \frac{\gamma_{z} \xi}{c_{0} \gamma_{x} + \xi},$$

(13)

whereas for the local gradient Richardson number,

$$Ri_g = \frac{\partial \langle b \rangle / \partial z}{\langle \partial (u) / \partial z \rangle} \simeq Ri_f Pr_{t},$$

(14)

where $Pr_{t}$ is the ratio of turbulent viscosity to turbulent diffusivity (turbulent Prandtl number). The fact that the structure functions for mean momentum and turbulent momentum flux are very close to the functions describing mean buoyancy and turbulent buoyancy flux lends strong support to the Reynolds momentum buoyancy analogy for these flows (Incropera and DeWitt 1990).

Equation (13) shows that the efficiency within an M–O layer is not constant. Close to the boundary of an M–O layer, the efficiency becomes proportional to the distance from the boundary scaled with the M–O length, $Ri_f = \xi / c_{0} + O(\xi^2)$, vanishing at the boundary, while for $\xi \gg c_{0} \gamma_{x}$, the flux Richardson number relaxes to a constant value $\gamma_{x}$. Note that M–O layers cannot extend to arbitrarily large values of $\xi$. Rather, for $\xi$ larger than a value $\sim 3$, the structure functions $\phi_{ma}$ and $\phi_{ba}$ approach a constant value in the large-$\xi$ limit (the so-called $z$-less regime; see Foken 2008). This can be readily justified if we consider that, in an M–O layer, local production of TKE asymptotes to a constant value. Thus, the production of TKE integrated from the wall to $\xi$ increases as $\xi$. Such production is supported by the work per unit mass and per unit area done by the stress applied at the wall (it is convenient to switch to a frame of reference moving with velocity $U$, in which the flow away from the wall is stationary), which is $Ur$, independent on the thickness of the layer. Thus, the work done on the system can support production of TKE at an asymptotically constant rate only over a finite region. For this reason, the large $\xi$ limit must be understood as $\xi = O(3)$.

Over 40 yr of measurements in atmospheric boundary layers show that $\gamma_{x}$ ranges between 1/8 to 1/5, with a recommended value of 1/6, whereas $c_{0} = O(1)$ [see, e.g., appendix A4 in Foken (2008)]. Thus, away from the boundary (but still within the M–O layer), the efficiency approaches the standard value used in oceanography. This is not a mere coincidence, as Osborn (1980) relied partly on atmospheric measurements to set an upper bound for the efficiency.

We now consider how the buoyancy Reynolds number varies across the layer. Following Lozovatsky and Fernando (2013), we can write the local buoyancy Reynolds number as

$$Re_b = \left( \frac{k}{\nu} \right) \left( \frac{\epsilon}{N^{2}} \right) \left( \frac{U'}{v} \right),$$

(15)

where the subscript $t$ refers to turbulent quantities. The first factor on the right-hand side is the inverse of the turbulent diffusivities ratio

$$Pr_{t} = \frac{\phi_{ba} \phi_{mt}}{\phi_{ma} \phi_{bt}},$$

(16)

while the second is $(1 - Ri_{f})/Ri_{f}$ for equilibrium turbulence. Finally,

$$\frac{\nu}{\nu} = k |z|^{+} \phi_{ma} \phi_{mt} = kL_{MO}^{+} \phi_{ma} \phi_{mt} = L_{MO}^{+} [kRi_{f} + O(1/|z|^{+})].$$

(17)

Using decomposition (15) together with (13), we obtain

$$Re_b = kL_{MO}^{+} (1 - Ri_{f}) \phi_{ba} \phi_{mt} \phi_{bt}^{3},$$

(18)

or

$$\frac{Re_b}{L_{MO}} = O(1),$$

(19)

since across the M–O layer, $(1 - Ri_{f})$ varies no more than 20% and $\phi_{ba} \approx \phi_{mt}$, while $\phi_{bt} \approx \phi_{mt} = 1$.

Taken together, these results indicate that, within an M–O layer, the flux Richardson number and the gradient Richardson number are not constant. They increase from 0 near the bottom of the layer to an asymptotic value that is reached when the distance from the boundary, measured in units of the M–O length, is greater than $O(1/4)$. The buoyancy Reynolds number, on the contrary, is approximately constant within the span of an M–O layer and equal to the M–O length expressed in wall units.

This precludes the existence of a universal $Ri_f(Re_b)$ relationship within an M–O layer.

b. Bounded layers

We now consider layers that have an external length scale, for example, the total height of the layer is set by physical constraints. The direct numerical simulation
(DNS) discussed below support the hypothesis that the structure functions still achieve complete similarity in $L_{MO}$ and retain the same functional dependence on $\zeta$, except that the slope and intercept are now functions of $\zeta_h$. Therefore, the analysis of the preceding section remains valid, provided we understand $c_0 = c_0(\zeta_h)$ and $\gamma = \gamma(\zeta_h)$ and further assume that, as $\zeta_h \to 3$ (the large $\zeta$ limit), they relax to the unbounded M–O values.

Let us define the core flux Richardson number $\text{Ri}_{fc}$ as the ratio of the layer-averaged buoyancy flux to the layer-averaged TKE production. If $\zeta_h$ is large, then most of the layer mixes with an efficiency close to the asymptotic efficiency $\gamma_\infty$, and $\text{Ri}_{fc} \simeq \gamma_\infty$. If $\zeta_h$ is not large, the average of (13) provides, via Jensen’s inequality (the average of a convex function is greater than the function that arises from integrating the production term), a convenient way to formulate an upper bound for $\text{Ri}_{fc}$:

$$\text{Ri}_{fc} \leq \frac{1}{\zeta_h} \int_0^h \frac{\gamma_\infty \zeta}{c_0 \gamma_\infty + \xi} \, d\xi = \gamma_\infty \left[ 1 - \frac{c_0 \gamma_\infty}{\zeta_h} \ln \left( 1 + \frac{\zeta_h}{c_0 \gamma_\infty} \right) \right].$$

That is,

$$\text{Ri}_{fc} \begin{cases} \simeq \zeta_h/c_0 + O(\zeta_h^2) & \text{if } \zeta_h \ll 1, \\ \simeq \gamma_\infty & \text{if } \zeta_h \gg 1. \end{cases}$$

Simple algebra shows that $\zeta_h$ can be written as

$$\zeta_h = k \text{Ri}_c C_d^{-3/2} \text{St},$$

where $\text{Ri} = \Delta bh/U^2$ is the bulk Richardson number, defined in terms of the buoyancy contrast across the layer and the velocity at the edge of the layer; $C_d = u_\infty^2/\Delta b U$ is half the drag coefficient, and $\text{St} = -B/\Delta b U$ is the Stanton number. To investigate the large $\text{Re} = Uh/v$ limit appropriate for environmental applications, we note that the Reynolds analogy assumes that the transport of buoyancy is analogous to the transport of momentum in a boundary layer so that, when properly rescaled, they are described by the same functions. In the present situation, the analogy is manifested by $\phi_{\text{mod}} \simeq \phi_{\text{mod}}$ and requires that the ratio of the Stanton number to drag coefficient is a function of the Prandtl number alone (typically $\text{St}/C_d \sim \text{Pr}^{-2/3}$). For the present purposes, the ratio can be considered close to unity. Then,

$$\zeta_h \simeq \frac{\text{Ri}_c}{\sqrt{C_d}}.$$  

In the spirit of Kolmogorov zeroth law (Frisch 1995), let us assume that, for large values of the Reynolds number, the drag coefficient asymptotes to a value that depends only on the Richardson number, and since it is sensible to assume that for small $\text{Ri}$ the drag coefficient approaches a finite positive value, we have $\zeta_h \sim \text{Ri}$ when $\text{Ri} \ll 1$. But this implies that in the limit $\text{Re} \to \infty$, $\text{Ri} \to 0$, the core efficiency within an externally bounded stratified turbulent boundary layer is no greater than an amount proportional to the bulk Richardson number.

c. Rotation

Rotation introduces an additional length scale

$$L_f = \frac{u_b}{f}.$$  

We can express the ratio

$$\frac{L_f}{L_{MO}} = -\frac{k B}{u_b f} \frac{\text{St}}{C_d} \text{Ri Ro},$$

where $\text{Ro} = U/h f$ is the Rossby number. In view of the Reynolds analogy discussed earlier, under the assumption that $\text{Ro} \gg \text{Ri}^{-1}$, rotation can be neglected.

3. Application to observations

In the preceding section, we have shown that in an M–O stratified boundary layer, a universal relationship between a buoyancy Reynolds number and either local or core efficiency should not be expected. This is due to the tight coupling between momentum transfer and work done on the system on one hand, and momentum and buoyancy transfer on the other (Reynolds analogy) that exists in these layers.

Observations in boundary layers are normally collected by instruments that are placed at fixed elevations (Lozovatsky and Fernando 2013). Thus, for a given instrument pod, $\zeta$ will vary as the M–O length changes in response to changes in stratification and/or forcing. Under weak stratification or strong forcing, the normalized elevation $\zeta$ is small (the Monin–Obukhov length increases). For sufficiently small values, according to (13), $\text{Ri}_f = \zeta/c_0$. Therefore, it is not surprising that inasmuch as the M–O theory provides a reasonable model for the dynamics of the layer, measurements of the flux Richardson number $\text{Ri}_f$ within such layers can span a range of values and in particular can be very low.

\footnote{We formulate the problem using Jensen’s inequality in order to avoid the nuisance represented by the weak logarithmic singularity that arises from integrating the production term.}
Consider the values of $R_i f$ reported by Lozovatsky and Fernando (2013, their Fig. 7; reproduced in Fig. 1). The data were collected in a stably stratified nocturnal boundary layer, with production locally balanced by dissipation and mixing. During the first six nights of the deployment, nighttime conditions were rather constant from day to day. The exception was the last two nights of the deployment, during which a strong synoptic system substantially increased the momentum flux causing $\zeta$ at the instrument pod to decrease by an order of magnitude (Fig. 1, bottom panel). Lozovatsky and Fernando (2013) used strict criteria to include only data collected during near-steady (as measured by variations in fluxes) conditions. The reported $(Re_b, R_i f)$ pairs are clustered in three tiers. The first tier (red circles in the top panel of Fig. 1) is characterized by a narrow range of $Re_b$, with large flux Richardson numbers that decline rapidly with increasing $Re_b$. The second tier (magenta circles) was observed under weaker stratification (mean $R_i f = 0.07$), and the $R_i f$ values are largely scattered across oceanographic canonical values, without a clear trend despite the fact that across the tier the buoyancy Reynolds number increases by an order of magnitude. The last tier (cyan circles) corresponds to even weaker stratification (mean $R_i f = 5 \times 10^{-3}$) and, with the exception of a single point, shows a clear drop with the increasing buoyancy Reynolds number, though over a somewhat smaller range of values. A relationship of type (4) with $C_5 50$ and $a = 1/2$ is found when all three tiers are considered together. The theoretical prediction based on (13) and (19) is in excellent agreement with the efficiencies measured in tiers two and three (Fig. 1, black dots). Note that the theory relies only on the turbulent fluxes, which were graciously provided to us by I. Lozovatsky. Other than averaging nighttime fluxes over 10-min intervals, we did not apply any other smoothing or correction. As for the first tier, the measured efficiencies are higher than the expected asymptotic efficiency of an M–O layer. It groups data collected during stronger stratification when the MO theory is not expected to apply, for example, because of intermittency in turbulence.

The hidden scale

We endeavor now to show that an asymptotic scaling law a la (4), linking measurements taken under different conditions within layers described by the M–O theory, requires the existence of a dimensional quantity that remains constant over the duration of the experiment, even as the buoyancy and momentum turbulent fluxes change. In order not to show itself when measurements are taken at a constant elevation, the hidden scale must be expressible as a length scale $L_{MO}$. Such scale can be determined from the exponent $a$.

Consider an M–O layer described by an arbitrary pair $(u_{b0}, B_0)$ and the ensemble of M–O layers that is generated applying the dilation group (Cantwell 2002)
where \( \alpha \) is the parameter) is an invariant under \( T^\text{dil} \). In an M–O layer, to leading order, the Taylor expansion of (13) gives \( \zeta = \epsilon_0 R_\text{el} \) and \( L_\text{MO}^+ \sim R_\text{el} \). In view of the invariant nature of (28), in the large \( R_\text{el} \) regime, we will observe (4) when considering measurements at a fixed location. Provided \( p = (1 - a)/(3 - 4a) \), the length scale

\[
L^+_\text{MO} \sim T^\text{dil}/A
\]

(28)

and that the 1-parameter family of curves

\[
(L_{\text{MO}},|z|^{1/2},\alpha L_{\text{MO}},\alpha^{-a} \zeta)
\]

(27)

and

\[
T^\text{dil}: \{\bar{u}_b = \alpha^{1-a}u_{\text{sys}}, \bar{B} = \alpha^{3-4a}B_0, \alpha > 0\}; \quad (26)
\]

all other dimensional quantities are held fixed. Recalling that \( L_{\text{MO}}^+ = |z|^1/\zeta \), we leave to the reader the simple exercise to show that under the action of the group

\[
(L_{\text{MO}},|z|^{1/2},\alpha L_{\text{MO}},\alpha^{-a} \zeta)
\]

(27)

and that the 1-parameter family of curves

\[
L^+_\text{MO} \sim T^\text{dil}/A
\]

(28)

where \( A \) (the parameter) is an invariant under \( T^\text{dil} \). In an M–O layer, to leading order, the Taylor expansion of (13) gives \( \zeta = \epsilon_0 R_\text{el} \) and \( L_{\text{MO}}^+ \sim R_\text{el} \). In view of the invariant nature of (28), in the large \( R_\text{el} \) regime, we will observe (4) when considering measurements at a fixed location. Provided \( p = (1 - a)/(3 - 4a) \), the length scale

\[
L^+_\text{MO} \sim T^\text{dil}/A
\]

(28)

is invariant and is the hidden length scale that remains constant over the ensemble. Thus, an ensemble of M–O layers generated by the application of \( T^\text{dil} \) is characterized by the pair \( (p, L_H) \), and it follows that the constant \( C \) in (4) must satisfy

\[
C = C(p, |z|/L_H).
\]

(30)

If we set \( a = 1/2 \), as suggested by field observations and DNS, we obtain \( p = 1/2 \), whereas using \( a = 2/3 \), as suggested by a meta-analysis of laboratory experiments (Barry 2002), yields \( p = 1 \). The difference may be due to different forcing mechanisms. Sensu stricto, what we have proven is that the existence of a hidden scale requires (4). While in principle it is possible that a set of observations out of a completely random ensemble (i.e., where \( L_H \) is not constant over the ensemble) of M–O layers could yield (4), it appears not likely.

An experimental test of this theory would require taking simultaneous measurements with pods located at different elevation and, for each height, fit a power law algorithm (4). If our theory is correct, then (i) the coefficient \( C \) would be height dependent and (ii) the range of validity of the fit would extend to lower values of \( R_\text{el} \) the closer the pod is to the ground, since departure from the 1/5 asymptotic efficiency occurs when \( \zeta \) is small. This is illustrated in Fig. 1, where, in addition to the efficiencies predicted at the actual location of the Vertical Transport and Mixing (VTMX) instrument pod (4.5 m above ground), we have also shown the efficiencies at a hypothetical pod located 100 m above ground (blue dots). Note how the power-law tail is shifted to the right for the virtual pod at higher elevation. Finally, in Fig. 1 (bottom panel), we show a time series of \( \zeta \)’s measured at the elevation of the actual pod versus the same elevation normalized with the hidden length scale. Whereas \( \zeta \) varies over two orders of magnitude, the distance normalized with \( L_H \) is more narrowly constrained.

There appear to be a shift to a different \( L_H \) during the last 2 days.

4. Numerical simulations

Numerical simulations can shed further light on the impact of an externally imposed length scale as well as the effect of a hidden scale, though limited to much lower buoyancy Reynolds numbers.

We consider a flow in a channel periodic in the spanwise and streamwise directions forced by moving the upper and lower walls relative to each other (stratified Couette flow) so that the relative velocity is \( \Delta U \). More specifically, at the upper wall the velocity is fixed at \( (\Delta U/2, 0, 0) \), while at the lower wall it is fixed at \( (-\Delta U/2, 0, 0) \). Likewise, Dirichlet conditions are applied to the buoyancy field, which is fixed at \(-\Delta b/2 \) at the bottom wall and \( \Delta b/2 \) at the top wall to maintain a stable stratification. The equations are solved numerically on a staggered grid, using a standard second-order discretization in space and a second-order projection scheme in time [for details, see Scotti (2015)]. The numerical grid is uniform in the horizontal directions with grid spacing \( O(10) \) wall units, while the vertical resolution increases near the wall to fit 10 grid points within the first 10 wall units.

Two nondimensional parameters, in addition to the aspect ratio of the channel, completely specify the parameter space of the problem. Table 1 lists the parameters of the five simulations considered here. Because of the symmetry around the midchannel plane, we have two M–O layers. We take advantage of this fact to double the statistical size of the sample. We set the outer
dimensions of the fit would extend to lower values of \( R_\text{el} \) the closer the pod is to the ground, since departure from the 1/5 asymptotic efficiency occurs when \( \zeta \) is small. This is illustrated in Fig. 1, where, in addition to the efficiencies predicted at the actual location of the Vertical Transport and Mixing (VTMX) instrument pod (4.5 m above ground), we have also shown the efficiencies at a hypothetical pod located 100 m above ground (blue dots). Note how the power-law tail is shifted to the right for the virtual pod at higher elevation. Finally, in Fig. 1 (bottom panel), we show a time series of \( \zeta \)’s measured at the elevation of the actual pod versus the same elevation normalized with the hidden length scale. Whereas \( \zeta \) varies over two orders of magnitude, the distance normalized with \( L_H \) is more narrowly constrained.

There appear to be a shift to a different \( L_H \) during the last 2 days.

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We consider a flow in a channel periodic in the spanwise and streamwise directions forced by moving the upper and lower walls relative to each other (stratified Couette flow) so that the relative velocity is \( \Delta U \). More specifically, at the upper wall the velocity is fixed at \( (\Delta U/2, 0, 0) \), while at the lower wall it is fixed at \( (-\Delta U/2, 0, 0) \). Likewise, Dirichlet conditions are applied to the buoyancy field, which is fixed at \(-\Delta b/2 \) at the bottom wall and \( \Delta b/2 \) at the top wall to maintain a stable stratification. The equations are solved numerically on a staggered grid, using a standard second-order discretization in space and a second-order projection scheme in time [for details, see Scotti (2015)]. The numerical grid is uniform in the horizontal directions with grid spacing \( O(10) \) wall units, while the vertical resolution increases near the wall to fit 10 grid points within the first 10 wall units.

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in turn implies that when the efficiency measured at a fixed location is plotted against $Re_b$, we expect an asymptotic scaling with a $-1$ slope. The inset in Fig. 1 (bottom) shows the calculated $Ri_f$ at a fixed location near the base of the layer versus $Re_b$. The slope is steeper than the one observed during VTMX, but not as steep as the theory predicts. However, the Reynolds numbers in these simulations is not very large because of numerical resolution constraints. Since we need to choose a location for which $|z| > 55$ in all experiments (to be outside the viscous sublayer), the minimum attainable $\zeta$ is determined by the overall stratification. In practice, $\zeta$ small enough to access the $Ri_f \sim \zeta$ regime can be obtained only for the runs at the weakest stratification considered. However, we clearly see that the $(Re_b, Ri_f)$ curve shifts to the right when increasing the bulk Reynolds number, while keeping the bulk Richardson number constant. The core efficiency decreases in the runs with the increasing bulk Richardson number together with $\zeta_h$ (Table 1), consistent with our analysis, whereas little effect is seen when $Re_b$ is increased, keeping $Ri$ constant.

The DNS data support the Reynolds analogy. The turbulent Prandtl number ranges between 0.79 and 0.95 in all the flows analyzed in this study (see Table 1). Likewise, the ratio of the M–O structure functions for the turbulent buoyancy and momentum fluxes remains close to one. To study how the structure functions change with $\zeta_h$, we plot the structure function $\phi_{ma}$ calculated from the DNS in Fig. 2a. In all cases, $\phi_{ma}$ is well approximated by a linear function of $\zeta$, though with slopes that depends on $\zeta_h$ (Table 1). The structure function of the buoyancy profile yields similar results. The inverse of the slope of $\phi_{ma}$ (the asymptotic mixing efficiency) is 0.126 in the weakly stratified case ($\zeta_h = 0.4$) and increases with $\zeta_h$. In the high $\zeta_h$ limit, the asymptotic efficiency is within the range of values found in atmospheric boundary layers (Foken 2008) and within 15% of the 1/6 value used by oceanographers. Note that because of the nature of the $Ri_f(\zeta)$ relationship within an M–O layer, in the weakly stratified case, the efficiency over most of the layer is actually much smaller than what is suggested by the asymptotic efficiency of the profile, since $\zeta \approx \zeta_h$ and to achieve asymptotic efficiency $\zeta$ must be much larger than $\gamma_w$ (Fig. 2b). The relative size of the logarithmic region vis-à-vis the constant shear region is determined by $\zeta_h$. In weakly stratified flows (small $\zeta_h$), the former dominates, whereas as the stratification increases, it is the latter that dominates.

Consistent with (18), the ratio $Re_b/L_{MO}$ is $O(1)$, varying less than 20% across the layer (Fig. 3c). Contrast this with the local efficiency and gradient Richardson number that vary by an order of magnitude (Fig. 3d).

### Table 1. DNS parameters

<table>
<thead>
<tr>
<th>Run</th>
<th>$Ri$</th>
<th>$Re$</th>
<th>$Re_b$</th>
<th>$Pr_f$</th>
<th>$\zeta_h$</th>
<th>$\gamma_w$</th>
<th>$Ri_f$</th>
<th>$\zeta_{pod}$</th>
<th>Domain</th>
<th>Grid size</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>0.03</td>
<td>14 250</td>
<td>470</td>
<td>0.79</td>
<td>0.393</td>
<td>0.126</td>
<td>0.07</td>
<td>0.04</td>
<td>12 × 6 × 1</td>
<td>512 × 512 × 256</td>
</tr>
<tr>
<td>A2</td>
<td>0.06</td>
<td>18 000</td>
<td>166</td>
<td>0.78</td>
<td>1.018</td>
<td>0.159</td>
<td>0.12</td>
<td>0.10</td>
<td>12 × 6 × 1</td>
<td>512 × 512 × 256</td>
</tr>
<tr>
<td>A3</td>
<td>0.1</td>
<td>28 000</td>
<td>66</td>
<td>0.91</td>
<td>2.225</td>
<td>0.188</td>
<td>0.16</td>
<td>0.22</td>
<td>12 × 6 × 1</td>
<td>512 × 512 × 256</td>
</tr>
<tr>
<td>B1</td>
<td>0.03</td>
<td>27 500</td>
<td>860</td>
<td>0.79</td>
<td>0.422</td>
<td>0.122</td>
<td>0.15</td>
<td>0.04</td>
<td>12 × 6 × 1</td>
<td>1024 × 1024 × 512</td>
</tr>
<tr>
<td>B2</td>
<td>0.1</td>
<td>55 000</td>
<td>120</td>
<td>0.89</td>
<td>2.512</td>
<td>0.177</td>
<td>0.07</td>
<td>0.25</td>
<td>12 × 6 × 1</td>
<td>1024 × 1024 × 512</td>
</tr>
</tbody>
</table>

The Monin–Obukhov theory provides a robust framework to characterize the efficiency within a specific class of stratified turbulent boundary layer. For M–O layers that extend several M–O lengths in the vertical, the core efficiency approaches a value close to 1/6, which happens to be the value commonly employed in ocean mixing parameterization (St. Laurent et al. 2002). For layers thin relative to the M–O length (weak stratification or strong forcing), the core efficiency is close to the bulk Richardson number calculated in terms of the buoyancy and velocity jump across the layer, and the system does not mix efficiently. A universal $Ri_f(Re_b)$ relationship is incompatible with the M–O analysis, yet many field studies where the M–O assumptions are verified to hold suggests the existence of a power-law relationship (Lozovatsky and Fernando 2013; Barry et al. 2001). To explain this apparent discrepancy, we need to admit the existence of a hidden scale. Namely, the field experiments sample an ensemble of M–O layers, measuring the efficiency at fixed location(s) $z$ in space. This generates efficiencies measured over a range of values for $|z|/L_{MO}$, the height from the boundary (which is constant) normalized by the Monin–Obukhov length (which varies over the ensemble). The existence of an asymptotic power-law relationship $Ri_f = CRe_b^{\gamma}$ between the flux Richardson number $Ri_f$ and the...
buoyancy Reynolds number $Re_b$ implies that, over the ensemble of M–O layers, there is a constant length scale $L_H$ such that the constant $C$ is a function of $|z|/L_H$. The manner in which $L_H$ depends on $B$, $u_*$ and $\nu$ can be determined if we know the scaling $a$. A natural question that arises is how universal the nature of $L_H$ is [i.e., its dependence on the turbulent fluxes and how it relates to other possible scales (see, e.g., Mater and Venayagamoorthy 2014a,b)]. The prevalence of a $-1/2$ power law (with some scatter) across datasets captured in different environmental conditions (Barry et al. 2001) hints that it may well be universal, but this can only be answered with a systematic analysis of existing data, which is beyond the scope of the present paper. Another interesting question, which is left for the future, is how to adapt the analysis to layers where a scale other than the M–O scale can be introduced, for example, the rotating scale $L_f$ or when a different length scale related to the

Fig. 2. DNS of stratified Couette: averaged momentum structure functions; the red lines are the corresponding linear interpolants. (b) The flux Richardson number as a function of $\zeta$, the distance from the wall normalized with the MO length $L_{MO}$; the red lines are the predicted value based on the interpolants from (a). Symbols are as follows: crosses, $Ri = 0.03$; circles, $Ri = 0.06$; and stars, $Ri = 0.1$. 
stratification can be defined, such as in Taylor and Sarkar (2008), who show that in a stratified bottom Ekman layer a role similar to the M–O length scale is played by $u^* N$, where $u^*$ is the friction velocity and $N$ is the stratification frequency. Finally, it is encouraging that several turbulent closures employed in atmospheric and oceanic simulations are consistent with the findings of this paper. For example, Burchard et al. (1998) show that even relatively simple closures such as $k^2 \approx \rho_0$ obtain structure functions that are very close to the ones observed (and thus reproduce the same profiles of mixing efficiency). Canuto et al. (2001) and Baumert and Peters (2004) show that a one-point closure model reproduces the linear relationship between flux and gradient Richardson number observed in Monin–Obukhov layers.

Last, we would be remiss if we did not stress one more time that these results apply only to stratified boundary layers that are described by the M–O theory. For such systems, provided the nondimensional thickness of the layer is not small, an efficiency equal to 1/6 appears to be appropriate.

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