On the Role of Bottom Pressure Torques in Wind-Driven Gyres

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ABSTRACT: Previous studies have concluded that the wind-input vorticity in ocean gyres is balanced by bottom pressure torques (BPT), when integrated over latitude bands. However, the BPT must vanish when integrated over any area enclosed by an isobath. This constraint raises ambiguities regarding the regions over which BPT should close the vorticity budget, and implies that BPT generated to balance a local wind stress curl necessitates the generation of a compensating, nonlocal BPT and thus nonlocal circulation. This study aims to clarify the role of BPT in wind-driven gyres using an idealized isopycnal model. Experiments performed with a single-signed wind stress curl in an enclosed, sloped basin reveal that BPT balances the winds only when integrated over latitude bands. Integrating over other, dynamically motivated definitions of the gyre, such as barotropic streamlines, yields a balance between wind stress curl and bottom frictional torques. This implies that bottom friction plays a nonnegligible role in structuring the gyre circulation. Nonlocal bottom pressure torques manifest in the form of along-slope pressure gradients associated with a weak basin-scale circulation, and are associated with a transition to a balance between wind stress and bottom friction around the coasts. Finally, a suite of perturbation experiments is used to investigate the dynamics of BPT. To predict the BPT, the authors extend a previous theory that describes propagation of surface pressure signals from the gyre interior toward the coast along planetary potential vorticity contours. This theory is shown to agree closely with the diagnosed contributions to the vorticity budget across the suite of model experiments.

KEYWORDS: Ocean dynamics; Pressure; Topographic effects; Wind stress; Isopycnal coordinates; Ocean models

1. Introduction

Gyres are the most prominent feature of the global surface circulation and have motivated decades of research aimed at understanding their dynamics (Pedlosky 1990). Gyres are understood as primarily wind-driven features that respond to the curl of the surface stress generated by the wind, rather than the wind’s absolute strength and direction (Sverdrup 1947; Stommel 1948; Rhines 1986; Luyten and Stommel 1986). The importance of the vorticity balance was recognized in early theories of gyre circulation, in which the net wind stress curl was balanced in a western boundary current by either frictional drag at the sea floor (Stommel 1948), or by lateral stresses associated with an “eddy” viscosity (Munk 1950). These early theories were posed in an ocean basin with flat bathymetry, or for fluid above the main thermocline that did not interact dynamically with the sea floor (Pedlosky 2013).

It has subsequently been shown that bathymetry, and particularly the continental slopes along western boundaries, can substantially reshape gyre circulations (Salmon 1994; Kubokawa and McWilliams 1996; Becker 1999; Vallis 2017). In particular, Holland (1973) first found that baroclinic flow over variable-depth topography in idealized numerical experiments established a dominant area-integrated barotropic vorticity balance between the wind stress curl and the bottom pressure torque, BPT = −J(p_b, η_b). (1)

Here \( \eta_b(x, y) \) denotes the sea floor elevation, \( p_b = p_z = \eta_b \) denotes the pressure at the sea floor, and \( J \) is the Jacobian operator. Note that throughout this study we use the term “barotropic vorticity budget” to refer to the curl of the depth-integrated momentum equations, as opposed to the depth-integral of the vorticity equation. The barotropic vorticity budget must also be distinguished from the curl of the depth-averaged momentum equation, which includes no contribution due to pressure in a barotropic fluid (Niiler 1966). In a baroclinic fluid, the effect of topography manifests in this equation as the joint effect of baroclinicity and relief (JEBAR) term (Sarkisyan 1971; Mellor 1999). We focus only on the barotropic vorticity budget because the JEBAR term may be large even in situations where there is no topographic forcing of the flow (Cane et al. 1998).

The degree to which the wind stress curl is balanced by bottom pressure torques has been investigated by various, more recent studies. Hughes and de Cuevas (2001) showed that bottom pressure torques balance the wind stress curl consistently over −3°-wide latitude bands throughout a 1/4° global ocean model simulation, and showed that this is a consequence of wind stress approximately balancing topographic form stress in the zonal momentum balance. Jackson et al. (2006) further confirmed the primary vorticity balance between wind stress curl and bottom pressure torque in idealized model simulations, but additionally noted that bottom friction...
is also required to form recirculating flows that cross the mean potential vorticity gradient. More recently, Schoonover et al. (2016) showed that the balance between wind stress curl and bottom pressure torque in the North Atlantic subtropical gyre holds in several comprehensive simulations conducted using three different circulation models, when integrated around appropriately chosen barotropic streamlines. Note that our results (see section 3) indicate that the wind stress curl-bottom pressure torque balance not generally hold in the interior of wind-driven gyres, with bottom frictional stresses balancing the wind stress curl (see also Le Corre et al. 2020).

However, there remains some ambiguity about the establishment of bottom pressure torques and their specific role in balancing the vorticity balance of wind-driven gyres. Consider the integral of the bottom pressure torque (1) over the area $A$ enclosed by a contour of constant ocean depth, i.e., $\nabla \eta_b \cdot ds = 0$ on $\partial A$, where $s$ is an along-contour coordinate and $\partial A$ denotes the boundary of $A$. Using the Stokes theorem, it follows that

$$\int_A J(p_b, \eta_b) dA = \oint_{\partial A} p_b \nabla \eta_b \cdot ds = 0. \quad (2)$$

Thus, the net bottom pressure torque must vanish over any area enclosed by an isobath or between any pair of isobaths, including any entire enclosed ocean basin, or indeed the entire World Ocean.\(^1\) This highlights a key difference between bottom pressure torques and friction/viscosity in the context of gyre-scale vorticity balances: there is no such global constraint on western-boundary frictional or viscous torques that develop to balance the zonally integrated wind stress curl.

One implication of (2) is that although we expect bottom pressure torque to balance the wind stress curl when integrated over narrow latitude bands, they cannot balance when integrated over a closed ocean basin (Hughes and de Cuenas 2001). Thus, the area-integrated vorticity balance of a gyre depends on how one chooses to define the areal extent of the gyre; it is not obvious that an integral around the gyre’s barotropic streamlines (Schoonover et al. 2016) should yield a balance between wind stress curl and bottom pressure torque, for example. Another implication is that if a single-signed wind stress curl over one part of the ocean is locally balanced by a bottom pressure torque, then an equal and opposite bottom pressure torque must be established elsewhere, along with the geographical structure that hinders dynamical interpretation (e.g., Schoonover et al. 2016; Le Corre et al. 2020); indeed, we diagnose complex structures even in our very simple geometry (see Fig. 1b and section 3). Additionally, using a lower-complexity model allows us to perform a large number of model experiments, and thus explore a larger parameter space than would be afforded, for example, by comprehensive regional models. As the adiabatic, wind-driven circulation is our primary focus, we select a two-layer, rigid-lid, isopycnal model (see Fig. 1b), which allows a representation of the gyre’s baroclinicity while preserving the model stratification exactly over long integrations. Below we describe the model configuration in detail.

For reference, key parameters summarized in Table 1. We simulate the flow of two isopycnal layers on a Northern Hemisphere midlatitude $\beta$ plane. Specifically, we solve the isopycnal momentum equations,

$$\frac{\partial u_k}{\partial t} + h_k q_k \hat{x} \times u_k + \nabla \cdot (M_k + \frac{1}{h_k} \eta_k) = \delta_{s,k} \tau \rho_0 h_k \hat{x} - W_k \frac{\partial \eta_k}{\partial t} \hat{u}_k + \frac{1}{h_k} \nabla \cdot \sigma_k, \quad (3)$$

for $k = 1$ (upper layer) and $k = 2$ (lower layer). The volume in each isopycnal layer is conserved according to

$$\frac{\partial h_k}{\partial t} + \nabla \cdot (h_k u_k) = 0. \quad (4)$$

Here $u_k$ is the horizontal velocity, $h_k$ is the layer thickness, $\hat{x}$ and $\hat{z}$ are unit vectors, and $\rho_0$ is a constant reference density. The potential vorticity $q_k$ is defined as

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\(^1\) Note that, by the symmetry of the Jacobian operator, this result also holds for any area enclosed by a contour of constant bottom pressure.
and the Montgomery potential $M_k$ is defined as
\[ M_k = \pi + \delta_{s_k} g' \eta_k - \delta_{s_k} H_{s_k}. \]

The flow is forced by a meridionally varying zonal wind stress $\tau(y)$, loosely motivated by the zonal-mean winds over the Northern Hemisphere subtropical and subpolar gyres (e.g., Large and Yeager 2009), and constructed as follows:
\[ \tau(y) = \tau_{\text{wind}} \begin{cases} 
-1 & y \leq -L_{\text{wind}}, \\
\cos \left( \frac{\pi y}{L_{\text{wind}}} \right) & -L_{\text{wind}} \leq y \leq L_{\text{wind}}, \\
-1 & y \geq L_{\text{wind}}.
\end{cases} \]

This profile is illustrated in Fig. 1a. The flow is damped by friction that depends on the near-bottom velocity $u_b$ with shallowest parts of the model domain, without destabilizing the model or creating large spurious thermal wind shears. Note that this term does not influence the bottom pressure torque (see section 3a).
and zero viscous stress. The bathymetry vertical walls, at which we apply conditions of no normal flux additionally include a hyperviscous term \( \kappa \), such that
\[
\frac{\partial \mathbf{u}}{\partial z} = -\kappa \mathbf{u}
\]
where \( \kappa \) is the hyperviscous coefficient. This term is added to the momentum equations to dissipate kinetic energy, potential energy, and potential enstrophy. All simulations discussed below were initialized with zero horizontal velocity and a uniform upper-layer thickness \( H_{\text{shelf}} \), as illustrated in Fig. 1b. Note that the bathymetric depth on the continental shelf must be nonzero to avoid numerical instabilities; we found that setting \( H_{\text{shelf}} = 100 \text{ m} \) was sufficient to allow bottom pressure torque to dominate the vorticity balance, without producing spurious behavior over the continental shelf (cf. Jackson et al. 2006). The shape of the bathymetry is defined precisely in appendix A in the interest of reproducibility.

To integrate (3)–(6) we use the AWSIM model (Stewart and Dellar 2016). We discretize the model equations on a uniform horizontal grid of \( 512^2 \) points, corresponding to a grid spacing of approximately 10 km. We use the energy- and potential enstrophy-conserving spatial discretization scheme of Arakawa and Lamb (1981), and we use the third-order Adams–Bashforth scheme to step forward in time (Durran 1991). There is no explicit evolution equation for the surface pressure \( \pi \), so we compute this field diagnostically at each time step in such a way as to guarantee nondivergence of the depth-integrated velocity field, following Zhao et al. (2019).

All simulations discussed below were initialized with zero horizontal velocity and a uniform upper-layer thickness \( H_{\text{pyc}} \) (see Fig. 1c) set to 500 m in our reference simulation. Together with \( g' = 3 \times 10^{-2} \text{ m} \text{s}^{-2} \) and \( f_0 = 8.5 \times 10^{-4} \), this results in a baroclinic deformation radius of \( L_d = \sqrt{g' h_2 / (\beta_1 + \beta_2)} \approx 45 \text{ km} \) that is typical of the ocean’s subtropics (Chelton et al. 1998). Each experiment is “spun up” at low resolution (20-km grid spacing) for 20 years, before being interpolated to high resolution (10-km grid spacing) and continued for a further 20 years. This is sufficient to achieve a statistically steady state in all of our runs, based on time series of the domain-integrated kinetic energy, potential energy, and potential enstrophy. All diagnostics presented here are drawn from averages calculated “online” during the last 10 years of integration at high resolution.

Figure 2 illustrates the circulation in our reference simulation, referred to as DOUBLEGYRE, which is run using the parameters listed in Table 1. Consistent with previous baroclinic, quasigeostrophic double-gyre models (see, e.g., Poje and Haller 1999; Berloff 2005), the simulation develops cyclonic and anticyclonic gyre circulations in the northern and southern portions of the model domain, respectively. These gyres may be regarded as idealized equivalents of the Northern Hemisphere subpolar

### Table 1. List of parameters used in our Reference simulation. Italics in the description column indicate parameters that are varied and/or covaried in our parameter sensitivity experiments in section 4.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_x \times L_y )</td>
<td>5000 × 5000 km</td>
<td>Horizontal dimensions</td>
</tr>
<tr>
<td>( H )</td>
<td>4000 m</td>
<td>Maximum ocean depth</td>
</tr>
<tr>
<td>( N_x \times N_y )</td>
<td>512 × 512</td>
<td>Horizontal grid size</td>
</tr>
<tr>
<td>( \Delta_x, \Delta_y )</td>
<td>~10 km</td>
<td>Horizontal grid spacing</td>
</tr>
<tr>
<td>( \Delta_t )</td>
<td>217 s</td>
<td>Time step size</td>
</tr>
<tr>
<td>( f_0 )</td>
<td>8.5 × 10^{-5} s^{-1}</td>
<td>Coriolis parameter at domain center</td>
</tr>
<tr>
<td>( B )</td>
<td>1.8 × 10^{-11} m^{-1}s^{-1}</td>
<td>Meridional gradient of Coriolis parameter</td>
</tr>
<tr>
<td>( \rho_0 )</td>
<td>1000 kg m^{-3}</td>
<td>Reference density</td>
</tr>
<tr>
<td>( h_{\text{sal}} )</td>
<td>5 m</td>
<td>“Salmon” layer thickness</td>
</tr>
<tr>
<td>( h_{\text{sm}} )</td>
<td>50 m</td>
<td>Surface boundary layer thickness</td>
</tr>
<tr>
<td>( h_{\text{bl}} )</td>
<td>50 m</td>
<td>Bottom boundary layer thickness</td>
</tr>
<tr>
<td>( \Lambda_{\text{Smag}} )</td>
<td>4</td>
<td>Smagorinsky biharmonic viscosity coefficient</td>
</tr>
<tr>
<td>( C_d )</td>
<td>2.5 × 10^{-3}</td>
<td>Quadratic bottom drag coefficient</td>
</tr>
<tr>
<td>( g' )</td>
<td>3 × 10^{-2} m s^{-2}</td>
<td>Wind stress amplitude</td>
</tr>
<tr>
<td>( \tau_{\text{wind}} )</td>
<td>0.1 N m^{-2}</td>
<td>Reduced gravity</td>
</tr>
<tr>
<td>( H_{\text{shelf}} )</td>
<td>100 m</td>
<td>Continental shelf depth</td>
</tr>
<tr>
<td>( W_{\text{slope}} )</td>
<td>150 km</td>
<td>Continental slope width</td>
</tr>
<tr>
<td>( H_{\text{pyc}} )</td>
<td>500 m</td>
<td>Pycnocline depth</td>
</tr>
<tr>
<td>( L_{\text{wind}} )</td>
<td>2000 km</td>
<td>Meridional width of wind stress curl</td>
</tr>
<tr>
<td>( R_{\text{slope}} )</td>
<td>500 km</td>
<td>Continental slope corner radius of curvature</td>
</tr>
<tr>
<td>( D_{\text{slope}} )</td>
<td>250 km</td>
<td>Distance from continental slope center to domain edge</td>
</tr>
</tbody>
</table>
and subtropical gyres, respectively. Both gyres develop strong western boundary currents that meet close to \( y = 0 \). These features are visible in the sea surface elevation, calculated here as

\[ \eta = \pi/g, \]  

(8)
in Fig. 2b. The boundary currents then separate from the western boundary and produce an energetic eddy field, visible in Fig. 2a. The separated boundary current is associated with recirculation gyres (Hogg et al. 1986; Jayne et al. 2009; Waterman and Jayne 2011) that have a strong barotropic component, and thus are pronounced in the barotropic streamfunction,

\[ \Psi = \int_y^{L/2} \sum_k h_k h_k^* dy', \]  

(9)

shown in Fig. 2c. Finally, the continental slope produces a southward “tail” close to the western boundary in both the northern and southern gyres, consistent with previous calculations of gyre structure over continental slopes via theoretical approaches (Kubokawa and McWilliams 1996; Wise et al. 2018) and numerical simulations (Salmon 1994; Becker 1999; Vallis 2017).

3. Role of bottom pressure torque in the vorticity balance

a. Vorticity budget in a reference simulation

To provide context for experiments discussed in later sections, we first briefly examine the barotropic vorticity budget in the DOUBLEGYRE reference simulation shown in Fig. 2. Taking a vertical integral of (3), and then averaging the result in time and taking its curl, yields the barotropic vorticity equation,

\[ \frac{\partial}{\partial t} \nabla \times \sum_k h_k \mathbf{u}_k + \mathbf{u}_k \nabla \times \mathbf{h}_k = -\mathbf{u}_k \cdot \nabla \mathbf{h}_k + \mathbf{u}_k \nabla \cdot (\mathbf{h}_k \mathbf{u}_k) - \mathbf{u}_k \nabla \cdot (\mathbf{h}_k \mathbf{u}_k) + \mathbf{u}_k \sum_k \mathbf{h}_k \nabla \mathbf{M}_k + \nabla \sum_k \mathbf{h}_k \nabla \mathbf{M}_k. \]  

(10)

Here an overbar denotes a time average, and summation is taken over \( k = 1 \) and \( k = 2 \). Note that the term labeled as “bottom pressure torque” in (10) differs from the form given in section 1. Substituting (6) into the bottom pressure torque term in (10) yields Eq. (1), after some manipulations, using \( p_b = \rho_0 (\pi + gh_1 + (g' + g^2)h_2) \) and \( \eta_b = -(h_1 + h_2) \). In particular, the bottom pressure torque is not influenced by the Salmond (2002) modification of the Montgomery potential, even where the lower layer becomes very thin. Note also that the units of (10) differ from (1) by a factor of \( \rho_0 \).

We diagnose the terms in (10) from the DOUBLEGYRE simulation, and subsequent experiments, by online averages of the momentum equation (multiplied by \( h_k \)) over the last 10 years of the simulation. This implies that we compute the bottom pressure torque in the form given in (10), rather than in the form (1). This approach exactly follows the model’s discretization of the momentum equation (see Stewart and Dellar 2016), and therefore minimizes discretization errors, which can be particularly pronounced in the computation of the bottom pressure torque (e.g., Schoonover et al. 2016; Stewart et al. 2019). However, some temporal discretization errors remain due to differences between the computed momentum tendency over each time step, which is first-order accurate in time, and the actual evolution in the momentum by the model’s third-order Adams–Bashforth scheme. We compute terms in (10) from the time-averaged momentum budget via the area-averaged discrete circulation around each horizontal grid point. This ensures that the integral of any term in the vorticity budget over a specified area is exactly equal to the discrete barotropic circulation tendency around that area. A caveat to this approach is that it hinders further decompositions of the terms in (10), e.g., to partition the advection into contributions due to mean flows and eddies (e.g., Marshall 1984; Waterman and Jayne 2011).

In the following analysis we will integrate the terms in (10) over various areas within our model domain. By the Stokes theorem, the integral of (10) over any area is equivalent to an integral of the circulation tendency around the boundary of that area. For example, for the bottom pressure torque, the area integral is

\[ \int_A \frac{\partial}{\partial t} \nabla \times \sum_k h_k \mathbf{M}_k dA = \oint_{\partial A} -\sum_k h_k \nabla \mathbf{M}_k \cdot ds. \]  

(11)

Here \( \partial A \) denotes the boundary of \( A \) and \( ds \) is an infinitesimal line element tangential to \( \partial A \), oriented such that the path of the integral is counterclockwise around \( A \). In cases for which the area \( A \) reaches the boundary of the model domain, the area integrals are taken over all vorticity points within the model domain, equivalent to a circulation integral that passes through the grid points along the edge of the model domain. A consequence of this approach is that the area-integrated barotropic vorticity budget north of a latitude line \( y = y_0 \) does not coincide with the zonal momentum budget along \( y = y_0 \), as it does when the ocean depth approaches zero at the domain boundaries (see Hughes and de Cuevas 2001; see also appendix D). Our area integrated barotropic vorticity balances therefore do not translate directly to the zonal momentum balance: for example, in cases where the bottom pressure torque does not balance the wind stress curl, the zonal wind stress is still primarily balanced by zonal pressure gradient forces (not shown).

Figure 3 maps each of the terms in (10), excluding the tendency term, which we found to be negligibly small everywhere in the model domain. The vorticity balance largely conforms to our expectations of topographically balanced gyre circulations.
(e.g., Wise et al. 2018): in the flat basin interior, the relatively weak wind stress curl is balanced by advection (Figs. 3a,b), primarily via advection of planetary vorticity advection by the mean flow ($\rho \sum \overline{\mathbf{u}} \cdot \mathbf{u}$, not shown). Around the basin’s continental slopes, particularly along the western boundary and within the recirculation gyres, there are much larger local contributions from advection, bottom stress curl, and the bottom pressure torque (Figs. 3b–d). Viscous torques are negligible almost everywhere except for thin strips of alternating sign along the western continental shelf break (Fig. 3e). We will show in section 3b that the integrated contribution of viscous torques is negligible. Finally, the residual (Fig. 3f) is nonzero but small compared to all other terms in the vorticity balance.

b. Experiments with single-signed wind stress curl

We now address questions of where the bottom pressure torque balances the wind stress curl and of the manifestation of nonlocal bottom pressure torques, discussed in section 1. We now pose two perturbation experiments, NORTHGYRE and SOUTHGYRE, in which the wind stress is nonzero only in the northern and southern halves of the model domain, respectively. Figure 4 illustrates the wind stress profiles, time-mean sea surface elevations and time-mean barotropic streamfunctions for each of these two experiments. Note that we have shifted the latitudinally averaged wind stress in each case such that the wind stress is only nonzero in the hemisphere with nonzero wind stress curl. Including a nonzero wind stress in the hemisphere with zero wind stress curl further amplifies the nonlocal response to the wind forcing.

The structures of the northern and southern gyre circulation that result are qualitatively similar to those in the DOUBLEGYRE simulation (Fig. 2), but with visible differences in the locations of the boundary current separation in to the basin interior and of the recirculation gyres. Below we focus our attention on the SOUTHGYRE case because it is the analog of the North Atlantic’s subtropical gyre, and so may be most closely compared with the results of Schoonover et al. (2016). In appendix B we present almost identical diagnostics for the NORTHGYRE experiment, which are qualitatively similar and thus indicate that the SOUTHGYRE results are not an accident of the geometry or model parameters.

In Fig. 5 we again map the terms in the barotropic vorticity budget (10), but for the SOUTHGYRE experiment. Like the mean circulation shown in Figs. 4e–f, the vorticity budget maps in Fig. 5 are visually similar to the lower halves of the vorticity maps for the DOUBLEGYRE experiment in Fig. 3. This suggests that the SOUTHGYRE experiment replicates and isolates the dynamics of the southern gyre in the DOUBLEGYRE experiment.

We now investigate overall vorticity balance of the southern gyre by integrating (10) over different coordinate systems. Figure 6a shows the result of integrating (10) over areas bounded by ocean depths $d$ greater than a reference depth $d_{ref}$. Consistent with (2), the bottom pressure torque is zero for all reference depths $d_{ref}$. The wind stress curl is balanced by advection in the basin interior, transitioning to a balance with friction when integrated over the full model domain. This
FIG. 3. Maps of terms in the barotropic vorticity budget in the reference DOUBLEGYRE experiment, drawn from a 10-yr integration period in statistically steady state. (a) Wind stress curl, (b) curl of momentum advection terms, (c) bottom stress curl, (d) bottom pressure torque, (e) curl of viscous forces, and (f) residual after summing (a)–(e). See (10) for formal definitions of each term.
is consistent with the balance between along-coast surface and bottom stresses that was postulated by Hughes and de Cuevas (2001).

In Figs. 6b and 6c we select two dynamically motivated coordinate systems: one defined by contours of sea surface height $H$ and another defined by contours of the barotropic streamfunction $C$. Here find that bottom pressure imposes a torque on the fluid in the same direction as the wind stress curl. This result is contrary to what one might have expected from some previous studies (e.g., Jackson et al. 2006; Schoonover et al. 2016), but a similar result was found by Holland (1973) for a baroclinic wind-driven gyre over a western boundary continental slope. The combined wind stress curl/bottom pressure torque is balanced by advection in the core of the gyre ($\eta_{ref} \approx 0.45$ m and $\Psi_{ref} \approx 10$ Sv; $1$ Sv $= 10^6$ m$^3$ s$^{-1}$), transitioning to bottom friction at the outer edge of the gyre.

Finally, in Fig. 6d we simply integrate the vorticity budget over the area north of a reference latitude $y_{ref}$. In this coordinate system we recover the balance between bottom pressure torque and wind stress curl that is expected over latitude bands (Hughes and de Cuevas 2001), only transitioning to a frictional balance when $y_{ref}$ approaches the southern boundary and the entire domain is included in the integration. To reinforce this point, in Fig. 7 we plot the zonally integrated vorticity budget as a function of latitude in the DOUBLEGYRE, NORTHGYRE, and SOUTHGYRE runs. Although there are substantial local deviations, the integrated vorticity budgets over the northern gyre ($0 \leq y \leq 2000$ km) and the southern gyre ($-2000 \leq y \leq 0$ km) show a close (within 10%–20%) balance between wind stress curl and bottom pressure torque. In contrast, at the northern ($2000 \leq y \leq 2500$ km) and southern ($-2500 \leq y \leq -2000$ km) boundaries the dominant balance is between bottom pressure torque and bottom stress curl.

Taken together, the diagnostics presented in Fig. 6 show that of the coordinate systems examined here, only integrating across latitude bands consistently yields an area-integrated balance between wind stress curl and bottom pressure torque (Hughes and de Cuevas 2001). An implication of this finding is that because friction balances the wind stress integrated along streamlines (within sea surface elevation or barotropic stream-function contours), the circulation around those streamlines should be sensitive to the bottom friction. For example, although the gyre transport remains constrained by Sverdrup balance in the interior, the speed and separation behavior of the western boundary currents should depend not only on the bathymetry, but also on the formulation of the bottom friction.
This could also be inferred from the potential vorticity budget (Jackson et al. 2006): friction allows the mean flow to cross isolines of mean potential vorticity, and thus must play a role in structuring the gyre circulation.

Finally, we use our diagnostics to address the nonlocal manifestation of bottom pressure torque. As discussed above, in the SOUTHGYRE experiment the negative wind stress curl is approximately balanced by a positive bottom pressure...
torque over the latitudinal range of each gyre (Figs. 6d and 7c). However, the bottom pressure torque must integrate to zero in a domain average, and in the SOUTHGYRE experiment the compensating negative bottom pressure torque manifests along the southern continental slope, close to the 500-m isobath. This may be understood as follows: (2) implies that the bottom pressure torque must integrate to zero over the area enclosed by any two isobaths, $\eta_{b1}$ and $\eta_{b2}$,

$$\int_{\eta_{b1} < \eta < \eta_{b2}} J(p_b, \eta_b) \, dA = 0.$$  

(12)

In our idealized domain with approximately constant topographic slopes along each isobath, the bottom pressure torque should therefore approximately integrate to zero along each isobath

$$\int_{\eta_{b} = \text{const}} J(p_b, \eta_b) \, ds = \int_{\eta_{b} = \text{const}} \frac{\partial p_b}{\partial s} \, \frac{\partial \eta_b}{\partial n} \, ds = 0,$$

(13)

where $n$ is an isobath-normal coordinate.

Thus, an interpretation of the nonlocal bottom pressure torque is that establishing an along-isobath pressure gradient, and thus bottom pressure torque, within the southern gyre ($-2000 \leq y \leq 0$ km) necessarily produces an opposite along-isobath pressure gradient elsewhere in the domain. A consequence of this nonlocal pressure gradient/bottom pressure torque is the formation of a circulation, albeit a relatively weak one, around the entire basin in both the NORTHGYRE and SOUTHGYRE cases (see Figs. 4b,d). An alternative interpretation of the nonlocal bottom pressure torque is that the nonzero domain-integrated wind stress curl corresponds to a nonzero forcing of the circulation around the domain boundary, via (11). This forcing can only be balanced by bottom stresses (or viscous forces) acting tangentially to the domain boundary, due to (2), and thus there is necessarily a bottom stress curl in the vicinity of the domain boundaries (Fig. 6c). This bottom stress curl generates a locally compensating bottom pressure torque, which exactly balances the bottom pressure torque generated within the gyre (Fig. 6d).

4. Transition to frictional and inertial gyres

To provide further insight into the dynamics of topographically balanced gyre circulations, we now perform a suite of experiments to examine the transition from topographically balanced to frictionally or inertially balanced vorticity budgets. We first perform a series of experiments that follow the DOUBLEGYRE configuration, but with successively deeper continental shelf depths $H_{\text{shelf}}$. These experiments range from $H_{\text{shelf}} = 100$ m, i.e., the DOUBLEGYRE simulations, to $H_{\text{shelf}} = H = 4000$ m, in which the bathymetry is flat and the flow may be expected to resemble the results of quasigeostrophic double-gyre calculations (e.g., Poje and Haller 1999; Berloff 2005). For each experiment, we compute the time-mean contributions to the barotropic vorticity budget (10) over latitude bands corresponding to the “northern gyre,” defined

![Fig. 6. Balance of terms in the area-integrated barotropic vorticity budget in different coordinate systems, computed from the SOUTHGYRE experiment over 10 years in statistically steady state. (a) Sea floor depth coordinates, (b) sea surface height coordinates, (c) barotropic streamfunction coordinates, and (d) latitudinal distance coordinates.](image-url)
as $0 \leq y \leq 2000 \text{ km}$, and the “southern gyre,” defined as $-2000 \leq y \leq 0 \text{ km}$. These latitude ranges are selected because they define areas over which bottom pressure torque does balance the wind stress curl in our NORTHGYRE and SOUTHERGYRE experiments (see section 3b).

In Fig. 8 we plot the dependence of the terms in the vorticity budget on $H_{\text{shelf}}$. In both the northern and southern gyres, the vorticity balance exhibits two transition: for shelf depths $H_{\text{shelf}}$ between zero and the pycnocline depth $H_{\text{pyc}} = 500 \text{ m}$, there is a transition from a bottom pressure torque-dominated balance to a balance in which bottom friction most strongly opposes the wind stress curl. Then when $H_{\text{shelf}} \sim H_{\text{pyc}}$ there is a relatively rapid transition to balance between wind stress curl and advection, with a small (10%–20%) contribution from viscous torques. This implies that the gyres transition through a Stommel (1948)-like regime as the shelf approaches the pycnocline depth ($300 \text{ m} \leq H_{\text{shelf}} \leq 500 \text{ m}$), but for shelves deeper than the pycnocline ($H_{\text{shelf}} \geq 500 \text{ m}$) the wind-input vorticity is primarily resolved via an advective vorticity transport between the two gyres, with a weak Munk (1950) viscous western boundary layer.

As discussed in section 3a, the area-integrated contributions to the barotropic vorticity budget shown in Fig. 8 correspond to circulation integrals around the boundaries of the northern and southern gyres. These area-integrated terms therefore quantify contributions to the barotropic circulation budget/vorticity budget within the model domain. This is consistent with formulation of the vorticity budgets in classical theories of wind-driven gyres (Stommel 1948; Munk 1950). An alternative formulation would include “delta functions” of the terms in (10) around the model’s vertical walls (see Hughes and de Cuevas 2001; see also appendix D). In this alternative formulation, the
gyre-integrated vorticity budgets considered here reduce to the difference between the zonal momentum budgets along the northern and southern edges of the gyres, which consistently exhibit a balance between the zonal wind stress and the zonal pressure gradient force (not shown).

While the presence of three dynamical regimes is visually clear in Fig. 8, it remains unclear whether this is an accident of our particular choice of reference parameters, and how these transitions are influenced by other aspects of the model forcing, stratification and geometry. We therefore perform additional sensitivity experiments, this time perturbing DOUBLEGYRE configuration with $H_{\text{shelf}} = 450$ m, for which friction plays the strongest role in the gyres' vorticity balances. We perform a suite of simulations in which we separately perturb the wind stress maximum $t_{\text{max}}$, the coefficient of friction $C_d$, the stratification/reduced gravity $g$, the width of the continental slope $W_{\text{slope}}$. We perform a separate set of experiments in which we perturb $H_{\text{pyc}}$, starting from the reference DOUBLEGYRE experiment ($H_{\text{shelf}} = 100$ m). In these experiments we varied $g'$ inversely with $H_{\text{pyc}}$ to preserve a constant Rossby radius of deformation $L_d = \sqrt{g H_{\text{pyc}}}$. We also varied $\tau_{\text{max}}$ linearly with $H_{\text{pyc}}$ so that the western boundary current (assumed to be on the order of $L_d$) has similar speeds, and so that $\eta$ undergoes similar fractional changes across the gyre, between experiments.

Figure 9 shows the partitioning of the vorticity budget across this suite of simulations. All of the parameters examined here produce modest (up to 30% of the wind stress curl) adjustments in the partitioning of the vorticity budget, with the exception of the pycnocline depth. Shallowing $H_{\text{pyc}}$ produces a qualitatively similar effect as deepening $H_{\text{shelf}}$ (see Fig. 8), with a gradual switch from a bottom pressure torque-dominated balance to a friction-dominated balance as $H_{\text{pyc}}$ shallows toward the shelf depth $H_{\text{shelf}} = 100$ m. This suggests that the relative depths of the shelf and pycnocline play a primary role in setting the dynamical regimes of the gyres.

Motivated by this finding, we pose a theory for the fractional importance of bottom pressure torque in balancing the vorticity budget. We make the assumption that the gyre circulation develops primarily above the pycnocline. This is consistent with the transport simulated by our model, which is primarily confined to the upper isopycnal layer (not shown), with the exception of the recirculation gyres. This corresponds to idealizing the circulation as an equivalent-barotropic model, with a single isopycnal layer that incrops into the continental slope around the domain boundaries. In appendix C we derive an explicit solution for the surface pressure field within the latitude band of the gyre. In the basin interior the mean layer thickness is assumed to be constant, $\bar{\eta}_{\text{in}} = H_{\text{pyc}}$, and Sverdrup balance is assumed to hold (Sverdrup 1947). Where the pycnocline incrops into the continental shelf and slope, the upper layer occupies the full depth of the water column (see Fig. 1c and appendix C), $\bar{\eta}_{\text{in}} = \eta(x)$, and the flow is assumed to follow $f \bar{\eta} = -\eta(x)$. The theory predicts that bottom pressure torque balances the fraction of the wind stress curl that lies south of a critical latitude $y^*$, i.e.,

![Figure 8](image_url)

**Fig. 8.** Partitioning of terms in the barotropic vorticity budget in the DOUBLEGYRE configuration over a range of perturbation experiments with varying continental shelf depths ($H_{\text{shelf}}$), averaged over 10 years in statistically steady state. All terms are integrated over the area of (a) the northern gyre ($-2000 \text{ km} < y < 0 \text{ km}$) and (b) the southern gyre ($0 \text{ km} < y < 2000 \text{ km}$), and normalized by the corresponding wind stress curl integrated over the same areas, respectively. The reference experiment introduced in section 2 corresponds to $H_{\text{shelf}} = 100$ m.
\[
\int_{y=y_{\text{north}}}^{y=y_{\text{south}}} \mathbf{BPT} \, dA = \int_{y=y_{\text{north}}}^{y=y_{\text{south}}} \frac{1}{\rho_0} \frac{\partial \tau}{\partial y} \, dA = \frac{L_0}{\rho_0} \left[ \tau(y^*) - \tau(y_{\text{south}}) \right].
\]

(14)

Here \(y_{\text{north}}\) and \(y_{\text{south}}\) denote the northern and southern edges of the gyre, i.e., between which the wind stress curl is single-signed (e.g., the red or blue regions in Fig. 3). The critical latitude \(y^*\) is defined the northernmost latitude for...
which \( f/H \) contours connect the gyre interior to the western boundary,

\[
f(y^*, H_{pyc}) = \frac{f(y_{south}, H_{shelf})}{H_{shelf}},
\]

where \( f \) is the Coriolis parameter. Note that in this theory \( y^* \), and thus also the total bottom pressure torque, are strictly functions of \( H_{shelf}/H_{pyc} \) by construction.

In Fig. 10 we plot the diagnosed bottom pressure torque, normalized by the wind stress curl, against the normalized shelf depth, \( H_{shelf}/H_{pyc} \), across our suite of perturbation experiments. The theoretical prediction, (14) and (15), approximately captures the dependence of the bottom pressure torque on the normalized shelf depth. Though the theory fails to capture the \~30\% variations in the bottom pressure torque due to changes in the wind stress, drag coefficient, slope width and stratification, Fig. 10 shows that even these perturbation results remain close to the theoretically predicted curves. We also note that the theory overpredicts the bottom pressure torque in the northern gyre for small normalized shelf depths. This is somewhat surprising because one might expect the theory to hold more accurately in the northern gyre than the southern gyre. Specifically, the northern gyre exhibits a southward tail at the western boundary (see Fig. 2; see also Kubokawa and McWilliams 1996) that might be expected to contribute to the area-integrated bottom pressure torque over the southern gyre’s latitude band. Despite this discrepancy, Fig. 10 suggests that our theory largely captures the dynamics of our numerical simulations, and thus yields insight into the development of bottom pressure torques over the western boundary continental shelf and slope.

5. Discussion and conclusions

The overarching motivation of this study is to clarify the role played by bottom pressure torque in balancing the vorticity budget of wind-driven gyres. We seek to provide this clarification with the aid of idealized two-layer isopycnal model simulations, selected because they represent the key process with minimum complexity, and thus offer insights more readily than comprehensive global or regional ocean models.

In section 1 we note that the bottom pressure torque must integrate to zero within any closed isobath or ocean basin (2). This constraint raises some ambiguity regarding the vorticity balance: what definitions of the gyre should yield an area-integrated balance between bottom pressure torque and the wind stress curl? Integrating the time-averaged vorticity budget across various coordinate systems in NORTHGYRE and SOUTHGYRE experiments (see Fig. 4 and section 3b), we find that bottom pressure torque approximately balances the wind stress curl only when integrated over latitude bands (Fig. 6), consistent with the findings of Hughes and de Cuevas (2001). This implies that although the gyre transport is constrained by Sverdrup balance in the gyre interior, the formulation of frictional stresses should also play a role in structuring the gyre circulation, particularly in the western boundary current. This is consistent with the role of friction (in addition to eddy potential vorticity fluxes) in permitting recirculatory flow across mean potential vorticity contours (Jackson et al. 2006), and may be of relevance to model representations of western boundary current flow and separation (Schoonover et al. 2016).

Equation (2) also implies that any single-signed bottom pressure torque generated to balance a wind stress curl must be compensated nonlocally to satisfy (2). Our NORTHGYRE...
and SOUTHGYRE experiments develop bottom pressure torques that balance the wind stress curl within the latitude band of the wind forcing, and compensating bottom pressure torques elsewhere along the continental shelf break (Fig. 5). For simple geometries the bottom pressure torque not only integrates exactly to zero over the area within closed isobaths [Eq. (2)], but should also approximately vanish in a line integral along closed isobaths [Eq. (13)]. In such geometries the nonlocal impact of bottom pressure torques can be interpreted in terms of along-isobath pressure gradients, which necessarily integrate to zero around a closed isobath. Our interpretation is therefore that wind-driven gyres generate bottom pressure torques/along-slope pressure gradients at the western boundary within the latitude band of the wind forcing, and that compensating, remote along-slope pressure gradients necessarily form as a result. These pressure gradients are associated with along-slope flows that are relatively weak compared to the wind-driven gyre circulation (see Fig. 4). The remotely generated bottom pressure torques are balanced by frictional torques, marking a transition to a balance between wind stress and bottom friction around the coast (see Hughes and de Cuevas 2001).

Finally, in section 4 we further probe the mechanics of bottom pressure torque generation by performing experiments spanning the transition from topographically balanced gyres to frictionally and advectively balanced gyres. In our numerical experiments, frictional boundary currents and frictional torques tend to develop while the shelf remains shallower than the pycnocline, whereas viscous and advective torques tend to develop when the shelf becomes deeper than the pycnocline (Fig. 8). In appendix C we pose a theory to explain the variable importance of bottom pressure torque: deepening the continental shelf, or shallowing the pycnocline, allows geostrophic streamlines from the gyre interior to propagate along planetary potential vorticity contours and reach the western boundary. Where this occurs, bottom friction, viscosity and/or nonlinear vorticity advection must become important close to the western wall, reducing the net contribution of bottom pressure torque to the gyre-integrated vorticity budget. Consequently, the fraction of the wind stress curl that is balanced by bottom pressure torque is largely a function of the ratio of the shelf depth to the pycnocline depth (Fig. 10).

While our results offer insight into the role of bottom pressure torques in gyre circulations, they carry a number of caveats. The simplicity of our model facilitates interpretation, but also produces results that conflict with diagnostics from regional models of the North Atlantic: Schoonover et al. (2016) found a leading balance between wind stress curl and bottom pressure torque by integrating within appropriately chosen barotropic streamlines. Additional experiments that bridge the gap in model complexity may be required to reconcile these findings, for example by including a more complete vertical discretization of the flow, irregular coastlines and seaamounts, surface and bottom mixed layer processes, and a buoyancy-driven component of the circulation (Luyten and Stommel 1986). The model’s fully enclosed basin geometry is also rather artificial: all ocean basins are connected together globally, and an isolated basin in the Northern Hemisphere allows for the development of circum-basin “free mode” circulations (Read et al. 1986). We experimented with extended domains that crossed the equator, and which included a circumpolar channel, but did not find qualitatively different results and so favored the simplicity of the model geometry shown in Fig. 1.

Caution is required in extrapolating our sensitivity experiments to the ocean: a tempting inference from Fig. 10 is that shallowing the global stratification in a warmer climate (decreasing pycnocline depth) would weaken bottom pressure torques and favor frictional torques in gyre-scale vorticity balances. While this is implied by the vorticity budget as represented in our idealized model domain, the result would change if we included delta functions of bottom pressure torque and other vorticity budget terms around the domain’s vertical sidewalls (see Hughes and de Cuevas 2001; Wise et al. 2018). If the full transition to the coast could be represented in the model then the vorticity budget integrated between the bounding latitude bands of each gyre (section 4) would coincide with the difference between the zonal momentum budgets across those latitude bands (see appendix D), which consistently comprises a balance between the zonal wind stress and pressure gradient force. Similarly, in our theoretical framework (appendix C), a vanishing ocean depth at the coast \((h = 0 \text{ at } x = x_{\text{west}})\) implies that no planetary potential vorticity contours connect the western boundary to the gyre interior, implying that bottom pressure torque exactly balances the total gyre-integrated wind stress curl. Therefore the transition of the vorticity budget from a wind stress curl/bottom pressure torque balance to a Stommel (1948)-like balance to an advective/Munk (1950)-like balance in Fig. 8 may be interpreted as a shift of the bottom pressure torque from the resolved continental shelf and slope to a “coastal” bottom pressure torque (Wise et al. 2018). There are regions in the high latitudes where coastal “walls” do exist in the form of ice shelf faces. However, the applicability of our results to the subpolar and polar regions is unclear, because the weak stratification may allow the wind stress curl to be balanced by friction or bottom pressure torques in the gyre interior (Su et al. 2014; Le Corre et al. 2020).

The vorticity balance of wind-driven gyres has been contrasted with that of the Antarctic Circumpolar Current (Jackson et al. 2006), where zonal boundaries are entirely absent and the vorticity balance is again primarily between the wind stress curl and bottom pressure torque (Thompson and Naveira Garabato 2014). This vorticity balance is implicit in the long-established momentum balance between zonal wind stress and topographic form stress (Munk and Palmén 1951; Tréguier and McWilliams 1990; Masich et al. 2015; Stewart and Hogg 2017). The vanishing of the integrated bottom pressure torque over areas enclosed by isobaths [Eq. (2)] therefore also implies that topographic form stress cannot balance the wind stress acting along circumpolar isobaths, which occur close to and over the Antarctic continental slope (Stewart et al. 2019).

The clarification of the role of bottom pressure torque offered in this study still leaves much to be understood. Bottom pressure torque remains theoretically elusive because it is closely linked to the pressure field, which is a globally
determined dynamical variable in quasigeostrophic dynamics. A necessary step is to establish direct relationships between the structure of the flow and the bottom pressure torque (e.g., Eden and Olbers 2010; Greatbatch and Hughes 2011). Insights might be gained by clarifying the link between bottom pressure torque and the generation of horizontal vorticity (Molemaker et al. 2015; Gula et al. 2015, 2016), or equivalently with the potential vorticity budget (Jackson et al. 2006). Given the issues associated with inferring bottom pressure torques from measurements of bottom pressure (Hughes et al. 2018), it is likely that further progress toward these goals must be made with the aid of process-oriented simulations that can capture both the large-scale induction of pressure torques from measurements of bottom pressure and the dynamics of the ocean’s bottom boundary layer.

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Data availability statement. The AWSIM model code used in this study can be obtained from https://github.com/andystew7583/AWSIM.

APPENDIX A

Model Bathymetry

The shape of the continental shelf and slope is defined as a function of a slope-normal coordinate s, following (Stewart and Thompson 2015; Stewart 2019):

\[
\eta_{\text{slope}}(s) = \frac{H + H_{\text{shelf}}}{2} + \frac{H - H_{\text{shelf}}}{4(1 + 4\gamma_b)^{1/2}} \times \left[ \sqrt{(1 - \delta)^2 + 4\gamma_b \delta^2} - \sqrt{(1 + \delta)^2 + 4\gamma_b \delta^2} \right]. \tag{A1}
\]

Here \( \delta = (s - s_{\text{mid}})/(1/2)W_{\text{slope}} \) is the normalized cross-slope coordinate, \( s_{\text{mid}} \) is the location of the center of the slope, and \( W_{\text{slope}} \) the width of the slope. The bathymetric structure is difficult to infer directly from \( \text{(A1)} \), and so is plotted in Fig. 1c. This structure has the properties that the bathymetry is symmetric about the middle of the slope \( \eta_{\text{slope}}(0) = -(H + H_{\text{shelf}})/2 \), and that the bathymetry asymptotes to the nominal depths of the basin and the continental shelf \( \eta_{\text{slope}} \to -H_{\text{shelf}} \) as \( s \to -\infty \) and \( \eta_{\text{slope}} \to H \) as \( s \to +\infty \). Additionally, the slope steepness is approximately equal to the change in depth divided by the nominal slope width \( W_{\text{slope}} \). \( d\eta_{\text{slope}}(s)/ds \) is \( H_{\text{shelf}}/W_{\text{slope}} \) for \( \gamma_b \ll 1 \). In all of our simulations we set the parameter \( \gamma_b \) to 0.004, for which the bathymetry closely approximates a linear slope between the continental shelf and the basin. The slope center \( s_{\text{mid}} \) lies parallel to and \( D_{\text{slope}} = 250 \) km from each of the domain boundaries. The slopes along the domain boundaries are joined by quarter-circular segments in the domain corners. We select a radius of curvature of these corners, \( R_{\text{slope}} = 500 \) km, that is much larger than the slope width to ensure similar Rossby wave propagation (Stewart et al. 2011, 2014) and unstable linear growth (Solodoch et al. 2016; Wang and Stewart 2020) as in the straight sections of the continental slope.

APPENDIX B

Northern Gyre Diagnostics

In this appendix we present vorticity budget diagnostics drawn from the northern gyre in our NORTHGYRE experiment and in our parameter sensitivity experiments. Figures B1–B3 are identical to Figs. 5, 6, and 9, respectively, in the main text.

APPENDIX C

Theoretical Prediction of Western Boundary Bottom Pressure Torque

In this appendix we present a detailed derivation of the theoretical prediction for the bottom pressure torque in our numerical experiments, used in section 4. Our formulation of the dynamical equations follows that of Kubokawa and McWilliams (1996) and Wise et al. (2018), to which the reader is referred for additional detail. We use these dynamical equations to obtain an explicit solution for the gyre pressure field, and then evaluate the bottom pressure torque induced by this pressure field.

We approximate the dynamics of our baroclinic wind-driven gyres using as a single isopycnal layer above the main pycnocline, as sketched in Fig. C1. Flow below the pycnocline is assumed to be negligibly small due to the great depth of the fluid there, i.e., \( \nabla(\pi + g'\eta) \approx 0 \), so the dynamics are governed by

\[
\frac{\partial}{\partial t}(hu) + \nabla \cdot (huu) + f \times hu + h\nabla\pi = \frac{\tau_b}{\rho_0} - \frac{\tau_b}{\rho_0} + \nabla \cdot \nabla \phi. \tag{C1}
\]

and

\[
\frac{\partial h}{\partial t} + \nabla \cdot (hu) = 0. \tag{C2}
\]

These equations follow directly from (3) and (4). We use \( \phi \) to denote viscous terms in (C1) and subsequent manipulations thereof. The layer only feels frictional drag where it is in contact with the sea floor, so we write the drag term in (C1) as

\[
\tau_b = \begin{cases} 
-C_s |u|, & x < x_{\text{incrop}}, \\
0, & x > x_{\text{incrop}}.
\end{cases} \tag{C3}
\]

We now seek a steady solution \( (\partial_t = 0) \) of (C1)–(C3) to simplify the form of the solution. An almost identical derivation
follows if we instead take a time average of the equations, as in the main body of this paper. We assume that the pycnocline is perfectly flat (Fig. C1a), corresponding to the limit of strong stratification, such that the upper-layer thickness is given by

\[
\begin{align*}
    h &= \begin{cases} 
        -\eta_p(x), & x \leq x_{\text{incrop}} \\
        H_{\text{pyc}}, & x \geq x_{\text{incrop}}
    \end{cases} 
\end{align*}
\] (C4)

Here we have additionally assumed that the bathymetry \( z = \eta_p(x) \) varies only in \( x \), which is true in our simulations over the
latitude ranges of both the northern and southern gyres, and we neglect the contribution from the eastern boundary slope (Wise et al. 2018).

With these assumptions, (C1) and (C2) may be written as
\[
\frac{\partial}{\partial \tau} \left[ \frac{h u}{f} \right] + \frac{\partial}{\partial \varphi} \left( \frac{h \varphi}{f} \right) = \frac{1}{\rho_0} \left( \frac{\tau}{f} \right) - \nabla \times \frac{\tau_b}{\rho_0 f} + \nabla \times A,
\]
and
\[
\nabla \cdot (h u) = 0,
\]
where we have divided (C5) by \( f \). We use \( A \) to denote advective terms in (C5) and subsequent manipulations thereof. Taking the curl of (C5), we obtain
\[
J \left( \frac{h}{f}, \varphi \right) = -\frac{1}{\rho_0} \left( \frac{\tau}{f} \right) - \nabla \times \frac{\tau_b}{\rho_0 f} + \nabla \times A,
\]
where we have used (C6) to eliminate the curl of the first term in (C5).

In the basin \( (x > x_{\text{incrop}}) \), (C3) implies that the drag curl term in (C7) is zero. We further assume that the Rossby number is very small and the Reynolds number is very large, and thus neglect the nonlinear advection terms \( A \) and the viscous terms \( \nabla \) (e.g., Pedlosky 2013). With these approximations, (C7) reduces to
\[
H_{\text{ave}} J \left( f^{-1}, \varphi \right) = -\frac{1}{\rho_0} \left( \frac{\tau}{f} \right) \varphi.
\]
(C8) may be integrated with respect to \( x \) to obtain the standard Sverdrup (1947) solution
\[
\pi(x_{\text{incrop}}, y) = \pi_{\text{east}} + \frac{f L_x}{H_{\text{ave}}} \beta_b \rho_0.
\]
(C9)

Here \( \pi_{\text{east}} \) is the pressure at the eastern boundary, which must be a constant to ensure zero geostrophic flow through the eastern boundary. We have further assumed that the western continental slope is much narrower than the basin, approximating the width of the basin as \( L_x \).

Over the continental shelf and slope \((x < x_{\text{incrop}})\), we assume that the viscous, nonlinear advection, drag curl and wind stress curl terms in (C7) are all negligible. This assumption is justified in the asymptotic limit of a narrow western slope (see Kubokawa and McWilliams 1996). Thus, (C7) reduces to
\[
J \left( \frac{h}{f}, \varphi \right) = 0 \Rightarrow \beta \pi_x - \beta \pi_y = 0,
\]
where
\[
\beta_b = \frac{f h}{h}
\]
is the “topographic” beta parameter (e.g., Rhines and Bretherton 1973). Equation (C10) states that \( \pi \) is invariant along characteristics that follow planetary potential vorticity contours, i.e., \( f \pi / h = \text{constant} \),
\[
\pi(x, y) = \pi[x_{\text{incrop}}, \gamma_0(x, y)].
\]
(C12)
Here \( y_0(x, y) \) is the latitude at which the \( f/h \) contour intersecting \((x, y)\) meets the incrop longitude, and is implicitly defined by

\[
\frac{f(y)}{h(x)} = \frac{f(y_0)}{H_{pyc}}. \tag{C13}
\]

An implication of this solution is that (C10) cannot hold everywhere in \( x_{west} < x < x_{incrop} \), as this would imply geostrophic wall-normal flow where \( f/h \) contours connect \( x = x_{west} \) with \( x = x_{incrop} \). Instead, viscous, nonlinear or drag effects must become important in a boundary layer at the western wall. However, we will not explicitly consider the form of this boundary layer, instead assuming that (C10) holds for \( x_{west} < x < x_{incrop} \) to obtain a theoretical estimate of the bottom pressure torque in this region.

We now consider the area-integrated bottom pressure torque acting within the latitude range of an isolated gyre. Specifically, we assume that \( \tau_y \) is single-signed and nonzero for \( y_{south} < y < y_{north} \), and zero for \( y > y_{north} \) and \( y < y_{south} \) (see Fig. C1b). The area-integrated bottom pressure torque is

\[
\int [dA \text{BPT}] = \int_{x_{west}}^{x_{incrop}} \int_{y_{south}}^{y_{north}} dy J(\pi, h) = -\int_{x_{west}}^{x_{incrop}} dx \int_{y_{south}}^{y_{north}} dy \pi_y h_y = \int_{x_{west}}^{x_{incrop}} dx [\pi(x, y_{south}) - \pi(x, y_{north})] h_y. \tag{C14}
\]
We now use (C12) and (C9) to evaluate the pressure terms in (C14):

$$\pi(x, y_{\text{south}}) - \pi(x, y_{\text{north}}) = \pi[x_{\text{incrop}}, y_0(x, y_{\text{south}})] - \pi[x_{\text{incrop}}, y_0(x, y_{\text{north}})] = \frac{L_x}{H_p \rho_0} \int_{y_0(x, y_{\text{south}})}^{y_0(x, y_{\text{north}})} \tau(Y) dY.$$  \hspace{1cm} (C15)

Here we have used the fact that \( y_0(x, y) > y \) in the Northern Hemisphere, provided that \( h_x > 0 \) everywhere. This ensures that \( \pi[x_{\text{incrop}}, y_0(x, y_{\text{north}})] = \pi_{\text{east}} \) for all \( x_{\text{west}} \leq x \leq x_{\text{incrop}} \), as we have assumed that \( \tau_y = 0 \) for \( y \geq y_{\text{north}} \).

To evaluate the integral with respect to \( x \) in (C14), we introduce the shorthand \( Y(x) = y_0(x, y_{\text{south}}) \) and use the transformation

$$\tau_{Y}(Y(x)) = \frac{d}{dY} \tau(Y(x)) = \frac{d}{dY} \tau(Y) \left(\frac{dY}{dx}\right)^{-1}.$$  \hspace{1cm} (C16)

To evaluate the coordinate transform term \( dY/dx \) we write

$$\int \int dA \text{BPT} = - \oint_{y_{\text{south}}}^{y_{\text{north}}} dx \int_{y_{\text{south}}}^{y(x)} dy \left( \frac{\tau}{\rho_0} \right) - \frac{\beta L_x}{\rho_0} \int_{y_{\text{south}}}^{y(x)} \frac{\tau(Y)}{\beta} dY.$$  \hspace{1cm} (C17)

$$Y(x) = y_0(x, y_{\text{south}}) = y_{\text{south}} + \int_{x_{\text{west}}}^{x} dx \frac{dy}{dx} \frac{\beta(x)}{\beta} = y_{\text{south}} - \int_{x_{\text{west}}}^{x} dx \frac{\beta(x)}{\beta}.$$  \hspace{1cm} (C18)

Finally, substituting (C13), (C15), and (C18) into (C14), we obtain after some manipulations

$$\int \int dA \text{BPT} = - \oint_{y_{\text{south}}}^{y_{\text{north}}} dx \int_{y_{\text{south}}}^{y(x)} dy \left( \frac{\tau}{\rho_0} \right) - \frac{\beta L_x}{\rho_0} \int_{y_{\text{south}}}^{y(x)} \frac{\tau(Y)}{\beta} dY.$$  \hspace{1cm} (C19)

Here \( y^* = Y(x_{\text{west}}) = y_0(x_{\text{west}}, y_{\text{south}}) \) is the critical latitude, defined in section 4. Thus, the area-integrated bottom pressure torque exactly opposes the wind stress curl integrated over the range of latitudes \( (y_{\text{south}} < y < y^*) \). In this region \( \beta h \) contours originating from the basin at \( x = x_{\text{incrop}} \) reach the southern limit of the gyre’s latitude band \( (y = y_{\text{south}}) \) before they reach the western wall \( (x = x_{\text{west}}) \), as illustrated in Fig. C1b.

The assumption that wind stress curl dominates the planetary vorticity gradient \( (\tau/\beta \gg \beta/f) \) may be relaxed in (C9). Following similar manipulations as discussed above, this yields to a modification of the area-integrated BPT

$$\int \int dA \text{BPT} = - \oint_{y_{\text{south}}}^{y_{\text{north}}} dx \int_{y_{\text{south}}}^{y(x)} dy \left( \frac{\tau}{\rho_0} \right) - \frac{\beta L_x}{\rho_0} \int_{y_{\text{south}}}^{y(x)} \frac{\tau(Y)}{\beta} dY.$$  \hspace{1cm} (C20)

This additional contribution slightly increases the theoretical magnitude of the bottom pressure torque in both the northern and southern gyres. However, the result is almost indistinguishable from using (C19), and so we have not included it in Fig. 10.

APPENDIX D

Relationship Between Integrated Vorticity and Momentum Balances

This appendix addresses the relationship between the area-integrated vorticity balance and the zonal momentum balance, in the presence or absence of vertical sidewalls around the model domain. Hughes and de Cuevas (2001) derive this relationship formally for the case of vanishing ocean depth around the domain boundaries, but only qualitatively discuss the implications for the case of a vertical sidewall. Our aim here is to clarify the latter case, which pertains to the present study.

For simplicity we consider a barotropic fluid in an ocean of variable depth. The equations of motion are identical to (C1) and (C6), though for mathematical convenience we rewrite (C1) in the form

$$\frac{\partial}{\partial t}(hu) + h^2 q \hat{\chi} \times \mathbf{u} + h \nabla \left( \frac{\tau}{\rho_0} \right) = \frac{\tau}{\rho_0} - \frac{\tau_h}{\rho_0},$$  \hspace{1cm} (D1)

where \( q = (\nabla \times \mathbf{u} + f)/h \) is the potential vorticity. We neglect viscous contributions to the momentum equation to simplify
the following discussion. Similar to section 3, we consider the area-integrated curl of (D1) over the area $A$, shown in Fig. D1a:

\[
\iint_A \left\{ \nabla \times (hu) + \nabla \cdot (h^2 qu) + J \left( h, \pi + \frac{1}{2} \bar{u}^2 \right) - \frac{\nabla \times \tau}{\rho_0} + \frac{\nabla \times \tau_b}{\rho_b} \right\} dA = 0.
\]  

(D2)

As discussed in section 3a, (D2) is equal to contour integral of the components of the terms in (D1) tangential to the bounding curve $C$ (see Fig. D1a), i.e.,

\[
\oint_C \left\{ \frac{\partial}{\partial t} (hu) + h^2 q \bar{z} \times u + h \nabla \left( \pi + \frac{1}{2} \bar{u}^2 \right) - \frac{\tau}{\rho_0} + \frac{\tau_b}{\rho_b} \right\} \cdot ds = 0.
\]  

(D3)

If the ocean depth vanishes around the domain boundaries, $h = 0$ on $\partial D$, then it follows that the left-hand side of (D1) vanishes on $\partial D$, and thus $\tau = \tau_b$ on $\partial D$. The terms in (D3) therefore vanish around the portion of the curve $C$ that coincides with $\partial D$, and thus (D3) reduces to

FIG. C1. Schematic of our theory for western boundary bottom pressure torque. (a) Depth/longitude view showing the bathymetry, the pycnocline (assumed to be perfectly flat), and the isopycnal incrop $x = x_{\text{incrop}}$. (b) Latitude/longitude view showing the northern and southern limits of the nonzero wind stress curl, $y_{\text{north}}$ and $y_{\text{south}}$, and the shape of a planetary potential vorticity contour emanating from the southwestern corner of the gyre’s latitude band. Our theory predicts that wind stress curl over the red shaded region should be balanced by bottom pressure torque. The model parameters used to construct these panels are $H_{\text{shelf}} = 350 \text{ m}$, $H_{\text{pyc}} = 500 \text{ m}$, $W_{\text{slope}} = 150 \text{ km}$, $y_{\text{north}} = 0$, and $y_{\text{south}} = -2000 \text{ km}$. 

FIG. D1. (a) Schematic of the circulation integral around a closed contour that follows a line of constant latitude, $y = y_0$, and then traverses the domain boundary. Black contours indicate isobaths, and match those shown in Fig. 1b. (b) Schematic of the delta function contributions to the barotropic vorticity budget that may be included in an area integral that encapsulates the vertical side-walls. The curl of terms integrated across the hypothetical gray shaded area does not vanish in the limit of a vertical sidewall ($e \to 0$).
\[
\int_{y=y_0} dx \left\{ (hu)_y - \nabla^2 qv + h \left( \pi + \frac{1}{2} u^2 \right)_x - \frac{\tau^{(x)}}{\rho_0} + \frac{\tau^{(y)}}{\rho_0} \right\} = 0.
\]

(D4)

This is simply the zonally integrated zonal momentum budget. However, if \( h \neq 0 \) on \( \partial D \), as in the model experiments described in this study, then (D2) no longer reduces to (D4). Thus, the area-integrated vorticity budget over \( A \) and the zonal momentum budget along \( y = y_0 \) are not equivalent in the presence of vertical sidewalls.

\[
\lim_{\varepsilon \to 0} \int^{x_{\text{west}}}_{x_{\text{west}} - \varepsilon} \left\{ \nabla \times (hu)_y + \nabla \cdot (h^2 qu) + \int \left( h, \pi + \frac{1}{2} u^2 \right) - \nabla \times \tau \frac{\tau_x}{\rho_0} + \frac{\tau_y}{\rho_0} \right\} dx = 0.
\]

(D5)

Here we have used the condition of zero normal flow on \( \partial D \), and we have again used \( \tau \to \tau_0 \) as \( h \to 0 \) from (D1). Equation (D5) can alternatively be derived by integrating the barotropic vorticity budget over an infinitesimal area bounded zonally by \( x_{\text{west}} - \varepsilon \) and \( x_{\text{west}} \), and bounded meridionally by \( y \) and \( y + \delta \).

\[
\int_{\partial A} \left\{ \nabla \times (hu)_y + \nabla \cdot (h^2 qu) + \int \left( h, \pi + \frac{1}{2} u^2 \right) - \nabla \times \tau \frac{\tau_x}{\rho_0} + \frac{\tau_y}{\rho_0} \right\} dA = \int_{\partial A} \left\{ \nabla \times (hu)_y + \nabla \cdot (h^2 qu) + \int \left( h, \pi + \frac{1}{2} u^2 \right) - \nabla \times \tau \frac{\tau_x}{\rho_0} + \frac{\tau_y}{\rho_0} \right\} dA = \int_{y=y_0} dx \left\{ (hu)_y - \nabla^2 qv + h \left( \pi + \frac{1}{2} u^2 \right)_x - \frac{\tau^{(x)}}{\rho_0} + \frac{\tau^{(y)}}{\rho_0} \right\} = 0.
\]

(D6)

Here we introduce the notation \( A^+ \) and \( A^- \) to distinguish between area integrals that do and do not include the delta function contributions from the vertical sidewalls, respectively. We denote the set of points that comprises \( \partial D \) as \( x_{\text{in}} \), and \( \hat{s} \) denotes a unit vector directed counterclockwise around \( \partial D \). The second equality follows from applying the Stokes theorem, as in the derivation of (D3) from (D2), and applying the no-flux boundary condition on \( \partial D \). Thus, when extended to include the sidewall delta function contributions, the area-integrated vorticity budget is exactly equal to the zonally integrated zonal momentum budget along \( y = y_0 \).

REFERENCES


