Response of an Equatorial Ocean to a Periodic Monsoon

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ABSTRACT

An infinitely deep stratified ocean on an equatorial beta plane is forced with a periodic wind system. The resulting linearized motion is shown to result in a deep cellular flow structure in rough agreement with recent observations. Because of the infinite depth, the vertical structure is dependent only on the horizontal structure and frequency of the wind-forced layer. The motion is a mechanism for carrying momentum downward from the surface. A western boundary is easily accommodated.

1. Introduction

It has become increasingly clear that equatorial region flows in the ocean exhibit a more complex and variable structure than is accounted for by existing models. Most of these models have concentrated upon the steady physics of the equatorial undercurrent (see Gill, 1975; Philander, 1973). For the most part, observations show that associated with the eastward moving undercurrents is a westward flow beneath, though the data are sparse and contradictory. Data of Taft et al. (1974) and Kort et al. (1966) suggest that in the Pacific the undercurrent might only be the upper cell of a multiple-cellular structure. But the most striking evidence that the equatorial circulation is much more complex than simply an undercurrent overlying ocean at rest beneath comes from the recent work of Luyten and Swallow (1976). Using a continuous acoustic profiling instrument in the equatorial Indian Ocean, they found a multiplicity of zonal currents as a function of depth. The bulk of the structure was confined to within about 1.5° of the equator, and it was difficult to assert that some particular portion of the flow was uniquely definable as the undercurrent.

It is the purpose of this paper to describe a very simplified time-dependent dynamical model which leads to a multiple-cellular structure confined to the equatorial region. The model cannot be defended in detail—it is linearized in a nonlinear flow regime, but there is a sufficient resemblance between model and some observations that one might suppose much of the linear physics persists into the fully nonlinear regime. In any event, one obtains some insight into the possibility of models which yield a considerable structure at depth, unlike the existing steady undercurrent theories.

The dominant feature of the oceanography and meteorology of the western Indian Ocean is the striking change that occurs with the annual cycle of the monsoon winds. Lighthill (1969), in an important paper, modeled the response of the Indian Ocean by postulating an abrupt onset of the southwest monsoon wind over the central portions of the ocean. He then was able to calculate the transient response of the ocean, leading in time to the formation of a western boundary current. Most discussions of this problem have proceeded along the lines outlined by Lighthill (see Moore and Philander, 1976).

But first hand, if casual, observation suggests that over parts of the Indian Ocean at least, the onset of the southwest monsoon is far from abrupt, occurring not over a few days, but rather over weeks to perhaps two months. If the transition and onset is indeed rather slow, occurring over a substantial fraction of the year, then at least part of the temporal spectrum is better described as a quasi-periodic process than as a transient. In fact of course, the spectrum undoubtedly lies somewhere between the two extremes of that of a process characterized by an annual periodicity and one characterized as a transient rich in overtones.

To make the point a little more concrete, consider Fig. 1, which shows eight years of wind record from Gan, Addu Atoll, Maldives Islands (0°18'N, 73°32'E). The raw record was averaged over two-week intervals before plotting. One sees a quasi-periodic character in the wind field, especially the meridional component (Fig. 1a) with a transition occurring over several weeks. Indeed the dominant period visible is the yearly one, though there are some harmonics of the year present. Fig. 1b (the zonal wind) is also roughly describable as periodic, rather than as a series of
“events,” though by contrast the dominant periodicity is semi-annual—a point to which we will have to return [time series of stress, computed by Knox (1976), show the same features]. The spectra of these respective times series (before averaging) are displayed in Figs. 2a and 2b, where it may be seen that each is describable as being dominated by a particular period (annual and semi-annual, respectively) with weaker overtones.

In this paper we take the opposite extreme from the wind model of Lighthill (1969) and study the linear response of a simple model ocean to a sinusoidally varying wind field of annual period (setting aside for now the question of whether a semi-annual period might not be more relevant). No claim of greater accuracy of description of the actual monsoon is made; but the resulting picture has some beguiling features. The choice of annual periodicity is merely to be specific; there is of course a spectrum of wind forcing and, in principle, the model could be applied to any frequency.

2. The models

a. Unbounded ocean

We begin with a resting ocean, buoyancy frequency $N(z)$, conventional coordinates $(x,y,z)$ and corresponding velocities $(u,v,w)$. The sea surface is $z=0$. Lighthill
(1969) took the ocean depth to be of finite depth $D$. The linear equations are separable into vertical and horizontal problems; Lighthill then solved the vertical problem for the normal modes satisfying the two boundary conditions $w=0$ on $z=0$, $-D$, and proceeded to examine the horizontal propagation problem.

Now if the onset of the wind forcing occurs over a few days, the vertical modal structure will be set up rapidly by internal gravity waves (G. Philander, private communication), and the procedure is a sensible one. In the Pacific Ocean over comparatively flat topography Wunsch and Gill (1976) showed that vertical normal modes were apparently set up at periods of 4–6 days. But if the forcing is assumed to be at an annual period, there is no reason to isolate the problem of vertical propagation from that of horizontal. Annual forcing at the sea surface will drive a wave of annual period in the vertical. If the vertical wavelength turns out to be a few hundred meters (and it will), then this is the distance the annual signal will travel in a year (phase and group velocity magnitudes are comparable). It is somewhat implausible that very much energy will travel from surface-to-bottom-to-surface in the presence of dissipation and rough bottom topography so as to set up classical normal modes. Therefore, we will assume at the outset that no energy is reflected from the bottom, and hence for simplicity, assume that the ocean is infinitely deep. As will be seen, the assumption changes the nature of the resulting response. It also has the very useful effect of making the problem nearly identical to the one studied by meteorologists in the tropical stratosphere, and attention is called to the monograph by Holton (1975). [Indeed, this paper was stimulated by reading Holton’s book on board the R/V Atlantis II, while the first continuous velocity profiles on the equator were being obtained by Luyten and Swallow (1976).]

1) DYNAMICS

We will study the response of an ocean to periodic wind forcing on an equatorial beta plane. The linear equations in conventional notation are

$$\partial_t u - \beta v = -\partial_z \rho + \partial_z G,$$
$$\partial_t v + \beta u = -\partial_z \rho + \partial_z H,$$
$$0 = -\partial_z ho - g \rho,$$
$$\partial_t w - (N^2/g)w = 0,$$
$$\partial_t \mu + \partial_z \nu + \partial_z \omega = 0,$$

where $\partial_z G$ and $\partial_z H$ represent the surface wind stress, applied as a near-surface body force, but we will not inquire into the details of how that stress is actually transferred. Lighthill (1969) and most subsequent investigators avoided a detailed discussion of the surface boundary layer by supposing the wind to be applied this way. Moore (1976) has studied the relationship between the time-dependent Ekman layer and the body-force assumption. Details of the time-dependent near-equatorial boundary layer are not very clear (see Kuo, 1975). We will also use this expedient of supposing that there is a well-behaved, equatorial region wind-forced boundary layer of unspecified physics. Its presence will be felt in the interior of the fluid solely through an associated horizontal divergence.

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1 Private communication.
Let the forcing be periodic with frequency $\sigma$. We suppose the buoyancy frequency is $N(\zeta)$, and let $a$ be the radius of the earth.

Using the nondimensional forms
\[
\omega = \alpha / \beta a, \quad (x,y) = (\xi, \eta), \quad z = z = L / \beta a, \quad N = N_0 = (z) \nonumber
\]
\[
\tau = \beta a^2, \quad \left( \partial_t G, \partial_t H \right) = \beta a^2 \phi \phi + \partial_t \phi \phi^* + \beta a^2 \phi \phi^*, \quad (a, b) = \phi \phi^* \phi^* N / \beta a^2 L, \quad \beta = 2 \alpha / \alpha \nonumber
\]
the resulting equation set may be written
\[
\begin{align}
-i \omega t - \eta \tau &= - \partial_t G, \quad \text{(2a)} \\
-i \omega t + \eta \tau &= - \partial_t H, \quad \text{(2b)} \\
2 \eta t &= (\eta / \beta a^2) \partial_t \phi \phi^* + (\beta a^2 \phi \phi^*) / \beta a^2 L \partial_t t, \quad \text{(2c)} \\
\partial_t \phi \phi^* + (\beta a^2 / \beta a^2) \partial_t \phi \phi^* &= 0, \quad \text{(2d)}
\end{align}
\]
where all variables have been supposed proportional to $e^{-\omega t}$ and the asterisks have been dropped.

If a vertical separation of variables were now assumed, the resulting equations governing the horizontal dependence would be solved most conveniently by eliminating in favor of the meridional velocity $\eta$. This leads to the simplest governing equation in the meridional coordinate (see Moore and Philander, 1976). In the present problem, we have found it convenient (because of the upper boundary condition and the simpler orthogonality condition) to work in terms of the pressure. Eqs. (2a)-(2d) may be combined to yield
\[
\nabla^2 \phi + \frac{\omega^2 + \eta^2}{\omega^2 - \eta^2} \partial_t \phi + \frac{2 \eta}{\omega^2 - \eta^2} \partial_t \phi - Q^2 (\omega^2 - \eta^2) \partial_t \phi - \frac{\eta^2}{\omega^2 - \eta^2} \partial_t \phi = 0, \quad \text{(3)}
\]
where
\[
T = \partial_t \left( \phi \phi^* / \omega^2 - \eta^2 \right) + \frac{\eta^2}{\omega^2 - \eta^2} \partial_t \phi + \frac{\eta}{\omega^2 - \eta^2} \partial_t \phi \phi^* + \frac{\eta}{\omega^2 - \eta^2} \partial_t \phi, \quad \text{and} \quad Q^2 = \beta a^2 / \beta a^2 L^2. \nonumber
\]
and $Q^2 = \beta a^2 / \beta a^2 L^2$. We will now make the assumption that
\[
\phi = e^{i t \tau} \sum \sum \mathbb{A}_n (\tau) \mathbb{P}_n (\eta)
\]
and, a crucial simplification, namely, that the forcing $T$ may be written in separable form as
\[
T = e^{i t \tau} \sum \sum \mathbb{A}_n \mathbb{P}_n (\eta) ; \quad (5)
\]
where $\bar{s}$ and $\omega$ are supposed given and fixed.

If (4) and (5) are substituted into (3) and using the orthogonality condition stated below, we find that
\[
\frac{d^2 \mathbb{P}_n}{d \eta^2} + \frac{2 \eta}{\omega^2 - \eta^2} \mathbb{P}_n + \left[ \frac{\omega^2 + \eta^2}{\omega^2 - \eta^2} - \bar{s}^2 + \varepsilon_n (\omega^2 - \eta^2) \right] \mathbb{P}_n = 0, \quad \text{(6)}
\]
where $\varepsilon_n = (2 \omega a^2 / \beta a^2) = \beta a^2 / (\beta a^2)$ is a separation constant. By definition $\bar{s}$ is the equivalent depth.

It is important to notice that the inhomogeneity due to the forcing occurs only in the vertical equation (7) and that (6) is the same homogeneous equation in $\mathbb{P}_n$ that one would obtain in the unforced finite depth models. In the finite depth ocean case, the separation constant $\varepsilon_n$ is determined from the homogeneous form of the vertical equation (7). In this problem it is determined from (6) instead and this makes a great difference. There is no reason in principal why we could not use the full spherical equations, with $\mathbb{P}_n$ being replaced by the Hough functions, except that we will later want to insert a meridional boundary into the problem, and this problem is analytically tractable with the Hough functions (see the related atmospheric tide problem treated by Chapman and Lindzen, 1970).

The solutions to (6) which are equatorially trapped are
\[
\mathbb{P}_n (\eta) = P_n (\varepsilon_n \bar{s}) = \varepsilon_n^{-1} \left[ \left( \frac{1}{2} \right)^{n} \frac{\bar{s} + \omega \varepsilon_n}{\bar{s} + \omega \varepsilon_n} \right] \left[ \left( \frac{1}{2} \right)^{n+1} \frac{\bar{s} + \omega \varepsilon_n}{\bar{s} + \omega \varepsilon_n} \right], \quad \text{(8)}
\]
where
\[
Y_n (\varepsilon_n \bar{s}) = \exp \left( -\frac{i}{2} \varepsilon_n \bar{s} \right) H_n (\varepsilon_n \bar{s}) / \left( 2n+1 \right)^{1/2} \nonumber
\]
are Hermite functions and $n = 0, 1, 2, \ldots, H_n$ are the Hermite polynomials. One must have
\[
\varepsilon_n - \bar{s}^2 - \omega^2 = (2n+1) \varepsilon_n = 0, \quad \bar{s} = \frac{n+1/2}{\omega} \left[ 1 + \sqrt{1 + \frac{\omega^4}{(n+1/2)^4} \left( \frac{\bar{s} + \omega \varepsilon_n}{\bar{s} - \omega \varepsilon_n} \right)^2} \right] \nonumber
\]
and, or
\[
\varepsilon_n = \frac{n+1/2}{\omega} \left[ 1 + \sqrt{1 + \frac{\omega^4}{(n+1/2)^4} \left( \frac{\bar{s} + \omega \varepsilon_n}{\bar{s} - \omega \varepsilon_n} \right)^2} \right] \nonumber
\]
The meridional structure associated with the velocity fields is
\[
V_n (\varepsilon_n \bar{s}) = i Y_n (\varepsilon_n \bar{s} \bar{s}), \quad \text{(9)}
\]
\[
U_n (\varepsilon_n \bar{s}) = \varepsilon_n^{-1} \left[ \left( \frac{1}{2} \right)^{n} \frac{\bar{s} + \omega \varepsilon_n}{\bar{s} + \omega \varepsilon_n} \right] \left[ \left( \frac{1}{2} \right)^{n+1} \frac{\bar{s} + \omega \varepsilon_n}{\bar{s} + \omega \varepsilon_n} \right], \quad \text{(9)}
\]
where
\[
\begin{align}
\mathbb{P}_n &= R_n (\tau) V_n (\varepsilon_j \bar{s}) \exp (i \bar{s} \bar{s} - i \omega \bar{s}), \\
\mathbb{U}_n &= R_n (\tau) U_n (\varepsilon_j \bar{s}) \exp (i \bar{s} \bar{s} - i \omega \bar{s}), \\
\mathbb{V}_n &= \mathbb{P}_n \mathbb{V}_n \exp (i \bar{s} \bar{s} - i \omega \bar{s}),
\end{align}
\]
etc.
In addition, one must append an \( n = -1 \) solution,
\[
\begin{align*}
P_{-1}(\eta) &= P_{-1}(e^{-1} \tilde{\eta}) = e^{-1} \exp(-\frac{3}{4} e^{-1} \eta^2), \\
U_{-1}(\eta) &= \exp(-\frac{3}{4} e^{-1} \eta^2), \\
V_{-1}(\eta) &= 0,
\end{align*}
\]
representing the forced Kelvin wave such that
\[
\epsilon_{-1} = \tilde{\eta} / \omega,
\]
\( \text{[} n = 0 \text{ must also be treated exceptionally in that only one root of (9) may be retained for } U_0 \text{ to be bounded at infinity].} \)

In deriving (6) and (7) we have used the result from standard Sturm-Liouville techniques, that two solutions to (6), \( P_n(\epsilon_n \eta), P_m(\epsilon_m \eta) \) are orthogonal:
\[
\int_{-\infty}^{\infty} P_n(\epsilon_n \eta) P_m(\epsilon_m \eta) d\eta = 0, \quad n \neq m. \tag{11}
\]
No normalization has been imposed. Also note that
\[
\int_{-\infty}^{\infty} (\omega^2 - \eta^2) Y_n(\epsilon_n \eta) Y_n(\epsilon_m \eta) d\eta = 0 \quad \epsilon_n \neq \epsilon_m.
\]

The behavior of these forced solutions (forced in the sense that because the ocean is infinitely deep there are no free waves) has been studied extensively elsewhere (Holton, 1975; Philander, 1976). Therefore, we will not go into their general properties but merely make a few observations relevant to the particular long periods we are examining.

For fixed \( \tilde{\eta}, \omega, \) as \( n \) grows, we have asymptotically that
\[
\epsilon_n \sim \frac{[(\tilde{\eta} / \omega) + \tilde{\eta}^2]}{2n+1} \to 0,
\]
so that the higher order modes are less equatorially confined, and one may do better (Lindzen, 1967) to consider the mid-latitude beta plane to represent them.

Eq. (9) has two roots corresponding to the two signs of the radical and, in principle, we should carry both solutions. However, choice of the positive root corresponds to vertical wavelengths in Eq. (7) that are so small, order tens of meters at most, that we can argue that they will die off rapidly because of friction. The positive roots correspond to forced internal gravity waves, the negative to forced Rossby waves. Internal gravity waves must have exceedingly small vertical wavelengths in order to achieve periods of a year, and we discount them. We will thus keep only the negative roots so that \( \epsilon_n \) implies the negative root has been taken in (9), except for \( n = -1, \) for which (10) applies.

It can be shown (D. Moore, 1976)\(^2\) that \( \epsilon_n \) can never become complex; the equatorial beta plane has no negative depth eigenvalues. However, \( \epsilon_n \) can become negative if \( \tilde{\eta} \) is positive [choosing as we do the negative root in (9)]. Such modes are not equatorially trapped, but grow exponentially toward the poles. They are thus not acceptable solutions in the context of the equatorial beta plane, but can be treated with the mid-latitude beta plane. The resulting solutions correspond to the negative depth eigenvalues there and are trapped near the surface at high latitudes (see Philander, 1976). Such solutions would not be detectable near the equator and we need not consider them. On the other hand, the Kelvin wave root (10) is only acceptable if \( \tilde{\eta} \) is positive. We can thus see that eastward moving wind fields can excite only the Kelvin wave, internal waves of exceedingly high vertical wavenumber, plus waves trapped at high latitudes. Westward moving disturbances can, however, excite the Rossby wave roots of moderate vertical wavenumber and the \( n = 0 \) mode, but not the Kelvin wave. For this reason, we will focus primarily upon the case \( \tilde{\eta} < 0. \)

If, as some data suggest, the monsoon is described as a standing wave system, then we can regard it as made up of equal parts of westward and eastward traveling waves. The response to the two components is then quite different.

With \( \epsilon_n \) fixed by (9), we can proceed to solve Eq. (7). The solution to this equation will have a homogeneous and an inhomogeneous part, which will depend upon the precise form of the nondimensional buoyancy frequency \( \tilde{N}(\tilde{\eta}) \). Both the constant \( N \) and exponential \( \tilde{N} \) cases are relatively straightforward. For pictorial purposes, the exponential \( N \) case resembles the “real” ocean a little more in giving an overall decrease of stability with depth, and we will use it. We thus suppose
\[
\tilde{N} = \tilde{\eta} = e^{\tilde{\eta}/L},
\]
so that the vertical scale parameter \( L \) has been set by the buoyancy frequency profile.

The homogeneous solution to (7) is then
\[
R_n^h(\eta) = C_n e^{b \eta L} / (e^{b \eta L} + \epsilon_n Q), \tag{13}
\]
where we have chosen the Hankel function (called \( Ha \) to avoid confusion with the Hermite polynomials) of the first kind in order to satisfy the radiation condition (Elissen and Palm, 1960). (Introduction of simple Rayleigh friction easily shows that this is the proper choice.) To be specific, we will make the assumption that \( \eta(\tilde{\eta}) \) has the simple form \( \gamma = e^{b \tilde{\eta}} - e^{b \tilde{\eta} L}. \) If \( b \) is sufficiently large, \( b \eta \gg 1, \) then the inhomogeneous solution to (7) is given approximately by
\[
R_n^h(\eta) = -A_n e^{b \eta L} / (b \eta L^2 + \epsilon_n Q). \tag{14}
\]

The exponential \( N(\tilde{\eta}) \) profile leads to one difficulty. As \( \tilde{\eta} \to -\infty, \) dimensional \( N \) eventually becomes smaller than any arbitrarily chosen wave frequency \( \sigma. \) When this occurs the wave motions reflect and one could

\(^2\) Private communication.
set up standing waves in the vertical even in an infinitely deep ocean. A standing wave solution $e^{itJ_1(\epsilon_n e^{it}/Q)}$ could have been chosen instead of (13). We must imagine that at sufficiently great depths the buoyancy frequency eventually becomes constant, permitting wave energy to be transmitted, and that we could match solutions in the regions of different $N$. Actually doing this will not change the solutions well above the transition region in any significant way.

The constants $C_n$ are determined from the boundary condition $w=0$ on $\xi=0$. From Eqs. (2c), (13) and (14) we have

$$\frac{i\omega}{N^2} \sum_n \{ C_n \left[ e^{iH_1(\epsilon_n e^{it}/Q)} + (\epsilon_n e^{it}/Q) \right] ; XH_1(\epsilon_n e^{it}/Q) \} - bLA \epsilon_n e^{it}/(b^2L^2 + \epsilon_n/Q^2) \} |_{\xi=0} = 0.$$  

We now define

$$w_*(\eta, t) = \frac{i\omega}{N^2} \sum_n \frac{bLA \epsilon_n e^{it}}{b^2L^2 + \epsilon_n/Q^2} \tilde{P}_n(\eta).$$  

This vertical velocity represents the convergence and divergence of the wind-driven surface boundary layer. It is only because such a vertical velocity is induced that wave modes are set up. In principle, we could postulate a wind stress field $G_\alpha, H_\alpha$ and deduce $w_*$.
from it directly. As noted above, the dynamics of the time-dependent near-equatorial surface boundary layer are far from clear and unless we invoke some form of lateral friction or other process, the boundary layer exhibits a number of apparent singularities (cf. Kuo, 1975). We find it convenient to specify a simple distribution of divergence, i.e., \( w_n \), setting aside the question of what wind field and what boundary layer dynamics might actually lead to it. The procedure is not completely satisfactory, but it will establish the physics of the interior flow.

We therefore assume the simple, minimally structured, latitudinally symmetric form \( w_n = e^{\beta L z} \omega \) and now can determine the constants \( A_n \). The algebra is tedious, but can be carried out with the repeated help of integrals in Gradshteyn and Ryzhik (1965); it is not reproduced here. We can now see that the actual form \( e^{\beta z} \) of the vertical dependence of the boundary layer is not vital because (15) which determines \( C_n \) is evaluated only at \( z = 0 \). Only the combination of variables \( b L A_n/(\beta^2 L^2 + \epsilon_n/Q^2) \) will occur. Because \( w_n \) is symmetric, only odd values of \( n \) will appear in the solution \( (Y_n \) is even for even \( n \), odd for odd \( n \).

It is now possible to write out all of the fields:

\[
\rho = \exp(i \tilde{\xi} - i \omega t) \sum_{n=1}^{\infty} \frac{A_n}{b^2 L^2 + \epsilon_n/Q^2} \times \frac{b L \text{He}_1(\epsilon_n^2 \omega_i/Q)}{\text{He}_1(\epsilon_n^2/Q) + (\epsilon_n^2/Q) \text{He}_1'(\epsilon_n^2/Q)} P_n(\epsilon_n^2 \eta) \quad (17a)
\]

\[
v = i \exp(i \tilde{\xi} - i \omega t) \sum_{n=1}^{\infty} \frac{A_n}{b^2 L^2 + \epsilon_n/Q^2} \times \frac{b L \text{He}_1(\epsilon_n^2 \omega_i/Q)}{\text{He}_1(\epsilon_n^2/Q) + (\epsilon_n^2/Q) \text{He}_1'(\epsilon_n^2/Q)} Y_n(\epsilon_n^2 \eta) \quad (17b)
\]

\[
u = -i \exp(i \tilde{\xi} - i \omega t) \sum_{n=1}^{\infty} \frac{A_n}{b^2 L^2 + \epsilon_n/Q^2} \times \frac{b L \text{He}_1(\epsilon_n^2 \omega_i/Q)}{\text{He}_1(\epsilon_n^2/Q) + (\epsilon_n^2/Q) \text{He}_1'(\epsilon_n^2/Q)} \left( \frac{(n/2)!Y_{n-1}}{3 + \omega \epsilon_n} \right) \left[ \frac{(n+1/2)!Y_{n+1}}{3 - \omega \epsilon_n} \right] \quad (17c)
\]

\[
w = \frac{i \omega}{N^2} \exp(i \tilde{\xi} - i \omega t) \frac{\partial}{\partial \tau} \rho \quad (17d)
\]

Here we have not included the portion that represents the detailed forced motion in the upper wind-driven layer, i.e., those terms multiplied by \( e^{\beta L z} \). The constants \( A_n \) are now known from (16).

2) RESULTS

In Figs. 3a–3c we display for the first term \( n = 1 \), only, the zonal, meridional and vertical velocity fields. We have chosen \( \delta = 13.2 \), \( \omega = 1.4 \times 10^{-8} \), \( L = 20 \), \( Q = 13.0 \), \( \kappa = 0.5 \). With these values, the figures may
be read as centimeters per second if dimensionally $w_* \sim 10^{-5}$ cm s$^{-1}$. In Fig. 3d the zonal velocity is displayed as though measured with a profiling instru-

ment on the equator. The same fields are in Fig. 4 but with seven terms out to $n=13$. The coefficient of the last term, $C_{13}$, is 10% of that of the dominant $n=1$ term. As expected, because $\epsilon_n$ decreases with increasing $n$, most of the vertical structure occurs in the fundamental meridional mode. The effect of the higher meridional modes is mainly to carry a weak zonal flow to higher latitudes, the motion tending to become less vertically structured as one moves poleward. This tendency is no more than a reflection of the fact that at mid-latitudes only the barotropic and lowest baroclinic modes have periods sufficiently short to be excited by annual forcing. (With the value $\kappa=0.5$ used, $w_*$ has fallen to only 0.995 of its equatorial value at $\eta=0.1$.)

Superficially at least, the vertical structure looks much like that observed by Luyten and Swallow (1976). There are also a number of features that look much like the “undercurrent” itself, notably in Figs. 3d and 4d, where a comparatively strong eastward flow underlies a westward surface flow (ignoring the question of the flow in the directly wind-driven layer which is not shown). As time goes on, the pattern tends to move upward with the phase velocity of the dominant $n=1$ mode, the group velocity being downward. The “undercurrent” thus evolves through the year (see Fig. 5) by “surfacing”—creating an eastward jet—then forming from below once again.
Because the vertical structure is controlled by Eq. (7), the horizontal velocities will tend to be strongest where \( N(z) \) has its maxima. One could solve (7) for an arbitrary \( N(z) \) either numerically or by a WKB approximation; for sufficiently large values of \( \epsilon_n \), the zonal and meridional velocities will tend to have their maxima at corresponding maxima of \( N \). Thus one automatically obtains the result that undercurrent-like features would be found in the thermocline [see the discussion of this point by Gill (1975)]. Munk and Moore (1968) tried to explain the Pacific equatorial undercurrent as rectified equatorial Rossby modes and noted that an appealing feature of their solution was that it puts the undercurrent in the thermocline [the model was subsequently shown to be incorrect; see Munk and Moore (1969)]. Here the Rossby wave is the undercurrent. With \( N(z) \) decreasing with depth, one sees other features deducible from the WKB approximation: an increase in vertical scale, an increase, roughly as \( N^{-1} \), of \( w \), and a decrease as \( N^1 \) of the horizontal velocity field.

Dominance of the \( n = 1 \) mode yields a tendency to a 180° phase reversal in \( u \) across the latitude 1.5°. Luyten (1976, private communication) sees a similar effect in the profiles.

We noticed in the Introduction that the zonal wind field at least at Gan tended to have a dominant semi-annual periodicity. In Fig. 6 we display the time history of the equatorial zonal flow with \( \epsilon_n \) corresponding to a semi-annual period. Notice the twice yearly appearance of an eastward jet as described by Wyrtki (1973) from observation. The full zonal field for the first seven meridional modes \( (n = 1, 3, 5, \ldots) \) at 6-months periodicity (not displayed) shows weak reversals just poleward of the equatorial jet [they appear also, as one would expect, in the numerical simulation by O'Brien and Hurlburt (1974) of a directly wind-forced eastward flow].

We are comparing observations with solutions computed for a specific value of \( \delta \), chosen to correspond to wind zonal wavelengths of the order of the width of the Indian Ocean. But because for the Rossby roots we have

\[
\epsilon_n = -\left[2(n + \frac{1}{2})\right]^{-1} \left[ (\delta/\omega) + \delta \right],
\]

the gross structure of the solutions will be comparatively insensitive to the actual choice of \( \delta \) as long as \( 1/\omega >> |\delta| \) (at the annual period \( \delta/\omega = 0.4 \times 10^9 \)). For example, the value of the latitude of the zero crossing of the zonal velocity for the \( n = 1 \) mode is dependent upon \( \epsilon_{n1} \) and varies only as \( \delta^3 \). On the other hand, if \( \delta \approx 0 \), the model would fail totally to describe the observations, but it seems implausible that the wind scale should much exceed the geo-
Figs. 5a, b. $u(0, 0, t)$ profile of Fig. 4d carried through a half-cycle for first seven odd modes. Notice upward movement of phase.

The dimensional magnitudes of the velocities displayed here are controlled by the amplitude of $w_0$ which, because we have not explicitly calculated the response to wind forcing, is not really known. But the predicted zonal velocities are not very different from those observed if $w_0 \sim 10^{-6}$ cm s$^{-1}$—a reasonable value.
It is clear, however, that these large velocities, that this model fails by any conventional measure of linearity, and it should be taken only as qualitative description of a subset of the full physics that must apply.

b. Western boundary

In the presence of boundaries, the solutions presented above must be modified. If we consider an ocean bounded on the west by a wall at \( \xi = 0 \), the changes required to insure that \( u = 0 \) there are comparatively straightforward. To proceed, we write

\[ u = \sum_{n=-\infty}^{\infty} R_n(\xi)[U_n + U_{n\text{Ref}}]e^{-\omega t}, \]

where \( U_n = e^{i\phi}U_n(\eta) \) is the solution (17c), given above for the unbounded ocean. The vertical structure \( R_n(\xi) \) must remain the same for both \( U_n \) and \( U_{n\text{Ref}} \). Therefore \( \epsilon_n \) must remain the same for both. Because (6) is homogeneous, the form of \( U_{n\text{Ref}} \) may be deducted from the solution given by Moore and Philander (1976) for reflection of a free wave from a western wall. Let

\[ U_{n\text{Ref}} = \sum_{q=1}^{n} C_q \exp(i\epsilon_q \xi) \left( \frac{Y_{q+1}(\epsilon_q \eta)}{s_q + \omega \epsilon_q} - \frac{(q+1)^{1/2}}{2} \right) \]

\[ \times \frac{Y_{q+1}(\epsilon_q \eta)}{s_q - \omega \epsilon_q} - C_{\text{Kelvin}} \exp(i\epsilon_q \eta) \theta(\epsilon_q \eta). \]

Then we have

\[ C_n = \frac{(\omega \epsilon_n^2 + s_q)}{\omega \epsilon_n^2 - s_q}, \]

\[ C_{n-2} = \frac{n}{(n-1)^{1/2}} \left( \frac{1}{\omega \epsilon_n + s_q} \right) \frac{C_n}{\omega \epsilon_n^2 + s_q}, \]

\[ C_{p-2} = \frac{p}{p-1} \left( \frac{1}{\epsilon_p^2 + s_p} \right) C_p, \quad p = n - 3, 1, \]

\[ C_{\text{Kelvin}} = C_1 / (\omega \epsilon_1^2 + s_1), \]

\[ s_q = -2(1/2\omega)^{1/2} [\omega^{-2} + 4\omega^2 \epsilon_q - 4(2q+1)\epsilon_q^2]. \]

The case \( n = 1 \) is an exception to the rules given by Moore and Philander. Then we have

\[ C_1 = \frac{-(\omega \epsilon_1^2 - s_1)}{(\omega \epsilon_1^2 - s_q)}, \]

\[ C_{\text{Kelvin}} = \frac{\sqrt{2} \omega \epsilon_1(s_1 - s_q) C_1}{(\omega \epsilon_1^2 + s_q)(\omega \epsilon_1^2 - s_q)} \]

For display purposes, we will confine ourselves to the case \( n = 1 \). The wave field is then in the form of an incident forced Rossby mode \( (n = 1) \), a reflected Rossby mode of higher wave number (also \( n = 1 \)) carrying energy to the east, plus a Kelvin wave also carrying energy to the east. The coefficients \( A_n \) in (16) need to be recomputed for \( w_n \) in terms of the new expansion functions \( P_n \) corresponding to the new full zonal velocity field, but because we are going to retain only \( n = 1 \), we will refrain from displaying what is only a tedious algebraic calculation.

Contours of the resulting surface zonal flow are shown in Fig. 7, demonstrating the rapidity with which the flow builds away from the western wall. The very small zonal scale comes from the presence of the reflected Rossby wave carrying energy eastward. One would expect that it would be quickly killed by friction and that only the longer westward-energy-carrying Rossby mode and the Kelvin wave would really be visible. Arguments given by Pedlosky (1965) suggest that intensification at the western boundary would result, owing to the preferential degradation of those Rossby modes at high wavenumber carrying energy eastward. The Kelvin wave would be relatively unaffected. This purely physical argument also suggests that a current much like the Somali current would be formed—apparently as a wave propagating from below, with vertical group velocity downward, as in the interior. Leetmaa (1972) noted that the Somali current seemed to respond largely to local winds, though strictly speaking we should increase \( \delta \) to model this process. We will not pursue the problem here, however.

As Moore (1968) notes, the problem of reflection from an eastern wall can also be solved explicitly. But the inability of a Kelvin wave to carry energy westward away from the wall makes the algebra much more complex.

c. Second-order effects

The wave field excited by the forcing carries momentum and energy downward from the surface. With dissipation or critical layers, the waves are capable of generating mean motion at depth. In the presence of a zonal mean flow it may be shown (e.g., Holton, 1975, p. 112) that the eddy acceleration of the zonal mean can be written (in nondimensional form) as

\[ F = -[\partial_x(u''') + Q\partial_t(u''') - \eta \partial_t(u'')/N^2]. \]

To be specific, consider the unbounded ocean in which \( n = 1 \) is the dominant response and \( N = \text{constant} \). Then it follows from (17b) and (17c) that \( u'' = 0 \), and then \( F = -Q\partial_t(u''') \). The second term is the latitudinal heat flux. Substituting the solutions (17), we find \( u'' / \omega < 0 \) and thus the \( n = 1 \) mode carries
eastward momentum downward. We also have
\[ \overline{v' \partial_t \rho'} < 0, \quad |v' \partial_t \rho'| > |u' \omega'|. \]

Thus the meridional heat flux overwhelms the momentum flux, and the net effect is for a forcing of westward flow at depth. If one invokes some dissipa-
tional mechanism, be it friction or critical levels, the surface forcing could accelerate the deep flow toward the west.

Similar computations for the Kelvin wave show a net eastward forcing at depth. Whether the flow at depth would tend to accelerate eastward or westward would depend upon the relative magnitudes of the Kelvin and Rossby modes which would depend in turn upon the relative magnitudes of eastward and westward moving components of the wind field, and finally, on the presence of critical layers in the mean flow. One could imagine an oceanic analog of the quasi-biennial oscillation of the atmosphere, but there are no data to support such a conjecture at this time.

3. Discussion

The model used here clearly yields a deep structured flow—in contrast to Lighthill’s solution which produces only a few zero crossings in the vertical. Without including nonlinear terms it is not possible to examine the behavior of our model in the limit \( \omega = 0 \) so as to study the setup of a mean flow. A closely related problem was, however, examined numerically by Gill (1975). Using the vertical normal mode expansion, he studied the spinup of the undercurrent from rest. He noticed that the formation of the undercurrent was associated with the appearance of a Kelvin wave arriving from the western coast, and his identification of the equatorial current system with the individual modes is very similar to the treatment here. Gill (1975), in the absence of observations, did not discuss the deeper flows.

By confining attention to an infinitely deep ocean, we obtain a simple relationship between the meridional structure of the forcing and the vertical scale of the resulting flow. The solution is thus comparatively insensitive to assumptions about the vertical structure of the wind-forced layer, depending only on its net divergence. The assumption that the ocean should be treated as infinitely deep is defensible for the lower modes, because their vertical wavelengths are quite small. For the higher meridional modes which become increasingly barotropic in nature, the assumption is a poor one. But observation (Luyten, private communication) suggests that these modes, which extend to high latitudes, may not be present. It is possible to construct a model in which the bottom is a partial absorber, resulting in a mixture of standing or traveling waves in the vertical.

The present model has a number of testable consequences. It suggests that unlike mid-latitudes, there should be a strong annual baroclinic signal at great depths owing to the availability of vertically propagating modes. The temporal correlation of \( u' \), \( v' \) should
We have swept under the carpet the question of wind direction and the sense of the resulting subsurface currents. Instead the motion is forced through \( \omega_* \) which results from the function \( T \) [Eq. (5)]. The potentially complicated relationship between \( \omega_* \) and the wind field implies that there need not be a simple relation as in most steady models between (say) eastward flow at depth and the presence of a westward surface stress. Indeed, Taft and Knauss (1967) found an undercurrent-like flow in the eastern Indian Ocean when the winds still had an eastward component.

If one is willing to adopt a more specific ad hoc model of the wind-driven boundary layer and an oversimplified wind field, then a relationship between \( \omega_* \) and the wind field can be deduced. For example, if the boundary layer is treated as quasi-steady, then following Gill (1971) one can remove the equatorial singularity by invoking (very large) lateral eddy friction. One then finds that a symmetric westward wind generates upwelling at depth within a latitude band about the equator, whose width is fractionally controlled, with downwelling beyond. The use of such a \( \omega_* \) would be more mode selective than the form we have adopted above. Indeed, it would tend to excite the \( n=1 \) mode more or less, depending upon the lateral eddy coefficient adopted.

A meridional wind will generate, in Gill's model, an antisymmetric \( \omega_* \) and hence an antisymmetric zonal flow, i.e., the \( n \) even modes. This result is particularly intriguing because the phase between \( \omega_* \) and \( u \) is then such that the “undercurrent” existing from the zonal winds could tend (though weakly) to move upwind with the addition of the antisymmetric components—a result seen in nonlinear steady models of the undercurrent.

The actual wind field over the Indian Ocean is very complicated. As noted already, at Gan the dominant periodicity is 6 months in the zonal wind, one year in the meridional; in the Seychelles further west, it is annual in both. Small spatial scales are clearly involved and, generally speaking, the ocean is subject to a complex frequency-wavenumber forcing spectrum about which little is known. Attempts to describe that spectrum, now underway, along with increased understanding of the surface boundary layer, will be required before one will be able to describe in detail the expected response in the ocean.

As pointed out above, the use of our model is not really permissible at annual and semi-annual periods because it neglects nonlinearity. Rossby numbers of the motion are \( O(1) \) at best, and any other measure such as particle/phase velocity is large. However, analogous atmospheric experience (Holton, 1975) suggests that features of the linear solutions persist well into the nonlinear range, where one had no real reason to expect them to be valid. One can hope that such will also be the case in the ocean. If a single mode

![Fig. 7. Contours of zonal flow at sea surface for incident \( n=1 \) mode with a wall at \( \xi=0 \).](image-url)
tends to be dominant, and there are weak data to support the hypothesis, then single modes of fixed wavenumber are finite-amplitude solutions (for constant \( N \) on a beta-plane) to the equations of motion, the nonlinear terms cancelling.

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