Rainfall Microphysics and Radar Properties: Analysis Methods for Drop Size Spectra

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ABSTRACT
Analyses are performed of experimental drop size spectra to explore the relationships among integral parameters for rain. The data used in this work were acquired with an airborne optical 2D precipitation probe in TOGA COARE during a 4-month period in 1992-93. It is assumed that the experimental size spectra can be described by a gamma drop size distribution (DSD) of the form \( N(D) = N_0 D^m \exp(-L D) \) involving three parameters \( (N_0, m, L) \), which are determined using a new method of truncated moments. The method allows for truncation of the DSD at the large-diameter end of the spectrum due in part to instrumental effects and also in part to the trajectory of the aircraft through a rain streamer that has been sorted by wind shear. An effect analogous to truncation can occur at the small-diameter end of the size spectrum due to evaporation. However, truncation of the spectrum at the small-diameter end is not considered in this work. It is found that spectra with small space and timescales display considerable fluctuations in all three of the DSD parameters. It is also shown that the method of truncated moments yields distributions of the DSD parameters that have smaller average and modal values than when using untruncated moments. The data are stratified separately into classes according to each of the two DSD parameters \( D_m \) (mass-weighted mean diameter) and \( m \). The latter parameter describes the shape of the distribution. Empirical analyses between the reflectivity factor \( Z \) and rainfall rate \( R \) are performed for the data in each of these classes, and it is found that the results are consistent with that predicted by theory. A synthesis of the results of these empirical analyses is presented in the form of a new rain parameter diagram, which allows for changes in DSD shape.

1. Introduction
In this work an extensive set of experimental raindrop size distributions (DSDs) is used to explore the relationships among the DSD parameters and integral parameters derived from them. The data were collected with an airborne 2D precipitation probe on board the NCAR Electra aircraft operating in the Tropical Ocean Global Atmosphere Coupled Ocean–Atmosphere Experiment (TOGA COARE) and all were collected in tropical rainfall systems. A description of the data-processing procedures for these data is given in Yuter and Houze (1997). No attempt is made to classify the data according to rainfall type [as in Tokay and Short (1996)] since the data are used here only to illustrate the effects of truncation on the results found for the gamma distribution parameters. In addition, although the experimental data used here are for tropical rain, the methods are general and apply to DSD data for any type of rain.

A central assumption in the present work is that the DSD can be approximated by a three-parameter gamma distribution of the form (Ulbrich 1983)

\[ N(D) = N_0 D^m \exp(-L D), \]

where \( N(D) \) (m\(^{-3}\) cm\(^{-1}\)) is the number of drops per unit volume per unit size interval, \( D \) (cm) is the drop diameter, and \( N_0 \) (m\(^{-3}\) cm\(^{-1}\)), \( m \), and \( L \) (cm\(^{-1}\)) are the three parameters. Other mathematical forms of the DSD have been employed by others in prior work (e.g., Best 1951; Feingold and Levin 1986), but the advantages of the distribution in Eq. (1) are that with three parameters it is general enough to describe the fluctuations of the DSD observed on small space and timescales and includes the exponential distribution as a special case. It is important that such small-scale variations be accounted for in analysis of the data used in this work since the temporal sampling interval employed is such that average sample volumes are only about 1 m\(^3\). It is notable that Sempere Torres et al. (1994) have shown that all the various forms of the DSD used in earlier work can be written in terms of a general formulation that uses a power-law relation between the raindrop fall speed and diameter. However, the approach emphasized...
in this work involves the method of moments using an excellent approximation to drop fall speeds.

A new method of determining the parameters \( N_s, \mu, \) and \( \Lambda \) (or the median volume diameter \( D_m \)) is described and used here that is a significant modification of the method of moments employed by Tokay and Short (1996). The present method accounts for truncation of the distribution at the large-diameter end of the spectrum due to such effects as instrumental truncation and, possibly, systematic drop size sorting relative to the position of the instrument in the rain shaft. The effects of truncation at the small-diameter end of the spectrum are shown to be negligible for the data used in this work. However, depletion of the numbers of small-diameter drops could occur due to evaporation. These effects are shown to be negligible for the data used in this work.

Almost all other integral parameters of interest can be approximated by a moment of the DSD defined by

\[
 p_m = \int D^n N(D) \, dD = \frac{\Gamma(n + 1 + \mu)N_0}{\Lambda_0^{n+\mu+1}}.
\]

(7)

Of interest in this work are \( p_2, p_4, \) and \( N_0, \) which, in the analyses that follow, will be used to determine the three DSD parameters \( \mu, \Lambda, \) and \( N_0, \) in a method that is known as the method of moments. In the present work it involves forming the ratio

\[
 \eta = \frac{p_2^2}{p_4} = \frac{[\Gamma(5 + \mu)]^2}{\Gamma(3 + \mu)\Gamma(7 + \mu)} = \frac{(3 + \mu)(4 + \mu)}{(5 + \mu)(6 + \mu)},
\]

(8)

which is a monotonically increasing function of \( \mu \) for \( \mu \geq -3.63. \) Equation (8) may be solved for \( \mu \) to yield

\[
 \mu = \frac{(7 - 11\eta) - [(7 - 11\eta)^2 - 4(\eta - 1)(30\eta - 12)]^{1/2}}{2(\eta - 1)};
\]

(9)

\( \Lambda \) is then found from

\[
 \lambda = \left[ \frac{4 + \mu(3 + \mu)p_4}{p_2} \right]^{1/2};
\]

(10)

and \( N_0 \) may be determined from substitution of the values of \( \mu \) and \( \Lambda \) in Eq. (6) for any of the three moments \( p_2, p_4, \) and \( p_6. \) The median volume diameter is found from \( \Lambda \) and \( \mu \) through

\[
 D_m = \frac{3.67 + \mu}{\Lambda}.
\]

(11)

Note that Eq. (8) is quadratic in \( \mu \) and should yield, therefore, two solutions for this parameter; the solution that is physically reasonable has been chosen in Eq. (9).

In the above it has been assumed that each of the integrals extends over all possible diameters (i.e., \( D_{\text{min}} = 0, D_{\text{max}} \rightarrow \infty \)). It has been shown in previous work (Ulbrich 1985, 1992, 1993) that such an assumption can produce significant errors in integral parameters and in results found from computer simulations of dual-measurement methods. To explore the extent to which the

Fig. 1. The ratio of \( L_D m \) to its limiting value for \( D_{\text{max}} \to \infty \) for a gamma distribution as a function of \( D_{\text{max}}/D_m \), where \( D_m \) is the diameter at which the distribution is truncated and \( D_m \) is the mass-weighted mean diameter of the truncated distribution. The limiting value is \( L_D m = 4 + \mu \).

Fig. 2. Variation of the ratio of moments \( \eta \) with \( D_{\text{max}}/D_m \) and \( \mu \). The different values of \( \mu \) are indicated by the different line types. The solid curve is a boundary below which a pair of values of \( \eta \) and \( D_{\text{max}}/D_m \) will not yield a solution for \( \mu \).

The effect of truncation of the DSD is important in the method of moments as employed in this work, it will be assumed that the minimum diameter \( D_{\text{min}} = 0 \), which is an excellent assumption for all of the experimental spectra used in this work, but that the maximum diameter \( D_{\text{max}} \) is finite. Following the work given in Ulbrich (1985), it may be shown with these assumptions that Eq. (7) must assume the form

\[
D_{\text{max}} = \alpha D_m \left( 1 - \frac{1}{e^x} \right),
\]

where \( \alpha = \Lambda D_m \), and \( D_m \) is the mass-weighted mean diameter defined by

\[
D_m = \frac{p_2}{p_1}. \tag{14}
\]

Note that in the case \( D_{\text{max}} \to \infty \) the quantity \( \Lambda D_m = 4 + \mu \). However, for finite \( D_{\text{max}} \) the behavior of \( \alpha \) as a function of \( \mu \) and \( D_{\text{max}} \) is shown in Fig. 1, where \( \alpha/(4 + \mu) \) is plotted versus \( D_{\text{max}}/D_m \) for several values of \( \mu \). It is clear from these results that it is important to account for variations in \( \alpha \) due to finite \( D_{\text{max}} \) when computing the theoretical values of the \( p_n \). Note that for a given value of \( \mu \) there is a minimum value of \( D_{\text{max}}/D_m \) below which values of \( \Lambda D_m \) are not defined.

Within these assumptions the ratio \( \eta \) is given in terms of \( \mu \) and \( D_{\text{max}}/D_m \) by the form

\[
\eta = \frac{p_2}{p_1} \frac{p_6}{p_4} \left[ \frac{\gamma \left( 5 + \mu, \alpha \frac{D_{\text{max}}}{D_m} \right)}{\gamma \left( 3 + \mu, \alpha \frac{D_{\text{max}}}{D_m} \right)} \right]^2, \tag{15}
\]

The behavior of \( \eta \) as a function of \( D_{\text{max}}/D_m \) and \( \mu \) is shown in Fig. 2. The region to the left of the solid curve is a forbidden region, that is, one where solutions for \( \mu \) are impossible. The dashed lines are isopleths of \( \mu \) and are labeled with the corresponding values. It is clear from this figure that allowance for truncation will reduce the values found for \( \mu \) from \( \eta \) relative to those found when it is assumed that \( D_{\text{max}} \to \infty \).

The implementation of the method of truncated moments to experimental drop size spectra follows from Eqs. (12), (14), and (15). This entails calculating from the experimental spectra the moments \( p_2, p_4, \) and \( p_6 \), as well as \( D_m \) and \( D_{\text{max}} \). The latter parameter can be defined as the diameter corresponding to the largest category of the drop size spectrum in which experimental counts are found. These data are then used to compute \( \eta \) and \( D_{\text{max}}/D_m \) from which \( \mu \) is found using Eq. (15). It is important to account for the dependence of \( \alpha \) on
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erence between the two diameters is within the accu-
$\Delta L / D_m$ to $\Delta L / D_m$ results from any of the 

$D_{\text{max}}$ together with the value of $D_{\text{max}}$ from Eq. (15) by multiplying the value of $D_{\text{max}}$ to find a calculated value of $D_{\text{max}}$.

In the case of large $D_{\text{max}}$ it is found that $D_{\text{max}} / D_m = F_D(\mu, \infty) = (3.67 + \mu)(4 + \mu)$. The function $F_D(\mu, D_{\text{max}} / D_m)$ represents the theoretical prediction of the dependence of the ratio $D_{\text{max}} / D_m$ on $\mu$ and $D_{\text{max}} / D_m$. It is used to find a calculated value of $D_{\text{max}}$ using the experimental values of $D_m$ and $D_{\text{max}}$ together with the value of $\mu$ found from Eq. (15) by multiplying the value of $D_{\text{max}}$ by $F_D$. The value of $D_{\text{max}}$ found by this method is not necessarily the same as the value deduced directly from the experimental spectra. The behavior of $F_D$ as a function of $D_{\text{max}} / D_m$ is shown in Fig. 3 for several values of $\mu$. Note that for $\mu > -2$ theory predicts that there is less than 10% difference between $D_m$ and $D_{\text{max}}$. Indeed, the difference between the two diameters is within the accuracy with which $D_{\text{max}}$ can be determined since it is usually found by interpolation between size categories, thus resulting in behavior that displays the quantization of these categories.

It should be noted that the method described above assumes that no drop size data exist for diameters greater than $D_{\text{max}}$, and does not therefore attempt to extend the DSD to larger values. It uses the available data in the range $0 \leq D \leq D_{\text{max}}$ to fit a gamma distribution that more accurately describes the data than if no account were taken of truncation. In the method $D_{\text{max}}$ and $D_m$ serve the same functions as the moments $p_{2n}$, $p_1$, and $p_{01}$, that is, they are all experimental quantities that are combined to deduce the values of the DSD parameters $N_0$, $\mu$, $\Lambda$, and $D_m$. In the present work, therefore, $D_m$ is an experimental quantity calculated from the drop size spectrum, whereas $D_{\text{max}}$ is a calculated DSD parameter deduced from $\eta$, $D_m$, and $D_{\text{max}}$ using the methods described above.

3. Distributions of DSD parameters

The data used in this work consist of numbers of drops per unit volume $n(D_0)$ (m$^{-3}$) having equivalent spherical diameters $D_0$ in 32 uniformly spaced size categories ranging from $D_{10} = 0.02$ to $D_{32} = 0.64$ cm. Values of the gamma distribution parameters for each of the data points have been determined using the method of moments described above. Computation of the moments $p_n$ from the spectral data involves using the finite difference equivalent of Eq. (7), that is,

$$p_n = \sum_{i=1}^{32} D_i n_i(D_i).$$

These were then used in Eqs. (9), (10), (11), and (7) (i.e., without allowance for truncation of the DSD) to find solutions for $\mu$, $\Lambda$, $D_{\text{max}}$, and $N_0$, respectively. The distributions of values for three DSD parameters are shown in Figs. 4a–d for $\mu$, $N_0$, $D_{\text{max}}$, and $\Lambda$, respectively, and are indicated by the plus signs. About 4% of the total number of the experimental spectra failed to yield solutions for the DSD parameters because the procedure described in the previous section was unable to determine $\mu$ from Eq. (9). The distribution for $\mu$ is sharply peaked at about $\mu = 4$ with the central 80% of the points lying in the range $-1 < \mu < 8$ and the central 60% in the range $0.5 \leq \mu \leq 7$. The distribution of $N_0$ values displays a well-defined peak at $\log_{10} N_0 = 8$ but extends to very large values, that is, values as large as $\log_{10} N_0 > 28$. The distribution for $D_{\text{max}}$ is also sharply peaked with a modal value of $D_{\text{max}} = 1$ mm and almost all of the points in the range $0.4 < D_{\text{max}} < 1.6$ mm. Finally, the $\Lambda$ distribution displays a peak near $\Lambda = 50$ cm$^{-1}$, a value not greatly different from that predicted by the Marshall and Palmer (1948) distribution. The average values and standard deviations of each distribution are shown in the figures.

It must be noted that the accuracy of results found for the DSD parameters depends on accuracy with which the diameters are measured using the 2D-P probe. If there exists a systematic error in the probe measurement of diameter, then this could propagate strongly throughout relations such as Eq. (12) when determining the DSD parameters. However, if this error involved, for example, a constant multiplicative factor for $D$ in each

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**Fig. 3.** Dependence of the ratio $F_D = D_m / D_n$ on $D_{\text{max}} / D_n$ for several values of the gamma DSD parameter $\mu$. 

\[ \Lambda = \frac{p_2 \gamma \left(5 + \mu, \alpha \frac{D_{\text{max}}}{D_m}\right)}{p_4 \gamma \left(3 + \mu, \alpha \frac{D_{\text{max}}}{D_m}\right)} \] 

and $N_0$ follows from any of the $p_n$ [Eq. (12)] together with the values of $D_{\text{max}} / D_m$, $\Lambda$, and $\mu$. Finally, $D_0$ may be found from

$$D_0 / D_m = F_D \left(\mu, \frac{D_{\text{max}}}{D_m}\right).$$


of the moments $p_2$, $p_4$, and $p_6$, then Eq. (12) indicates that the result found for $\mu$ should be insensitive to such systematic offset. The results found for $N_0$ and $\Lambda$ would, however, be sensitive to such an error. Random errors within size categories should have little effect on the estimates of the moments.

To assess the differences obtained by accounting for truncation in solving for the DSD parameters, the procedure described in the previous section, which employs the maximum diameter $D_{\text{max}}$ and the mass-weighted mean diameter $D_m$, has been applied to the data to find $\mu$, $N_0$, $D_0$, and $\Lambda$. The resultant distributions of $\mu$, $N_0$, $D_0$, and $\Lambda$ are shown in Figs. 4a–d and are indicated by the squares. The distribution for $\mu$ has somewhat narrower breadth than that found when truncation is not considered and has a modal value near $\mu = 0$. The
central 80% of the data lies in the range \(-2 \leq \mu \leq 5\) and the central 60% in the range \(-1.5 \leq \mu \leq 3\). Similar remarks apply to the distributions of \(D_0\), \(N_0\), and \(\Lambda\).

The fact that the distribution of \(N_0\) values extends to very large values is characteristic of this parameter and is, as has been shown in earlier work (Ulbrich 1983; Ulbrich and Atlas 1985; Chandrasekar and Bringi 1987), due to the fact that there exists a very strong correlation between \(N_0\) and \(\mu\). This is illustrated in Fig. 5, where the values of \(N_0\) and \(\mu\) as found by the method of truncated moments are plotted; the data plotted in this figure involve a correlation coefficient of 0.98. Also shown in the figure is an empirical least squares fit of the form \(N_0 = a' \exp(b'\mu)\), where \(a' = 6.73 \times 10^4\) and \(b' = 3.79\). It is clear from this figure that the large values of \(N_0\) are always associated with large values of \(\mu\). It may be concluded that the large values of \(N_0\) and \(\mu\) are associated with spectra that are sharply peaked and for which data may exist in only a few size categories. Presumably, with sufficient averaging these kinds of spectra will be eliminated and the ranges of the latter two DSD parameters will be reduced. As shown by Chandrasekar and Bringi (1987), this strong correlation between \(N_0\) and \(\mu\) is not based on a physical relationship but is due to the fact that the parameter \(N_0\) involves units that depend on the parameter \(\mu\).

Another consequence of applying the method of truncated moments to the 2D-P probe data is that the results display dependence on the number of drop size categories in which data reside, especially when the number of categories is small. In such cases the method often attempts to fit a sharply peaked distribution through the data with a large value of \(\mu\), whereas visual inspection of the data reveals that these cases could be equally described by an exponential distribution (\(\mu = 0\)). This is illustrated in Fig. 6, where the values of \(\mu\) found by the method of truncated moments are plotted versus \(N_{\text{cat}}\), the number of size categories in which data occur. (For the data used in this work almost all of the samples had data in contiguous categories; i.e., there were no gaps in the data. In addition, greater than 99.99% of the data had data in the first category so that \(D_{\text{min}} = 0\) for virtually all the data. There are no data in Fig. 6 for \(N_{\text{cat}} > 25\), which is indicative of the fact that when data are found in categories at the large-diameter end of the spectrum there are also gaps in the data; i.e., the data are not contiguous.) The squares indicate the average values of \(\mu\), and the error bars depict their standard

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**Fig. 5.** Plot of the values of \(N_0\) vs \(\mu\) found by the method of truncated moments used in this work. The coefficient and exponent shown on the diagram were determined from a least squares fit of an exponential function to the data. The data involve a correlation coefficient greater than 0.98.

**Fig. 6.** The average value of \(\mu\) (open squares) found by the method of truncated moments vs the number of size categories \(N_{\text{cat}}\) in which data occur. The error bars depict the standard deviation of the data for a given value of \(N_{\text{cat}}\).
deviations at a given value of $N_{\text{CAT}}$. Several effects are immediately apparent: 1) for $N_{\text{CAT}} > 9$ the average values of $\mu$ tend to cluster around a value of 0.5 (i.e., not greatly different from an exponential distribution), 2) the average value of $\mu$ tends to increase as $N_{\text{CAT}}$ decreases, and 3) the spread or standard deviation of the $\mu$ values also increases as $N_{\text{CAT}}$ decreases. It should be noted that the results shown in Fig. 6 were deduced using the method of truncated moments; when the method of moments is used (without accounting for truncation), then the average values of $\mu$ and their standard deviations are evenly more strongly dependent on $N_{\text{CAT}}$. The depiction in Fig. 6 of the effects described above illustrates one of the difficulties associated with applying the method of moments to drop size spectra.

4. Empirical $Z$–$R$ analyses

In this section empirical $Z$–$R$ analyses of the data are performed to find relations of the form $Z = AR^b$ by computing $Z$ and $R$ from the finite difference equivalents of Eqs. (6) and (2), namely,

$$R = 0.6 \pi \sum_{i=1}^{32} v(D_i)D_i^4/\eta(D_i) \quad (19)$$

and

$$Z = 10^6 \sum_{i=1}^{32} D_i^4/\eta(D_i). \quad (20)$$

It should be noted that rainfall rates in this section are calculated assuming that the particles have fall speeds equal to sea level values. It is a simple matter to correct the results found for the coefficient $A$ for other elevations. For example, Foote and duToit (1969) correct the $v(D)$ by multiplying by the factor $F_{\text{ZR}} = (\rho_0/\rho)^{1/4}$, where $\rho_0$ and $\rho$ are the air densities at sea level and at altitude, respectively. Therefore, the relationship between the sea level rainfall rate $R$ and the rainfall rate at altitude $R_i$ is $R_i = F_{\text{ZR}} \times R$. The relationship between $Z$ and $R_i$ at altitude can then be written

$$Z = A_i R_i^b = A_i F_{\text{ZR}}^b R_i^b = A_i (\rho_0/\rho)^{1/4} R_i^b = AR^b. \quad (21)$$

The correction to the sea level $Z$–$R$ relation $Z = AR^b$ to find the corresponding relation at altitude merely involves dividing the coefficient $A$ by $F_{\text{ZR}}^b$. It can be shown using the NACA Standard Atmosphere (List 1984) that the coefficient $A$ increases by 5% per kilometer of altitude. Other more accurate methods of correcting the fall speeds can be used (e.g., Beard 1976), but the qualitative behavior of the coefficient $A$ will be the same as the simpler procedure outlined here.

In the following, empirical $Z$–$R$ analyses are performed in which the data are restricted to specified ranges of DSD parameters. The motivation for pursuing this approach will be elaborated later in this section. To illustrate the results obtained by stratifying the data according to the value of $\mu$ found using the method of moments outlined above, the results found for $A$ and $b$ in $Z = AR^b$ for five of the experimental days are shown in Table 1. The important features are that the coefficient $A$ decreases as $\mu$ increases, whereas the exponent $b$ changes very little. This behavior is very well defined, as is illustrated in Fig. 7, for one of the experimental days in Table 1. Data are shown for two ranges of $\mu$, namely, $-2 < \mu < 0$ and $6 < \mu < 8$, with the squares and crosses representing the data in each of these categories, respectively. There is a clear separation between the data in the two categories with little overlap. This is typical of the results found for other ranges of $\mu$ and for other days.

Although the DSD parameter $\mu$ has been used in separating the data in these $Z$–$R$ analyses, the results are not dependent on the assumption of a gamma distribution. The parameter $\mu$ can be regarded as a proxy for the distribution shape as defined by the moments of distribution $p_2$, $p_4$, and $p_6$. An alternative shape parameter, $\sigma_D/D_m$, can be defined as the normalized standard deviation of the mass spectrum with respect to the mass-weighted mean diameter $D_m$, which can be written

$$\frac{\sigma_D}{D_m} = \left[ \frac{\int (D - D_m)^2 D^4/\eta e^{-AD} dD}{D_m^2 \int D^4/\eta e^{-AD}} \right]^{1/2} = \frac{1}{(4 + \mu)^{1/2}}, \quad (22)$$

where it has been assumed that $D_{\text{max}} \to \infty$. A plot of $\sigma_D/D_m$ versus $\eta$ is shown in Fig. 8; a scale of $\mu$ is also
Fig. 7. Empirical Z–R analyses for TOGA COARE day 930218 in which the data have been stratified according to the value of $m$. The open squares and crosses correspond to stratification of the data according to $-2 \leq \mu \leq 0$ and $0 \leq \mu \leq 8$, respectively. Also shown in the figure are the coefficient and exponent found for each of the fits.

shown at the top of the figure as well as the quantity $G$, the measure of distribution shape introduced by Atlas (1964), which is defined by

$$G = \frac{\int D^5 N(D) \, dD}{D_0^3 \int D^3 N(D) \, dD} = \frac{(6 + \mu)(5 + \mu)(4 + \mu)}{(3.67 + \mu)^3}.$$ (23)

Figure 8 shows that there is a one-to-one relation between the ratio of the moments $\eta$ and the shape of the drop size distribution. This relationship is not strongly dependent on $D_{\text{max}}$ even though $\eta$, $\sigma_D/D_m$, and $G$ are each strong functions of the ratio $D_{\text{max}}/D_m$. Although the gamma distribution has been used here to illustrate the mathematical interrelationships between $\mu$, $\eta$, $\sigma_D/D_m$, and $G$, it is clear from these considerations that the stratification with respect to $\mu$ in the above empirical Z–R analyses is a general classification according to spectral shape. It is also important to further emphasize that each of these parameters is a measure of spectral shape and not spectral width. The latter quantity is determined by quantities such as $\sigma_D$, which is proportional to $D_m$. Indeed, the width or breadth of the spectrum is more strongly controlled by the value of $D_m$ (or $D_0$) than by the shape factor $\mu$, as is apparent from examination of Eq. (22).

The results shown in Table 1 for the coefficient $A$ are consistent with those found for $N_0$ from the moments. To demonstrate that this is the case, assume that the rainfall rate can be approximated by a form in which the drop fall speed is given by a power law of the form proposed by Atlas and Ulbrich (1977), namely, $v(D) = \kappa D^{0.67}$, where $v(D)$ is in m s$^{-1}$, $D$ is in cm, and $\kappa = 17.67$ m s$^{-1}$ cm$^{-0.67}$. In that case Eq. (2) becomes

$$R = 0.6 \pi \kappa N_0 \int_0^\infty D^{3.67+\mu} \exp(-\kappa D) \, dD
= \frac{0.6 \pi \kappa N_0 \Gamma(4.67 + \mu)}{A^{4.67+\mu}}.$$ (24)

Elimination of $\Lambda$ between Eqs. (6) and (24) yields a Z–R relation of the form $Z = A' R^\beta$, where the exponent $\beta = (7 + \mu)/(4.67 + \mu)$ and the coefficient $A'$ is

$$A' = N_0^{1-\beta} \frac{10^\beta \Gamma(7 + \mu)}{[0.6 \pi \kappa \Gamma(4.67 + \mu)]^\beta} = N_0^{1-\beta} F_A(\mu),$$ (25)

where $F_A(\mu)$ is the factor in braces and is dependent only on $\mu$. This expression has been used to deduce values of $N_0$ by setting $A'$ equal to the values of $A$ listed in Table 1 while using the value of $\mu$ implied by the stratification. The results are shown in Fig. 9 as the filled circles; the value of $\mu$ at the midpoint of each category has been employed in plotting these data, and there are five data points plotted at each value of $\mu$ corresponding to the five days shown in Table 1. The results found for $N_0$ as a function of $\mu$ are in very good agreement with the dashed line, which is the empirical relation shown in Fig. 5 that was deduced from the moments. For the purposes of comparison, the empirical $N_0-\mu$ relation
found by Ulbrich (1983) from disdrometer data is also shown as the solid line. It should be noted that it is possible to derive values of \( \mu \) from the values of \( \beta \) in each row of Table 1 using the relation given by Ulbrich (1983), that is, \( \mu = (7 - 4.67\beta)/(\beta - 1) \). If this is done, then the values for \( \mu \) are not in agreement with the values used to stratify the data. The origins of this discrepancy are due to the fact that the value of \( \mu \) for each individual empirical spectrum is determined from the moments \( p_3, p_4, \) and \( p_n \) for that spectrum, whereas \( A' \) and \( \beta \) are found from a larger number of \((Z, R)\) pairs in a least squares fit. It may be demonstrated by simulation that if all of the data in each stratification with respect to \( \mu \) had exactly the same value of \( \mu \) and that \( N_0 \) for each of these samples was uniquely related to \( \mu \) through the form \( N_0 = a' \exp(x'\mu) \) (where, as before, \( a' = 6.73 \times 10^6 \) and \( b' = 3.79 \)), then the value found for \( \mu \) from \( \beta \) would be exactly equal to that used to stratify the data. However, it is apparent from Fig. 5 that there are large deviations of the data from the empirical \( N_0 - \mu \) relation and that these moment-to-moment deviations result in averaging of the data that produces an exponent \( \beta \) similar to that found for the entire dataset for a given day, as shown in the last row of Table 1. The implication is that it is improper to deduce a value for \( \mu \) from the exponent \( \beta \) because of variations in \( \mu \) and \( N_0 \) among the individual spectra within a given stratification class. Nevertheless, it may also be shown by simulation that regardless of the variations in \( \mu \) within a given class and whether the empirical \( N_0 - \mu \) relation is exactly obeyed, an increase in \( \mu \) is always associated with a decrease in the coefficient \( A \). Since it was shown earlier that \( \mu \) is directly related to spectrum shape, it may be concluded that the decrease in \( A \) with increasing \( \mu \), as apparent in Table 1, is due primarily to increasing spectral shape.

A similar analysis has been performed by restricting the range of values for \( D_m \) used in the empirical fits. The results for the five experimental days in Table 1 are shown in Table 2 and depicted in Fig. 10 for one of the days using three ranges of \( D_m \). The important features are that \( A \) increases sharply with increasing \( D_m \), whereas the exponent \( b \) remains fairly close to 1.0 for all values of \( D_m \). It is not surprising that \( b = 1 \) in this analysis since restricting \( D_m \) to a narrow range of values is equivalent to constraining the data to lie along an isopleth of \( D_m \) on the rain parameter diagram (Ulbrich and Atlas 1978); these isopleths have slopes of 1 on such a \( Z - R \) diagram. [Similar behavior is evident in the work of Atlas et al. (1997).]

The results can be explained again using the theoretical expressions for \( Z \) and \( R \) given by Eqs. (6) and

### Table 2. Parameters derived from empirical \( Z-R \) fits using 2D probe data for five experimental TOGA COARE days (symmdd) in which the data points have been restricted to have values of \( D_m \) between \( D_{\text{min}} \) and \( D_{\text{max}} \) (cm). The value \( N \) is the total number of points used in the fit; \( A \) and \( b \) are the coefficient and exponent in the empirical \( Z-R \) relation \( Z = AR^b \).

<table>
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<tr>
<th>( D_{\text{min}} )</th>
<th>( D_{\text{max}} )</th>
<th>( N )</th>
<th>( A )</th>
<th>( b )</th>
<th>( N )</th>
<th>( A )</th>
<th>( b )</th>
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<td>27 27.3 0.97</td>
<td>88 37.8 1.08</td>
<td>77 30.5 1.10</td>
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<td>127 95.8 1.15</td>
<td>251 95.7 1.14</td>
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<td>303 282.6 0.92</td>
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FIG. 10. Empirical $Z\pm R$ analyses for TOGA COARE day 930209 in which the data have been stratified according to the value of $D_m$. The open squares, crosses, and open circles correspond to stratification of the data according to $0.2 \leq D_m \leq 0.4$, $0.6 \leq D_m \leq 0.8$, and $1.2 \leq D_m \leq 1.4$ mm, respectively. Also shown in the figure are the coefficient and exponent found for each of the fits.

FIG. 11. The coefficient $C_R$ vs $C_Z$ in Eq. (19). Each of the points is labeled with the value of $\mu$ for which it was calculated.

(24). Dividing $Z$ by $R$ eliminates the parameter $N_0$ as well as $\mu$ in the exponent and yields (using $\Delta D_m = 4 + \mu$)

$$\frac{Z}{R} = \left[ \frac{C_Z(\mu)}{C_R(\mu)} \right] D_m^{2.33}, \quad (26)$$

where the coefficients $C_Z$ and $C_R$ are given by

$$C_Z = \frac{10^6 \Gamma(7 + \mu)}{(4 + \mu)^{7+\mu}}$$

and

$$C_R = \frac{0.6 \pi \Gamma(4.67 + \mu)}{(4 + \mu)^{4.67+\mu}}. \quad (27)$$

The factor in braces in Eq. (26) is therefore a function of $\mu$ only. A plot of $C_R$ versus $C_Z$ is shown in Fig. 11; the open squares represent values of the ordinate and abscissa for integral values of $\mu$ and are labeled with their corresponding values of $\mu$. This figure demonstrates that the $C_R$ is approximately proportional to $C_Z$ over most of the range of $\mu$, that is, $C_R \approx 7.1 \times 10^4 C_Z^{1.06}$, so that the ratio in Eq. (26) can be very closely approximated by $Z/R = 7.1 \times 10^4 D_m^{2.33}$. It may be concluded that, when the data are stratified according to $D_m$, $Z$ is directly proportional to $R$ and that the coefficient of proportionality increases as $D_m^{2.33}$. For $D_m \approx 0.5$ mm the values of $Z/R$ predicted by the latter expression are in good agreement with those found from the empirical analysis.

It should be noted that each of the above theoretical analyses is intended only as a means of demonstrating why the $Z\pm R$ relations behave as they do when stratified according to $\mu$ and $D_m$. The assumption of a power-law dependence of raindrop fall speed on diameter is not intended to be an accurate representation of the actual fall speeds. Rather, it is used to show that the results found for the coefficients and exponents in the empirical fits are in agreement with that which is predicted by theory. It is also noteworthy that similar explanations can be devised using the equations presented by Atlas (1964, 365).

The results of these empirical analyses can be synthesized in the form of a rain parameter diagram, which is shown schematically in Fig. 12. The diagram consists of isolopes of $D_m$ (dashed lines) and $\mu$ (solid lines). The $D_m$ isolopes are, of course, related to mean particle size and have slopes equal to 1, as indicated earlier. The isolopes of $\mu$ are related to spectral shape with small and large values corresponding to greater and lesser shape, respectively. The lower and upper values of $\mu$ are chosen such that their isolopes span all of the $Z\pm R$ data for the particular dataset under consideration. The slopes of the $\mu$ isolopes are equal to that of the empirical $Z\pm R$ relation found when all the data are included in the least squares fit. In accordance with the empirical results found above, the $\mu$ isolopes extend over the full range of the rainfall rates, whereas the $D_m$ isolopes range only over the region of the diagram defined by the lower and upper values of $\mu$. In view of these considerations, different datasets might have different slopes for the $\mu$ isolopes. Nevertheless, the qualitative behavior of the isolopes on this rain parameter diagram can be used to determine the general behavior.
of the DSD parameters for each of the Z–R points plotted on the diagram.

As an example of the use of the diagram, at a given \( R \), as \( Z \) increases, the mean particle size increases, while the spectral shape lessens. Conversely, as \( R \) increases at constant \( Z \), \( D_m \) decreases as the spectral shape increases. Finally, it is interesting to note that variations along a perpendicular to a \( \mu \) isopleth (i.e., perpendicular to the overall Z–R relation) show that as \( \mu \) increases, \( D_m \) decreases, thus demonstrating that the spectral breadth \( \sigma_m \) must increase or decrease with \( D_m \).

Generally speaking, data points at small \( R \) and small \( Z \) have the largest spectral shape and smallest mean particle size. As \( Z \) increases, while \( R \) remains small, the mean particle size increases as the spectral shape lessens. Similar remarks apply to data points at large \( R \) except the mean particle sizes are larger in all cases.

It should be noted that it is not possible to derive this form of the rain parameter diagram mathematically using the theoretical relations given earlier. Since the equations defining \( Z \) and \( R \) [Eqs. (4) and (6)] depend on three parameters, it is necessary to assume that one of them is constant or that a relation exists between two of them in order to display the relationships between integral rainfall parameters in two dimensions. An investigation has been made of the use of the empirical relation between \( N_o \) and \( \mu \) (as shown in Fig. 5) for such a purpose, but the results so obtained do not realistically depict the behavior observed empirically. The diagram in Fig. 12 is derived only from consideration of the results of the empirical analyses and not from mathematical analysis.

5. Conclusions

This paper has examined the relationships between parameters of the gamma drop size distribution, \( N_o \), \( D_m \), and \( \mu \), and integral parameters derived from them using a set of raindrop size spectra measured with an airborne 2D-P precipitation probe in the TOGA COARE experiment. A new method has been developed for deducing the values of the drop size distribution (DSD) parameters from the spectral moments, which takes account of the truncation of the experimental spectra at a maximum diameter \( D_{\text{max}} \). It is found that the effects of truncation of the DSD can be significant; when no allowance is made for truncation, the method of moments can produce values for the DSD parameter \( \mu \) that are on average about a factor of 2 too large. The resultant effects on the other DSD parameters are such that they are also overestimated. Although the methods are illustrated by applying them to experimental data for tropical rain, they are general and can be used with data acquired in any type of rainfall.

Distributions of the DSD parameters as found by the method of truncated moments are displayed where it is shown that the distribution for \( \mu \) is sharply peaked with a modal value of \( \mu = 0 \), a mean value of \( \mu = 1.67 \), and with the central 80% of the data occurring in the range \( -2 \leq \mu \leq 5 \). The distribution for \( \log_{10} N_o \) is also sharply peaked and involves a modal value \( \log_{10} N_0 = 4 \) and a mean of 7.5, although very large values of \( N_o \) are often found, that is, values at least as high as \( \log_{10} N_0 = 20 \). The modal value of \( N_o \) for the data used in this work is similar to that found by Marshall and Palmer (1948). It is concluded from these results that with sufficient averaging the distribution for TOGA COARE rainfall can be described approximately by an exponential distribution with a value of \( N_o \) a little less than that found by Marshall and Palmer (1948).

An investigation is made of the effect of experimental size spectra that are narrow, that is, spectra for which data occur in a small number of size categories. The distribution of estimates of \( \mu \) found by the method of truncated moments as a function of the number of size categories in which data reside, \( N_{\text{CAT}} \), shows that the average value of this parameter increases sharply as \( N_{\text{CAT}} \) decreases and that the range of values for a given \( N_{\text{CAT}} \) (or standard deviation of \( \mu \) values) also increases. The conclusion reached from this result is that as \( N_{\text{CAT}} \) decreases, the method of truncated moments at times attempts to fit a sharply peaked distribution through the data with a large value of \( \mu \). At other times it may fit a concave upward distribution with a large negative value of \( \mu \) even though the distribution does not extend to very large diameters. In fact, for most of these cases visual inspection of the data reveals that they might be equally well described by an exponential distribution. An investigation is currently being undertaken to determine whether alternative methods of analysis can

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**Fig. 12.** Schematic rain parameter diagram with isopleths of DSD parameters \( \mu \) and \( D_m \), illustrating the effects on variations in these parameters for (Z, R) data plotted in the diagram.
avoid the pitfalls inherent in the method of moments for these type of spectra.

Empirical $Z-R$ analyses of the data are performed in which the data are classified separately according to values of the parameters $\mu$ and $D_m$, the mass-weighted mean diameter. The latter parameter is shown to be close to the median volume diameter $D_m$. The coefficient $A$ and exponent $b$ in $Z = AR^b$ are determined for several values of $D_m$ and are shown to be in good agreement with theory. It is shown that classification with respect to $\mu$ is equivalent to stratifying the data according to spectral shape and that the values found for $A$ are consistent with the values for $N_o$ deduced from the moments. The results of these empirical analyses are summarized in the form of a new rain parameter diagram containing isopleths of $\mu$ and $D_m$, which is useful for describing the variations in DSD parameters $\mu$ and $D_m$ among the $Z-R$ data plotted on the diagram. As an application of the new diagram, one may consider the circumstance when precipitating generating cells happen to be located to one side of the aircraft sampling line over a considerable distance. Then truncation may occur on either end of the DSD systematically so that $\mu$ remains large (or small) over the entire path and the sample $Z$ values will remain low (or high) for the particular values of $R$ over that path. It is only when there is a sufficient population of generators that are disposed more or less randomly around the aircraft sampling path that the entire $\mu-D_m$ domain in Fig. 12 will be filled and the $Z-R$ relation will be representative of the untruncated DSDs (or grand average DSD) in the sampling domain.

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