Anisotropic Error Distributions in a Bistatic Doppler Radar System

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ABSTRACT

Anisotropic errors that contaminate two-dimensional wind vectors determined using a bistatic Doppler radar system composed of a transceiver (traditional Doppler radar) and a receiver are characterized. Two kinds of errors are identified and their influences on the retrieved wind vectors described. Type-1 errors are caused by the finite resolution of the Doppler velocity observed at the transceiver and receiver. Type-2 errors are associated with the finite resolution in detecting azimuth angles at the transceiver and ranges at the receiver.

The two errors show remarkably anisotropic behavior. Error vectors for type-1 or type-2 errors have lengths that are a unique function of direction. These error vectors that can appear with equal probability form an ellipse at observation points. The larger the ratio of the length of the longer axis to that of the shorter axis, the higher the anisotropy. For type-1 errors, the longer axis is approximately perpendicular to the radar beam; the shorter axis is parallel to it. The directions of the corresponding axes for type-2 errors depend on the location of the observation point and the wind there.

Retrieved wind vectors contaminated with type-1 and type-2 errors are simulated. Random noise is superimposed on a uniform wind with standard deviations of half the resolutions of four parameters. The resultant type-1 and type-2 error vectors both show anisotropic distributions consistent with the error ellipses mentioned above. This suggests that spurious divergence/convergence and spurious rotation dominate cases wherein the wind is parallel or perpendicular to the direction of the longer axis.

1. Introduction

Wurman and his colleagues proposed and developed the first full-scale but not permanent bistatic Doppler radar systems suitable for weather observation (Wurman et al. 1993; Wurman 1994; Wurman et al. 1994; Satoh and Wurman 1999). The J. S. Marshall Radar Observatory at McGill University (Montreal, Quebec, Canada) deployed the first permanent bistatic Doppler radar system for research in 1995 (Kilambi et al. 1997). Since then several institutes have adopted the system (Friedrich et al. 2000; Friedrich and Hagen 2001), and it has been further studied. De Elia and Zawadzki (2000, 2001) focused on sidelobe contamination, a serious problem with the system. Takaya and Nakazato (2002) related the root-mean-square error of the system to the finite resolution of the Doppler velocity, azimuth angle, and range. Data generated by the bistatic radar have been used in analyses, retrievals, and simulations (Protat and Zawadzki 1999; Protat et al. 2001; Montmerle et al. 2001).

This paper focuses on the anisotropic character of the error distributions. Namely, at each observation point, among the error vectors that can be realized with equal probability, there is an error vector that has the largest length pointing to a certain direction and also an error vector with the smallest length to another direction. The larger the ratio of the largest length to the smallest one, the higher the anisotropy is regarded to be. If the wind direction is perpendicular to the direction of the largest error vector, the main contamination by the errors will appear in the directions of the retrieved wind vectors. On the other hand, if the wind direction is parallel to the direction of the largest error vector, the main contamination will appear in the length of the retrieved wind speeds. These contaminations will provide spurious rotation and/or divergence/convergence to the retrieved wind field. Therefore, it is of practical importance to analyze the anisotropic character of the error vectors.

Among the possible sources of errors, this study is limited to errors originating from the system itself. Therefore, for example, the variance of Doppler velocities aroused from strong wind shear within a sample volume is not taken into consideration.

Section 2 reviews the nature of type-1 and type-2 errors. The anisotropic character of the error is discussed in section 3 and simulated in section 4. Section 5 summarizes the results.
2. The two error types

This section reviews the main results in Takaya and Nakazato (2002) in a new mathematical format.

a. Definition of two types of errors

Retrieval of a horizontal wind velocity vector by a bistatic Doppler radar system composed of a transceiver (traditional Doppler radar) and a receiver proceeds as follows. The transceiver sends azimuth and elevation angle information and Doppler velocity to a signal processor. The receiver sends range information on the distance from the radar to the receiver via a scatterer. The receiver also sends an apparent Doppler velocity (Protat and Zawadzki 1999) that is related to the real Doppler velocity by a factor that depends on the scattering angle defined below. Parameters from the receiver and transceiver yield the velocity vector of the scatterer.

Errors in determining the four parameters provide an error vector for this synthesized velocity vector. The error vector is partitioned in two. One part of the error arises from errors in determining the two Doppler velocities and is called type-1 error. The other error originates from errors in determining the azimuth angle and range, and is called type-2 error. In a previous paper (Takaya and Nakazato 2002) mathematical formulas that express these two error types were presented without proof. This paper includes a more detailed derivation with more elegant mathematical expressions for the relevant equations.

As in the previous paper, simplifying assumptions of very small elevation angles and a horizontal plane are made. The coordinate vector from the radar to a scatterer is \( \mathbf{r}_1 \), and that from the receiver to the scatterer is \( \mathbf{r}_2 \). (See Fig. 1; note that the transceiver is marked T/R and the receiver R in all figures.) These vectors are assumed to lie in a horizontal plane. The scattering angle \( \alpha \) of the transmitted pulse wave is

\[
\cos(\alpha) = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2},
\]

where \( d, r_1, \) and \( r_2 \) are the lengths of the baseline, vector \( \mathbf{r}_1 \), and vector \( \mathbf{r}_2 \), respectively. The baseline connects the transceiver and receiver. The unit vectors \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) are

\[
\mathbf{e}_i = \frac{\mathbf{r}_i}{r_i} \quad (i = 1, 2).
\]

The vectors \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) are parallel to \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \), respectively, and \( \mathbf{e}_1 \) is parallel to the transmitted beam. These unit vectors are also related to the scattering angle:

\[
\cos(\alpha) = (\mathbf{e}_1 \cdot \mathbf{e}_2).
\]

Here \( (\mathbf{A} \times \mathbf{B}) \) denotes the vector product of the two vectors \( \mathbf{A} \) and \( \mathbf{B} \). The \( x \) and \( y \) coordinates lie in the horizontal plane, and the \( z \) coordinate describes altitude. The vector product can also be used to relate unit vectors \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) to the scattering angle:

\[
\sin(\alpha) = (k \cdot (\mathbf{e}_1 \times \mathbf{e}_2)).
\]

In Fig. 1, we give angle \( \alpha \) with this definition.

The range \( s_2 \), which is the distance from the transceiver to the receiver via the scatterer, is

\[
s_2 = r_1 + r_2.
\]

The azimuth angle is denoted as \( \theta \) (see Fig. 1). From \( \theta \) and \( s_2 \), the location of the scatterer can be determined as the intersection of an ellipse, the two foci of which are the transceiver and the receiver, and a semi-infinite straight line starting from the transceiver with azimuthal angle \( \theta \). This geometry defines the coordinate vectors \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \), the scattering angle \( \alpha \), and the unit vectors \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \).

Protat and Zawadzki (1999) defined a unit vector \( \mathbf{e}_0 \) that is parallel to the Doppler velocity at the receiver.
Fig. 2. (a) Horizontal distribution of the root-mean-square of the type-1 error. The root-mean-square errors of the Doppler velocities of the transceiver and receiver are assumed to be 0.5 m s$^{-1}$. (b) The same as in (a) but for the type-2 error. The root-mean-square errors of the azimuth angles and ranges are assumed to be 0.5° and 10 m, respectively. The wind field is uniform northeasterly at 20 m s$^{-1}$. The length of the baseline is 30 km.

and is orthogonal to the ellipse of constant $s_2$ with the following equation:

$$
e_b = \frac{e_1 + e_2}{2 \cos(\alpha/2)}. \tag{7}$$

Two Doppler velocities $V_t, V_b$, azimuth angle $\theta$, and range $s_2$ give a Doppler velocity vector for a scatterer that is represented by

$$V = \frac{1}{(k \cdot [e_1 \times e_b])}(-V_t[k \times e_b] + V_b[k \times e_1]). \tag{8}$$

Doppler velocities $V_t$ and $V_b$ are expressed by the scalar product of velocity vector $V$ given in (8) with unit vectors $e_1$ and $e_b$, respectively:

$$V_t = (e_1 \cdot V), \tag{9}$$

$$V_b = (e_b \cdot V). \tag{10}$$

The apparent Doppler velocity of receiver $\hat{V}_b$ is the Doppler velocity detected at the receiver but is not a true component of the velocity $V$ (Protat and Zawadzki 1999). Using this apparent Doppler velocity, (8) can be rewritten as

$$V = -V_t \frac{[k \times e_b]}{(k \cdot [e_1 \times e_b])} + \hat{V}_b \frac{[k \times e_1]}{\cos(\alpha/2)(k \cdot [e_1 \times e_b])}. \tag{11}$$

If the error part of each term is expressed using $\delta$, the equation for the error in the synthesized vector becomes

$$\delta V = -\delta V_t \frac{[k \times e_b]}{(k \cdot [e_1 \times e_b])} + \hat{\delta V}_b \frac{[k \times e_1]}{\cos(\alpha/2)(k \cdot [e_1 \times e_b])} \tag{12}$$

where

$$\hat{\delta V}_b \left[ \frac{[k \times e_1]}{\cos(\alpha/2)(k \cdot [e_1 \times e_b])} \right]. \tag{13}$$

The first two terms represent the type-1 error. The third and fourth terms give the type-2 error (Takaya and Nakazato 2002).

Therefore, the type-1 error can be expressed as

$$\delta V_1 = -\delta V_t \frac{[k \times e_b]}{(k \cdot [e_1 \times e_b])} + \hat{\delta V}_b \frac{[k \times e_1]}{\cos(\alpha/2)(k \cdot [e_1 \times e_b])}. \tag{13}$$

The type-2 error, written in terms of errors in the azimuth angle and range, becomes

$$\delta V_2 = \frac{1}{\sin(\alpha)r_2} \{ ([k \times e_1] \cdot V)[k \times e_1]r_1 - ([k \times e_1] \cdot V)[k \times e_1]r_2 \} (\delta \theta)$$
The proof for Eq. (14) appears in appendix A.

b. Horizontal distribution of the root-mean-square error

This section focuses on horizontal distribution of root-mean-square errors. If we assume that the standard deviations of the two Doppler velocities are equal, that is,

\[
(\delta V) = \sigma_v,
\]

and that the correlation between \(\delta V\) and \(\delta t\) is zero, that is,

\[
\delta V \delta t = 0,
\]

then from (13) we derive the following for the mean-square type-1 error:

\[
(\delta V_1 \cdot \delta V_2) = \frac{1}{\sin^2(\alpha/2)} r_2^2 (\sigma_a)^2 + \frac{1}{4 \cos^2(\alpha/2)} r_2^2 (\sigma_a)^2,
\]

where \(\sigma_a\) and \(\sigma_r\) are the standard deviations of the azimuth angles and ranges, respectively. This equation appears in Takaya and Nakazato (2002) in a different format.

Figures 2a and 2b show the horizontal distribution of the root-mean square of these two error types. Throughout this paper, the standard deviation of the Doppler velocity \(\sigma_v\) is assumed to be 0.5 m s\(^{-1}\). This value was chosen from the following simple estimation. If the Nyquist velocity of the system is 16 m s\(^{-1}\) and the Doppler velocity is given by an FFT calculation with 32 pulses, the resolution of the Doppler velocity is 1 m s\(^{-1}\), or twice \(\sigma_v\). Doviak and Zrnic (1993) gave a more sophisticated estimation of the standard deviation in which similar values are derived.

The variance of the azimuth angle is assumed to be 0.5\(^\circ\). Estimation is done in the following example: 4 rpm scanning mode rotates an antenna 24\(^\circ\) s\(^{-1}\), and 600 PRF and 32-pulse sampling needs a dwell time of 0.0533 s, during which the antenna rotes 1.28\(^\circ\). A single azimuth angle is assigned to all the data during this time. In this example the variance of the azimuth angle is estimated to be about 0.64\(^\circ\). In the conventional bistatic Doppler radar system, the timing is adjusted by using a GPS that has a timing error of several tens of nanoseconds that corresponds to about a 10-m variance to the range determination. In this paper we assume range variance to be 10 m. In sharp reflectivity gradient regions, however, real ranging error can be larger, possibly reaching the length of the sampling volume because the dominant return may come from off-center high reflectivity.

We further assume that the baseline is 30 km long, pointing from south to north. The length is a scale parameter used to determine the absolute length of parameters with length dimensions. The wind is constant northeasterly at 20 m s\(^{-1}\). Wind information is necessary to compute type-2 error, as shown by (14) and (19). Figure 2a, which shows horizontal distribution of type-1 error, is similar to Fig. 3 of Takaya and Nakazato (2002). The domain of the minimum error lies on an arc around a scattering angle of approximately 100\(^\circ\). Figure 2b shows a complicated distribution of type-2 error. The magnitude is, as a whole, smaller than type-1 error, but around the baseline and receiver, larger values are observed. The pattern of the contour and the error values change as we adopt another wind field.

Figure 3 shows the root-mean-square error of the synthesized velocity vector \(\sqrt{(\delta V_1 \cdot \delta V_2)}\) in which both types of error are added in the following way:

\[
\sqrt{(\delta V \cdot \delta V)} = \sqrt{(\delta V_1 \cdot \delta V_1) + (\delta V_2 \cdot \delta V_2)}.
\]
3. Anisotropy of the error

a. Anisotropy of the type-1 error

If the errors of the Doppler velocities $\delta V_t$ and $\delta V_b$ are random with Gaussian probability distributions with a common standard deviation $\sigma_V$ with a magnitude that is half of the resolution of the Doppler velocity of the system, then the sum of the mean-square errors of the two velocities satisfies

$$\langle (\delta V_t)^2 + (\delta V_b)^2 \rangle = 2\sigma_V^2.$$  \hfill (21)

The combination of errors $\delta V_t$ and $\delta V_b$ that is realized with equal probability can be parameterized as

$$\delta V_t = \sqrt{2}\sigma_V \cos(\phi), \quad (0 \leq \phi \leq 2\pi),$$  \hfill (22)

$$\delta V_b = \sqrt{2}\sigma_V \sin(\phi), \quad (0 \leq \phi \leq 2\pi).$$  \hfill (23)

For each value of $\phi$, $\delta V_t$ and $\delta V_b$ in (22) and (23) will appear with equal probability. Therefore, the following error velocity vector will appear with equal probability for any values of $\phi$:

$$\delta V_r(\phi) = \sqrt{2}\sigma_V \left[ -\cos(\phi) \frac{[k \times e_f]}{(k \cdot [e_t \times e_b])} + \sin(\phi) \frac{[k \times e_f]}{\cos(\phi/2)(k \cdot [e_t \times e_b])} \right].$$  \hfill (24)

In Fig. 4, we show the point-by-point distribution of the error as $\phi$ varies from $0^\circ$ to $360^\circ$ in the horizontal plane containing the transceiver and receiver by thin arrows.
Fig. 6. The same as Fig. 4, but for the type-2 error. The parameters are the same as in Fig.
2. In this figure, error ellipses (left) and simulations (right) are shown separately. Depicted in
the lower-left corner is an enlarged view of the part of the figure marked by a circle, wherein
32 arrows jut out of the ellipse, showing quite nice agreement of the simulation with the prediction.

Note that simulated error vectors, which will be explained
in the next section, are superimposed on the ellipses.

The heads of the arrows form an ellipse at each point.
The length of each arrow means standard deviation of
the error at the direction of the arrow.

At points far from and very close to the baseline, the
eccentricity of the ellipses is close to 1, suggesting high
anisotropy in the error distribution.
The lengths of the long and short axes are functions
of the scattering angles $\alpha$ and $\sigma_s$. The directions of these
axes relative to beam direction $e_1$ are functions of the
scattering angle, as shown in appendix B. Figure 5
shows the change in length and direction of the axes as
a function of scattering angle $\alpha$. The direction of the
short axis varies from $0^\circ$ to nearly $14^\circ$ and is approximately parallel to the beam direction.
As the scattering angle approaches 0, the length of the short axis
approaches $0.5 \text{ m/s}$, which equals the standard deviation
of the Doppler velocity. In other words, the error represented by the short axis approaches the error at the transceiver by backscattering for targets far from
the radar system. The direction of the long axis approaches $90^\circ$
to nearly $10^\circ$ and is approximately perpendicular
to the beam. The length of the longer axis becomes
infinite as the scattering angle approaches $0^\circ$ or $180^\circ$;
information in the velocity vector perpendicular to the
beam direction becomes poor at points close to the base-
line or far from it. At these points the direction between
the two Doppler velocities approaches $0^\circ$ or $90^\circ$.

b. Anisotropy of the type-2 error

Errors in the azimuth angle $\theta$ and the range $s_2$ are
assumed to obey a Gaussian random distribution with
standard deviations of $\sigma_\theta$ and $\sigma_{s_2}$, respectively. For this
case, the errors satisfy

$$\left(\frac{\delta\theta}{\sigma_\theta}\right)^2 + \left(\frac{\delta s_2}{\sigma_{s_2}}\right)^2 = 2.$$  \hspace{1cm} (25)

As before, the errors that occur with equal probability
are parameterized as

$$\delta\theta = \sqrt{2\sigma_\theta} \cos(\phi), \quad (0 \leq \phi \leq 2\pi), \quad (26)$$

$$\delta s_2 = \sqrt{2\sigma_{s_2}} \sin(\phi), \quad (0 \leq \phi \leq 2\pi). \quad (27)$$

Therefore, the type-2 error velocity vector can be written as

$$\delta \mathbf{v}_2 = \frac{\sqrt{2}\sigma_\theta}{\sin(\alpha) r_2} [\mathbf{k} \times \mathbf{e}_1] \cdot \mathbf{V} [\mathbf{k} \times \mathbf{e}_1] r_1 - [\mathbf{k} \times \mathbf{e}_1] \cdot \mathbf{V} [\mathbf{k} \times \mathbf{e}_1] \frac{\cos(\phi)}{\sin(\alpha)} r_2 \cos(\alpha/2) r_2 - [\mathbf{k} \times \mathbf{e}_1] \cdot \mathbf{V} [\mathbf{k} \times \mathbf{e}_1] \frac{\sin(\phi)}{\sin(\alpha)} r_2 \cos(\alpha/2). \quad (28)$$

The left-hand part of Fig. 6 shows the point-by-point
distribution of the error vectors. The type-2 errors are
highly anisotropic, are larger near the baseline, and rapidly
decrease far from the radar system. The error distribu-
tion also strongly depends on the wind field through the
factors $[\mathbf{k} \times \mathbf{e}_1] \cdot \mathbf{V}$ and $[\mathbf{k} \times \mathbf{e}_1] \cdot \mathbf{V}$ in (28).
In Fig. 6, the wind field is assumed to be uniform, with
wind direction northeast and wind speed $20 \text{ m/s}$.

4. Simulation of the error

To confirm the anisotropic error distribution, simu-
lations of synthesized wind vectors contaminated with
Gaussian random noise, as noted above, were performed. This is the double-checking of theoretical results. Below, in sections 4a and 4b, for clarity, simulated data are plotted on the same grid points as for the error ellipses in Figs. 4 and 6, and 100 samples are produced on each grid point. Grid points close to the baseline are omitted because of the very large error magnitude there. Probability theory, as

\[
\text{Erf}(1/\sqrt{2}) \approx 0.68, \quad (29)
\]

predicts that 68% of the simulated error vectors lie within the ellipse. In other words, 32 arrows out of 100 are expected to jut out of the ellipse. In section 4c, simulated error vectors are superposed on the original uniform wind field on the realistic grid points.

a. Simulation of type-1 error

A type-1 error simulation gives arrows plotted by thick lines over the ellipses, shown in Fig. 4. Approximately, 30% of the arrows protrude out of the ellipse at each grid point. In the lower-left corner of Fig. 4, we give a magnification of the part of the figure marked by a circle. Close examination of the figure shows that the heads of 33 arrows are observed to lie outside of the ellipse, showing remarkable agreement of the simulation with the prediction.

b. Simulation of type-2 error

The right-hand side of Fig. 6 shows an example of type-2 error simulation. In type-2 error, because the length of the short axis of the ellipse is much smaller than the long axis, it is difficult to see if the 70% of the simulated error vectors are included in the ellipse. The enlarged figure of simulated error vectors and the ellipse given in the lower-left corner of Fig. 6 shows that the heads of 32 arrows are located outside of the ellipse, also showing strikingly good agreement with the probability prediction. The error magnitude is large around the receiver, as expected from Fig. 2b.

c. Superposition of error vectors on the original uniform wind

Figure 7a plots wind vectors obtained by linearly adding the simulated error vectors to the original uniform northeasterly wind shown in Fig. 7b on realistic grids that are located at every 2° of azimuth angle and every 3 km of range. Data contamination is obvious. Spurious divergence/convergence couplets and anomalous rotation are common. Especially in the northeast part of the figure, where the original wind is nearly perpendicular to the longer axis of the type-1 error, contamination appears in the wind direction, causing spurious rotation and spurious divergence/convergence. In contrast, as the wind directions are nearly parallel to the longer axis in the southeast part of the figure, the wind directions are not greatly perturbed.
However, the wind speed is contaminated, which also results in spurious rotation and spurious divergence/convergence. The idea about the spurious rotation and divergence/convergence is depicted in Fig. 8.

5. Conclusions and discussion

The anisotropic character of errors associated with the finite resolution of Doppler velocities at both the transceiver and receiver, of the azimuth angle at the transceiver, and of the range at the receiver have been discussed. The error is divided into two categories. Errors in the first category (type-1 errors) are linked to the finite resolution of Doppler velocities. Errors in the second category (type-2 errors) are associated with the finite resolution of azimuth angles and ranges. If the errors occur randomly with Gaussian error distributions, for which the standard deviations are half of the resolution of the four parameters, the resultant error in the determined velocity vector is shown to have an elliptic distribution.

The direction of the long axis of the ellipse for type-1 errors is approximately perpendicular to the radar beam. The direction of the short axis is approximately parallel to the beam. For type-2 errors, the directions depend on both the wind vector and the location of the observation point.

Wind vectors obtained by this system are contaminated by errors yielding spurious divergence/convergence and spurious rotation, especially where the wind direction around the observation point is parallel or perpendicular to the beam. The direction of the short axis is approximately parallel to the beam. The direction of the long axis of the ellipse for type-2 errors must be similar, because vector \( d \) does not include type-2 error. We can determine the locations of the transceiver and receiver far more precisely than the location of the target (e.g., to an order of centimeters by using GPS). Therefore, we can assume

\[
\delta r_1 - \delta r_2 = 0. \tag{A3}
\]

Using (2), (A3) can be rewritten as

\[
\epsilon_1 \delta r_1 + r_1 \delta \epsilon_1 - \epsilon_2 \delta r_2 - r_2 \delta \epsilon_2 = 0. \tag{A4}
\]

The error in \( \epsilon_1 \) comes only from the error of azimuth angle \( \theta \):

\[
\delta \epsilon_1 = -[k \times \epsilon_1] \delta \theta. \tag{A5}
\]

Since the length of unit vector \( \epsilon_2 \) cannot change as a result of type-2 errors, the following equation holds:

\[
(\epsilon_2 \cdot \delta \epsilon_2) = 0. \tag{A6}
\]

Thus, the error component of \( \epsilon_2 \) that is normal to \( \epsilon_2 \) is obtained from the scalar product of (A4) and \([k \times \epsilon_2] \):

\[
([k \times \epsilon_2] \cdot \epsilon_1 \delta r_1 + r_1 \delta \epsilon_1) = r_2 ([k \times \epsilon_2] \cdot \delta \epsilon_2), \tag{A7}
\]

Using (A5), (A7) can be rewritten as

\[
-(k \cdot [\epsilon_1 \times \epsilon_2]) \delta r_1 - (\epsilon_1 \cdot \epsilon_2) r_1 \delta \theta = r_2 (k \times \epsilon_2) \cdot \delta \epsilon_2. \tag{A8}
\]

Using (3), (5), and (A6), (A8) yields an equation for \( \delta \epsilon_2 \):

\[
\delta \epsilon_2 = -\frac{\sin(\alpha) \delta r_1 + \cos(\alpha) r_1 \delta \theta}{r_2} [k \times \epsilon_2]. \tag{A9}
\]

Computing the scalar product of \( \epsilon_2 \) and (A4) and using (2), (5), (A5), and (A6), we obtain an equation for \( \delta r_2 \):

\[
\delta r_2 = \cos(\alpha) \delta r_1 - r_1 \sin(\alpha) \delta \theta. \tag{A10}
\]

From (6), we have a relationship involving \( \delta r_1 \) and \( \delta r_2 \), and \( \delta s_2 \):

\[
\delta r_1 + \delta r_2 = \delta s_2. \tag{A11}
\]

Using (A10) and (A11), we can relate \( \delta r_1 \) to the type-2 error variables \( \delta s_2 \) and \( \delta \theta \):

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This equation is used later in this appendix.

From (3) and using (A5) and (A9), we can derive an equation for \( \delta \alpha \):

\[
\delta \alpha = -\frac{[e_1 \cdot \delta e_2] + (e_2 \cdot \delta e_1)}{\sin(\alpha)} 
= \delta \theta \left[ 1 - \frac{r_1}{r_2} \cos(\alpha) \right] - \frac{\sin(\alpha)}{r_2} \delta r_1. \tag{A13}
\]

Substituting (A12) into (A9) and (A13), we can derive equations for the type-2 error of the variables \( e_2 \) and \( \alpha \):

\[
\delta e_2 = -\frac{1}{r_1} \left[ \frac{\sin(\alpha)/2}{\cos(\alpha)/2} \delta s_2 + r_1 \delta \theta \right] [k \times e_1]. \tag{A14}
\]

\[
\delta \alpha = \left[ 1 - \frac{r_1}{r_2} \right] \delta \theta - \frac{1}{r_2} \frac{\sin(\alpha)}{\cos(\alpha)} \delta s_2. \tag{A15}
\]

By rearranging the terms in (11), we can derive a simpler expression for the synthesized velocity vector:

\[
V = \frac{1}{\sin(\alpha)} \left( -(e_1 \cdot V) [k \times e_2] + (e_2 \cdot V) [k \times e_1] \right). \tag{A16}
\]

From (A16), and considering that the Doppler velocities at the transceiver and receiver do not change as a result of type-2 error, we can derive the following equation for the type-2 error vector:

\[
\delta V_2 = -(e_1 \cdot V) \delta \left[ \frac{[k \times e_2]}{\sin(\alpha)} \right] 
+ (e_2 \cdot V) \delta \left[ \frac{[k \times e_1]}{\sin(\alpha)} \right]. \tag{A17}
\]

Here, we give useful equations for later use:

\[
e1 = (e_1 \cdot e_1) e_1 + (e_1 \cdot [k \times e_1]) [k \times e_1] 
= \cos(\alpha) e_2 - \sin(\alpha) [k \times e_2], \tag{A18}
\]

\[
e2 = (e_2 \cdot e_1) e_1 + (e_2 \cdot [k \times e_1]) [k \times e_1] 
= \cos(\alpha) e_1 + \sin(\alpha) [k \times e_1]. \tag{A19}
\]

The error part of the first term in (A17) is

\[
\delta \left[ \frac{[k \times e_2]}{\sin(\alpha)} \right] = -\frac{\cos(\alpha)}{\sin(\alpha)} [k \times e_2] \delta \alpha 
+ \frac{1}{\sin(\alpha)} [k \times \delta e_2]. \tag{A20}
\]

Substituting (A14) and (A15) into (A20) and using (A18) yields, after a lengthy calculation,

\[
\delta \left[ \frac{[k \times e_1]}{\sin(\alpha)} \right] = \frac{1}{\sin^2(\alpha)} \left[ \frac{r_1}{r_2} [k \times e_1] - \cos(\alpha) [k \times e_2] \right] \delta \theta 
+ \frac{1}{\sin^2(\alpha)} \frac{1}{r_2} \cos(\alpha/2) [k \times e_1] \delta s_2. \tag{A21}
\]

The error in the second term of (A17) is calculated similarly:

\[
\delta \left[ \frac{[k \times e_1]}{\sin(\alpha)} \right] = -\frac{\cos(\alpha)}{\sin^2(\alpha)} [k \times e_1] \delta \alpha 
+ \frac{1}{\sin^2(\alpha)} \left[ -\cos(\alpha) [k \times e_1] \delta \alpha 
+ \sin(\alpha) [k \times \delta e_1] \right]. \tag{A22}
\]

Substituting (A5) and (A15) into (A22) and using (A19) yields

\[
\delta \left[ \frac{[k \times e_1]}{\sin(\alpha)} \right] = \frac{1}{\sin^2(\alpha)} \left[ \frac{r_1}{r_2} \cos(\alpha) [k \times e_1] - [k \times e_2] \right] \delta \theta 
+ \frac{1}{\sin^2(\alpha)} \frac{1}{r_2} \cos(\alpha/2) \cos(\alpha) [k \times e_1] \delta s_2. \tag{A23}
\]

Substituting (A21) and (A23) into (A17) gives the following expression for type-2 error:

\[
\delta V_2 = -(e_1 \cdot V) \delta \left[ \frac{[k \times e_1]}{\sin(\alpha)} \right] 
+ (e_2 \cdot V) \delta \left[ \frac{[k \times e_1]}{\sin(\alpha)} \right] 
= -(e_1 \cdot V) \frac{r_1}{\sin^2(\alpha)} [k \times e_1] \delta \alpha 
+ \frac{(e_1 \cdot V)}{\sin^2(\alpha)} \left[ [k \times e_1] - \cos(\alpha) [k \times e_2] \right] \delta \theta 
+ \frac{(e_2 \cdot V)}{\sin^2(\alpha)} \left[ \frac{r_1}{r_2} \cos(\alpha) [k \times e_1] - [k \times e_2] \right] \delta \theta 
+ \frac{(e_2 \cdot V)}{\sin^2(\alpha)} \frac{1}{r_2} \cos(\alpha/2) \cos(\alpha) [k \times e_1] \delta s_2. \tag{A24}
\]

Again, by using (A18) and (A19), (A24) is simplified to

\[
\delta V_2 = \left[ \left[ \frac{(k \times e_1) \sin(\alpha) \cdot V}{\sin(\alpha)} \right] \frac{r_1}{r_2} [k \times e_1] - \left[ \frac{(k \times e_1) \sin(\alpha) \cdot V}{\sin(\alpha)} \right] [k \times e_1] \delta \alpha 
- \frac{1}{\sin^2(\alpha)} \left( (k \times e_1) \cdot V \right) r_1 [k \times e_1] 
- (k \times e_1) \cdot V r_2 [k \times e_2] \right] \delta \theta 
+ \frac{1}{\sin^2(\alpha)} \frac{1}{r_2} \cos(\alpha/2) \cos(\alpha) [k \times e_1] \delta s_2. \tag{A25}
\]

which is equivalent to (14).
APPENDIX B

An Elliptic Type-1 Error Distribution

In this appendix, we show that (24) gives an elliptic error distribution when angle $\phi$ varies from $0^\circ$ to $360^\circ$. For this purpose, we break the type-1 error vector into two components: one parallel to the beam direction and one normal to the beam. The component along the beam direction, $A_1$, is obtained from the scalar product of the error vector and the unit vector $e_1$:

$$A_1 = (\delta V_1(\phi) \cdot e_1) = \sqrt{2} \sigma_v \left[ -\cos(\phi) \frac{([k \times e_3] \cdot e_1)}{(k \cdot [e_1 \times e_2])} \right] = \sqrt{2} \sigma_v \cos(\phi). \quad (B1)$$

The component normal to the beam, $A_2$, is given by

$$A_2 = (\delta V_1(\phi) \cdot [k \times e_1]) = \sqrt{2} \sigma_v \left[ -\cos(\phi) \frac{([k \times e_3] \cdot [k \times e_1])}{(k \cdot [e_1 \times e_2])} + \sin(\phi) \frac{([k \times e_3] \cdot [k \times e_1])}{\cos(\alpha/2)(k \cdot [e_1 \times e_2])} \right]. \quad (B2)$$

Equations (B1) and (B2) can be written in matrix form as

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \sqrt{2} \sigma_v \begin{bmatrix} 1 & 0 \\ \cos(\alpha/2) & \sin(\alpha/2) \end{bmatrix} \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix}. \quad (B3)$$

The inverse of (B3) is given by

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix} \frac{1}{\sqrt{2} \sigma_v} \begin{bmatrix} 1 & 0 \\ \cos(\alpha/2) \sin(\alpha/2) \end{bmatrix}. \quad (B4)$$

Transposed, (B4) becomes

$$\begin{bmatrix} \cos(\phi) & \sin(\phi) \end{bmatrix} = [A_1 \ A_2] \begin{bmatrix} 1 \\ \cos^2(\alpha/2) \cos(\alpha/2) \sin(\alpha/2) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}. \quad (B5)$$

Then, the scalar product of (B4) and (B5) yields the following quadratic form of $A_1$ and $A_2$:

$$1 = [\cos(\phi) \sin(\phi)] \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix} = [A_1 \ A_2] \begin{bmatrix} 1 & \cos^2(\alpha/2) \cos(\alpha/2) \sin(\alpha/2) \end{bmatrix} \begin{bmatrix} 1 \\ \cos^2(\alpha/2) \cos(\alpha/2) \sin(\alpha/2) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

$$= [A_1 \ A_2] \begin{bmatrix} 1 + \cos^4(\alpha/2) \cos^2(\alpha/2) \sin^2(\alpha/2) & \cos^2(\alpha/2) \sin^2(\alpha/2) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}. \quad (B6)$$

The matrix in (B6) is symmetric, and after an appropriate orthogonal transformation,

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} e_{11} & e_{21} \\ e_{12} & e_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad (B7)$$

it can be diagonalized.

The diagonalized expression of (B6) can be written as

$$1 = \frac{1}{2\sigma_v^2} \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad (B8)$$

where eigenvalues $E_1$ and $E_2$ are

$$E_1 = \frac{1}{2} \left( 1 - \frac{\sin^2(\alpha/2)}{2} + \frac{\sqrt{4 - 8 \sin^2(\alpha/2) + 5 \sin^4(\alpha/2)}}{2} \right) \quad \text{and} \quad E_2 = \frac{1}{2} (\cos^2(\alpha/2) - 1) \quad (B9)$$
The corresponding normalized eigenvectors are represented by the following equations:

\[ e_{11} = \frac{1 - \sin^2(\alpha/2) - \sqrt{4 - 8 \sin^2(\alpha/2) + 5 \sin^4(\alpha/2)}}{2} \]

\[ e_{12} = \frac{2 \cos^3(\alpha/2) \sin(\alpha/2)}{\sqrt{2[1 - 2 \cos^2(\alpha/2) + 5 \cos^4(\alpha/2) + [1 - \cos^2(\alpha/2) + 2 \cos^4(\alpha/2)] \sqrt{1 - 2 \cos^2(\alpha/2) + 5 \cos^4(\alpha/2)}}} \]  
\[ e_{21} = \frac{2 \cos(\alpha/2) \sin(\alpha/2)}{\sqrt{2[1 - 2 \cos^2(\alpha/2) + 5 \cos^4(\alpha/2) + [1 - \cos^2(\alpha/2) + 2 \cos^4(\alpha/2)] \sqrt{1 - 2 \cos^2(\alpha/2) + 5 \cos^4(\alpha/2)}}} \]  
\[ e_{22} = \frac{1 - \cos^2(\alpha/2) + 2 \cos^4(\alpha/2) + \sqrt{1 - 2 \cos^2(\alpha/2) + 5 \cos^4(\alpha/2)}}{\sqrt{2[1 - 2 \cos^2(\alpha/2) + 5 \cos^4(\alpha/2) + [1 - \cos^2(\alpha/2) + 2 \cos^4(\alpha/2)] \sqrt{1 - 2 \cos^2(\alpha/2) + 5 \cos^4(\alpha/2)}}} \]

We can transform (B8) into the standard form of an equation for an ellipse:

\[ \frac{a_1^2}{(\sqrt{2} \sigma_v)}^2 + \frac{a_2^2}{(\sqrt{2} \sigma_v)}^2 = 1. \]  

(B15)

It is clear that \( E_1 \) is larger than \( E_2 \). The lengths of the short and long axes of the ellipse are

\[ l_1 = \frac{\sqrt{2} \sigma_v}{\sqrt{E_1}} = \sqrt{\frac{2 \sin^2(\alpha/2) + \sqrt{4 - 8 \sin^2(\alpha/2) + 5 \sin^4(\alpha/2)}}{2}} \]

(B16) and

\[ l_2 = \frac{\sqrt{2} \sigma_v}{\sqrt{E_2}} = \sqrt{\frac{2 \sin^2(\alpha/2) - \sqrt{4 - 8 \sin^2(\alpha/2) + 5 \sin^4(\alpha/2)}}{2}} \]

(B17)

From (B16), (B17), and (17), we derive the following interesting relationship:

\[ \frac{l_1^2 + l_2^2}{2} = 2\sigma_v^2 \left[ \frac{3 + \cos(\alpha)}{\sin^2(\alpha)} \right] = (\delta V_1 \cdot \delta V_1). \]

(B18)

Simultaneously, we obtain

\[ \frac{1}{2\pi} \int_0^{2\pi} (\delta V_1(\phi) \cdot \delta V_1(\phi)) \, d\phi = 2\sigma_v^2 \int_0^{2\pi} \left[ \cos(\phi) \frac{[k \times e_\perp]}{(k \cdot [e_\perp \times e_\parallel])} \right]^2 + \left[ \sin(\phi) \frac{[k \times e_\parallel]}{\cos(\alpha/2)(k \cdot [e_\parallel \times e_\perp])} \right]^2 \, d\phi \]

\[ = \sigma_v^2 \left[ \frac{[k \times e_\perp]}{(k \cdot [e_\perp \times e_\parallel])} \right]^2 + \left[ \frac{[k \times e_\parallel]}{\cos(\alpha/2)(k \cdot [e_\parallel \times e_\perp])} \right]^2 \]  
\[ = 2\sigma_v^2 \left[ \frac{3 + \cos(\alpha)}{\sin^2(\alpha)} \right]. \]

(B19)

The angles of the short and long axes anticlockwise from the beam direction are given by

\[ \tan(\angle_1) = \frac{e_{12}}{e_{11}}, \]  
\[ \tan(\angle_2) = \frac{e_{22}}{e_{21}}. \]

(B20)

(B21)

Equations (B16), (B17), (B20), and (B21) give Fig. 5.

The discussion of the type-2 error parallels the above discussion but is omitted because of extreme length.
REFERENCES