An Approximation to the Effective Beam Weighting Function for Scanning Meteorological Radars with an Axisymmetric Antenna Pattern

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ABSTRACT

To obtain statistically stable reflectivity measurements by meteorological radars, it is common practice to average over several consecutive pulses during which the antenna rotates at a certain angular velocity. Taking into account the antenna’s continuous motion, the measured reflectivity is determined by an effective beam weighting function, which is different from a single-pulse weighting function—a fact that is widely ignored in applications involving beam weighting. In this paper, the effective beam weighting function is investigated in detail.

The theoretical derivation shows that the effective weighting function is essentially a simple moving sum of single-beam weighting functions. Assuming a Gaussian shape of a single pulse, a simple and easy-to-use parameterization of the effective beam weighting function is arrived at, which depends only on the single beamwidth and the ratio of the single beamwidth to the rotational angular averaging interval. The derived relation is formulated in the “radar system” (i.e., the spherical coordinate system consisting of azimuth and elevation angles) that is often applied in practice. Formulas for the “beam system” (two orthogonal angles relative to the beam axis) are also presented.

The final parameterization should be applicable to almost all meteorological radars and might be used (i) in specialized radar data analyses (with ground-based or satellite radars) and (ii) for radar forward operators to calculate simulated radar parameters from the results of NWP models.

1. Introduction

The reflectivity measured by a meteorological radar is a weighted average over a certain measuring volume. To obtain statistically stable measurements of that echo power, the common technique is to average over several consecutive pulses during which the radar antenna continues its azimuthal or elevational rotation. Because of the antenna’s motion, the consecutive radar pulses do not illuminate the same radar volume—as in the case of pulses released in a fixed direction—but a larger one. The advantage of the continuous scanning is a considerable increase in the number of independent pulses at the expense of a measurement in an identical volume. Averaging over such consecutive measurements leads to a spatial broadening of the sampling volume. In this paper, a simple approximation of the corresponding broader beam weighting function (hereafter called the effective beam weighting function), which results from the consideration of a continuous motion of a radar antenna, is presented.

The effective beam pattern approximation derived here can be used for applications such as specialized radar data analyses and algorithms to calculate radar-measurable parameters from the output of atmospheric simulation models (“radar forward operators”) for four-dimensional variational data assimilation (4DVAR), for example. Note that at present it is customary to consider only the beam weighting function of a single pulse. Moreover, few investigators in the meteorological community seem to have recognized the problem at all. It is not considered in the radar forward operators of Xue et al. (2006) and Meetschen et al. (2000), among others. Also, in the specialized radar data analyses of Gosset and Zawadzki (2001), Delrieu et al. (1995), and Wood and Brown (1997)—just to name a few—which all involve consideration of the beam weighting function, the effective antenna pattern is not taken into account. To give an illustrative example, if, in a case of stratiform precipitation, the...
The effective beam weighting function has previously been investigated by Zrnić and Doviak (1976) and Doviak and Zrnić (1993, hereafter DZ93) but they merely considered the beam function as a one-dimensional function (one angle coordinate), which is more suited to analog averaging techniques than to the digital signal processing of today’s radars. In this paper, the full two-dimensional geometry of the beam pattern is considered.

2. General description of the problem

The power $P_r$ received from a single radar pulse, which is attributed to the so-called pulse volume around its center at locations ($r_0$, $\alpha_0$, $\epsilon_0$), is described by the radar equation

$$P_r(r_0, \alpha_0, \epsilon_0) = C \int_{r_0}^{r_0+c/4} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi/2} \eta(r, \phi, \theta) f^4(\phi, \theta) \frac{\exp\left[-\frac{4}{r^2} \cos\theta \cos\phi \cos\epsilon\right]}{r^2} \, dr \, d\phi \, d\theta,$$  \hspace{1cm} (1)

where $r_0$ is distance between the antenna and the center of the pulse volume, and $\alpha_0$ and $\epsilon_0$ are the antenna’s azimuth and elevation angles, respectively. Parameter $C$ contains the specifics of the radar (e.g., transmitted power, antenna gain, radar wavelength and system loss factors, and attenuation by radome) as well as the attenuation by atmospheric gases, which is a function of range (see, e.g., DZ93); $c$ denotes the speed of light, $\tau$ is the pulse duration, $\eta$ is the reflectivity, and $f_N$ is the attenuation factor resulting from hydrometeors present along the ray path.

As usual, the geometric dependence of quantities determining $P_r$ is described in terms of a spherical coordinate system centered around the radar itself with radial distance $r$ and angles $\phi$ and $\theta$ with respect to the beam center (“beam system”). This is different from the coordinate system ($r, \alpha, \epsilon$, with $\alpha$ and $\epsilon$ being the azimuth and elevation angles, respectively), which we call the “radar system.”

Figure 1 presents a simplified sketch of a single radar beam, the pulse volume, the coordinates $r_0$, $\alpha_0$, and $\epsilon_0$ of the beam center relative to the radar system and the coordinate directions $\phi$ and $\theta$ relative to the beam system. (Figure 3 is more rigorous and shows the exact coordinates, but it is more complicated to interpret. It will be explained in more detail in section 3.)

An essential part of Eq. (1) is the beam weighting function $f^2$ that describes the amount of power transmitted (and received) by a finite-sized radar antenna in a certain direction. Accordingly, its square is the weight at which local reflectivity (and attenuation) contributes to the received power. For meteorological applications, usually the main part of the power is confined to a narrow angular region near the beam center, which, in the antenna far-field, can be well approximated by (Probert-Jones 1962; Sauvageot 1992)

$$f^2(\phi, \theta) = \exp\left[-4 \ln 2 \left(\frac{\phi^2}{\phi_3^2} + \frac{\theta^2}{\theta_3^2}\right)\right].$$  \hspace{1cm} (2)

Here, contributions from sidelobes are neglected, which is justified in the context of this paper because of their usually very low power level.

For symmetric antennas, the $-3$-dB values $\phi_3$ and $\theta_3$ are equal, so—by taking into account that $\phi_3$ and $\theta_3$ typically have small values ($\approx 1^\circ$)—Eq. (2) reduces to the usual form

$$f^2(\theta) = \exp\left[-4 \ln 2 \left(\frac{\theta^2}{\theta_3^2}\right)\right].$$  \hspace{1cm} (3)

with $\theta_3^2 = \phi_3^2 + \theta_3^2$. However, in the following only Eq. (2) will be used. In the literature, the $f^2$ contribution is usually regarded as being restricted to angles smaller than $-3$ dB, the width of the radar beam where $\approx75\%$ of the transmitted power is concentrated.

In Eq. (1), the characteristics of the radar beam are contained in the term $f^2 \cos\theta$ regarding angular-dependent contributions and in a range weighting function $W(r)$ (see, e.g., DZ93) in the along-beam direction.
divided by $r^2$. To simplify notation, $W(r)$ has been approximated by a simple step function in Eq. (1) comprising the $r$ interval $[r_0 - c\sigma/4, r_0 + c\sigma/4]$, which is only valid for an infinite bandwidth receiver. In doing so, $W(r)$ does not appear explicitly in (1). To consider $W(r)$ in general form, one would have to add it as a multiplicative term to the integrand in (1) and extend the limits of integration for $r$ to [0, $\infty$] in (1) and in the follow-up integral equations below. A more realistic form of $W(r)$ (a “matched filter”) is treated in detail in textbooks (e.g., DZ93). Because range weighting and angular weighting are independent (multiplicative combination), the exact form of $W(r)$ is unimportant for the following considerations and derivations.

In the following discussion we neglect the fact that electromagnetic rays do not travel along straight lines but usually are bent toward the earth’s surface because of the decreasing refractive index of the atmosphere. However, that also does not affect the presented results very much as long as adjacent portions of the radar beam are similarly bent and the pulse does not “diverge.” The latter is known to happen only in the case of a strong inversion at the height of the radar antenna with $|\epsilon_0| \leq 1^\circ$. Also, attenuation by hydrometeors does not play a role in the beam weighting function and will be neglected (i.e., $I_p = 1$).

Let $Z_e^*$ be the “true” effective radar reflectivity field, $Z_e^{(R)}$ an instantaneous value derived by the radar processor from a single measurement of received power and $Z_e^{(R)}$ an average over several consecutive pulses, which finally is recorded as one measured value. Further, assume that the radar is scanning horizontally at constant elevation $\epsilon_0$, which relies on the common practice in setting up radar schedulers scanning azimuthally in a continuous mode with a discrete change in elevation after finishing a 360° revolution. Then, a detailed derivation presented in appendix A shows that the relation of the radar averaged $\overline{Z_e^{(R)}}$ and the true $Z_e$ field, involving the effective beam weighting function $\overline{I^4}$, is given by

$$\overline{Z_e^{(R)}}(r_0, \alpha_0, \epsilon_0) = \frac{r_0^2 16 \ln 2}{c^2 \pi \phi_3 \theta_3} \times \int_{0}^{r_0} \int_{-\pi + \alpha_0}^{\alpha_0 + \pi} \int_{-\pi/2 + \epsilon_0}^{\pi/2 + \epsilon_0} Z_e(r, \alpha, \epsilon) \left[ \frac{1}{N} \sum_{i=0}^{N-1} f^4(\alpha - \alpha^{(i)}_0, \epsilon - \epsilon_0) \right] r^2 \cos \epsilon \ d\alpha. \quad (4)$$

Note that the formulation has been switched to the radar coordinate system to describe the angular averaging correctly in terms of azimuth. It is emphasized that the formulation of Eq. (4) relative to the radar system is necessary because radar data are recorded and averaged with regard to this system. Because both radar and beam coordinate systems are spherical, the functional determinant $r^2 \cos \theta$ in Eq. (1) formally changes to $r^2 \cos \epsilon$, leading to the $\cos \epsilon$ term in Eq. (4). As described in appendix A, $\alpha^{(i)}_0$ denotes the azimuthal angle of the beam center of each single measured $Z_e^{(R)}$, where an equidistant distribution of $\alpha^{(i)}_0$ over the averaging interval $\Delta \alpha$ is assumed (see Fig. 2). The angle $\alpha_0$ henceforth denotes the center of the averaging interval. The dependence $f^4 = f^4(\alpha - \alpha^{(i)}_0, \epsilon - \epsilon_0)$ is an expression of the fact that the region providing contributions to the beam weighting function $f^4$ gradually shifts because of the radar’s motion from pulse to pulse. The arithmetic average $\overline{I^4}$ defines the effective beam weighting function of a scanning radar, which is also depicted in Fig. 2.

A more general formulation than Eq. (4) can be derived as follows: because of the folding of $\alpha$ into $[\alpha_0^{(i)} - \pi, \alpha_0^{(i)} + \pi]$ and $[\alpha_0 - \pi, \alpha_0 + \pi]$, it is exactly
 emphasis the fact that the azimuthally averaged measured $\overline{Z_e}$ has to be interpreted as a weighted average over $Z_e(r,\alpha,\epsilon)$ with the weighting function $\bar{f}^4 \cos \epsilon/r^2$. Note again that the range weighting function is assumed to be a simple step function here. This resembles the double overbar operator (A5), but with $\overline{f}^4$ replacing $\bar{f}^4$.

Equations similar to (4) and (5) may be applied as a radar forward operator to calculate radar reflectivity (or any other radar-measurable quantity) from the numerical output of atmospheric simulation models, using the radar or beam coordinate systems. From a rigorous standpoint, Eq. (5) should be the most general and unproblematic formulation because it contains no radar-specific constants.

After these quite general considerations, we now focus on the question of how to calculate $\bar{f}^4$. The basis is the formula for a single-beam pattern $f^4$. For narrow beams near the equator of the spherical coordinate system (low elevations), a simple Gaussian distribution, as in the beam system, can be used to describe $f^4(\alpha - \alpha_0)$. For large elevation angles this is no longer the case, which becomes obvious in the next section. Instead, in the case of beamwidths as small as those used for meteorological radars, it is possible to avoid the calculation of Fisher distribution values even for larger elevations; it suffices to use a Gaussian distribution for $f^4$ relative to the beam coordinate system (maximum of $f^4$ at the beam system equator) and transform it to the radar system using appropriate transformation formulas, as will be done in this paper. Following this method, rather simple approximate parameterization formulas for $\bar{f}^4$ are finally obtained based on an asymmetric Gaussian function, which is broadened in the direction of antenna rotation. Note that because of this asymmetry, it is not possible to use a Fisher distribution.

To proceed, transformation formulas $(\phi, \theta) \rightarrow (\alpha, \epsilon)$ are needed to explicitly express $f^4(\phi, \theta)$ in the radar system (as a general field function, $Z_e$ does not need to be transformed explicitly). A suitable form of the transformation will be given in the next section along with appropriate approximations, which are basically not new but seem to be seldom applied or elaborated in the literature.

### 3. Transformation between the beam system and the radar system

The required transformation between the beam system and the radar system can be determined, for example, with the help of Fig. 3, and applying the laws of sines and cosines of spherical trigonometry and neglecting beam refraction (adjacent beams in elevation usually bend similarly; therefore, the transformation remains valid in a relative sense). Figure 3 (top) depicts an arbitrary point S in space and its coordinates $(r, \phi, \theta)$...
relative to the beam center as well as \( r, \alpha, \epsilon \) relative to the horizontal plane through the radar site \( O \), where \( r \) is the radial distance from the radar site \( O \) and \( A, P, \) and \( Q \) are the corners of two auxiliary spherical triangles, which are also shown separately in the two lower figures. All solid circles are great circles and the dashed circle denotes the latitudinal circle in the beam system through point \( S \). For clarity, intersection points of circles are marked by black dots.

Both the beam and radar systems have the same origin, therefore only the angle coordinates have to be transformed. Without loss of generality (W.l.o.g.) computations are done on the unit sphere with radius 1. In triangle \( APS \) we have, with \( b = \frac{\pi}{2} - \phi, q = \theta, \)

\[
\cos p = \cos b \cos q + \sin b \sin q \cos \frac{\pi}{2} \Rightarrow p = \arccos(\sin\phi \cos\theta),
\]

\[
\cos q = \cos b \cos q + \sin b \sin q \cos\beta
\]

\[\Rightarrow \cos\beta = \frac{\cos\theta - \sin\phi \cos p}{\cos\theta \sin p}.\]

In triangle \( AQS \) we have, with \( a = \frac{\pi}{2} - (\alpha - \alpha_0), d = \epsilon, \)

\[
\frac{\sin d}{\sin(\epsilon_0 + \beta)} = \frac{\sin p}{\sin(\frac{\pi}{2})} \Rightarrow \epsilon = \arcsin(\sin\epsilon \sin(\epsilon_0 + \beta)),
\]

\[
\cos p = \cos d \cos a + \sin d \sin a \cos \frac{\pi}{2}
\]

\[\Rightarrow \alpha - \alpha_0 = \arcsin\left(\frac{\cos p}{\cos\epsilon}\right).\]

For \( \alpha \) and \( \epsilon \), as a function of \( \phi \) and \( \theta \), one obtains

\[
\alpha = \alpha_0 + \arcsin\left\{\frac{\sin\phi \cos\theta}{\cos\left[\arcsin\left(\sin[\arccos(\sin\phi \cos\theta)]\right) \times \sin[\epsilon_0 + \arccos\left(\frac{\cos\theta \cos\phi}{\sin[\arccos(\sin\phi \cos\theta)]}\right)]\right]}\right\}
\]

\[
\epsilon = \arcsin\left[\sin[\arccos(\sin\phi \cos\theta)] \sin[\epsilon_0 + \arccos\left(\frac{\cos\theta \cos\phi}{\sin[\arccos(\sin\phi \cos\theta)]}\right)]\right].
\]
Note that the arccos function here is needed for the codomain $[-\pi/2, \pi/2]$.

Because of symmetry, the reverse transformation $(r, \alpha, \epsilon) \rightarrow (r, \phi, \theta)$ results from formulas (6) and (7) by interchanging $\phi$ and $\alpha - \alpha_0, \epsilon_0$ and $-\epsilon_0$, as well as $\theta$ and $\epsilon$.

As a cross check, the Jacobian of the transformation can be computed in a lengthy but straightforward calculation from (6) and (7) to be

$$\frac{\partial (\alpha, \epsilon)}{\partial (\phi, \theta)} = \cos \theta \cos \epsilon,$$

which yields $\cos \epsilon \, d\alpha \, d\epsilon = \cos \theta \, d\phi \, d\theta$, as was previously applied in Eq. (4), and which was to be expected because both systems are spherical. It is helpful for this calculation to first substitute the denominator of the arccos term of (6) by $\cos \epsilon$ before calculating the derivatives of $\alpha$ and $\epsilon$ by applying chain and quotient rules.

The exact (and complicated) formulas (6) and (7) can be approximated for the case $\phi \ll 1$ (i.e., a narrow beam) by setting $\sin \phi \approx \phi \ll 1$, $\cos \phi \approx 1$ and considering the fact that the arccos of a small argument equals approximately $\pi/2$. Thus,

$$f^4(\alpha - \alpha_0, \epsilon - \epsilon_0) = \exp\left(-8 \ln2\left[\left(\frac{(\alpha - \alpha_0) \cos \epsilon}{\phi_3}\right)^2 + \left(\frac{\epsilon - \epsilon_0}{\theta_3}\right)^2\right]\right),$$

because, for realistic values of $\phi_3, \theta_3,$ and $\Delta \alpha$ (contained in the $\alpha_0^{(i)}$) of around 1° each, this Gaussian only differs significantly from 0 for $\phi, \theta < 5°$. The reference angle $\alpha_0$ in Eq. (10) has been replaced by $\alpha_0^{(i)}$, which, again, denotes the center angle of a single beam out of the averaging ensemble [see Eq. (A8) and Fig. 2] and may slightly differ from $\alpha_0$. The term $\cos \epsilon$ may be approximated by $\cos \epsilon_0$ and added to the horizontal beamwidth $\phi_3$, which leads to the above-mentioned ostensible broadening of the beam weighting function in the radar system. Note that the $\alpha$ coordinate has to be folded in the ranges $\lbrack \alpha_0^{(i)} - \pi, \alpha_0^{(i)} + \pi\rbrack$ (single beam) and $\lbrack \alpha_0 - \pi, \alpha_0 + \pi\rbrack$ (effective beam).

Equation (11) may now be used to explicitly calculate the (approximate) effective beam weighting function $\tilde{f}^4$ in Eqs. (4) or (5) as an arithmetic mean:

$$\tilde{f}^4(\alpha - \alpha_0, \epsilon - \epsilon_0) = \frac{1}{N} \sum_{i=0}^{N-1} f^4(\alpha - \alpha_0^{(i)}, \epsilon - \epsilon_0).$$

(12)

Note again that sidelobes are not considered here. However, because these secondary maxima of $f^4$ are usually less than 1/100 of the main lobe maximum (−20 dB), they would not contribute considerably to the shape of $\tilde{f}^4$. Taking into account the sidelobes in $\tilde{f}^4$ would lead to an effective pattern that also contains sidelobes in an averaged and broadened fashion, having the same relative power level to the main lobe as in the single-beam pattern.

Because Eq. (12) might still be too costly for practical applications, a simple approximation based on a single Gaussian function is derived in the next section. It is limited to the same constraints of elevation angle and beamwidth as formulas (9), (10), and (11) and also contains no sidelobes. Because sidelobes can be different for different radar systems, a parameterization formula for the effective beam pattern would be more complicated to formulate and would be only of minor practical significance for most applications.

4. Simple approximation to the effective beam weighting function

In this section, an approximation for $\tilde{f}^4$ is developed as a function of the parameters $\phi_3, \theta_3,$ and $\Delta \alpha$. Numeri-
of $\Delta \alpha$, $\theta_3$, and $\epsilon_0$ useable in Eq. (4), first some general features have to be investigated with the help of the above-mentioned numerical calculations of $\tilde{f}^4$.

First, the quarter-widths of $\tilde{f}^4$ in the radar coordinate system ($\alpha_{3,\text{eff}}$, $\epsilon_{3,\text{eff}}$) will be discussed, which correspond to the half-width $\theta_3$ of $f^2$ in cases where the Gaussian model is applied for both. W.l.o.g., $\alpha_{3,\text{eff}}$ is discussed in the following because azimuthal antenna rotation [planned-position indicator (PPI) scan] is assumed for the exemplary radar above, at which $\epsilon_{3,\text{eff}}$ is essentially the same as $\theta_3$. The application to RHI scans (elevation rotation) will be discussed at the end of this section.

Considerations of symmetries lead to the conclusion that at fixed $\epsilon_0$, the general shape of $\tilde{f}^4$ as a function of the dimensionless parameter $\alpha_{3,\text{eff}}/\theta_3$ depends only on the ratio $\Delta \alpha/\theta_3$. This could be verified by numerical calculations with meteorologically meaningful parameter sets different from the exemplary radar above, which always lead to the same functional relation for the same $\epsilon_0$. Figure 5 (left panel) shows $\alpha_{3,\text{eff}}/\theta_3$ as a function of $\Delta \alpha/\theta_3$ for different values of the elevation angle $\epsilon_0$. Note that the values at $\Delta \alpha/\theta_3 = 0$ correspond to a single ray. The middle panel depicts the dependence on $\epsilon_0$ at a fixed value of $\alpha_{3,\text{eff}}/\theta_3 = 0.1$, and the right panel shows the same at $\alpha_{3,\text{eff}}/\theta_3 = 1$. Here, the solid lines correspond to the numerically calculated $\tilde{f}^4$ and the dashed lines to the analytical approximation [Eq. (17)], which will be derived and discussed in the following. Numerical values of $\alpha_{3,\text{eff}}$ are given in Table 1 for $\epsilon_0 = 0^\circ$, $20^\circ$, and $40^\circ$ and for a wide range of $\alpha_{3,\text{eff}}/\theta_3$.

One can see that at an elevation of $0^\circ$ and $\Delta \alpha/\theta_3 = 1$, the effective quarter-width $\alpha_{3,\text{eff}}$ is larger by a factor of 1.45 than $\theta_3$. The dependence on elevation stems from the coordinate transformation of $f^2$ into the radar system, which is done before averaging to $\tilde{f}^4$. However, this effect is only important for small $\Delta \alpha/\theta_3$, which seems at first glance to contradict the $1/\cos \epsilon$ broadening of the single ray $f^4$. The cause is that $\alpha_{3,\text{eff}}$ is comprised of a part from $\theta_3$ and a part from $\Delta \alpha$. The $\Delta \alpha$ part decreases at nearly the same rate as the “coordinate transfer” broadening increases, which leads to compensation of both effects for higher $\Delta \alpha/\theta_3$.

An intermediate case is shown in Fig. 6 for the exemplary radar ($\Delta \alpha/\theta_3 = 1$) and $\epsilon_0 = 40^\circ$. Figure 6 differs from Fig. 4 only by $\epsilon_0$. The effect of the ostensible azimuthal broadening at higher elevations is clearly visible, but an approximation by a simple Gaussian is still applicable.

The numerical calculations revealed further that approximation by a Gaussian (as shown in Figs. 4 and 6) is justified if $\Delta \alpha/\theta_3$ is smaller than approximately 1.5. At
larger values, \( \tilde{f}^4 \) becomes more and more cuboidal (see Fig. 7, which is the same as Figs. 4 and 6, but for \( \Delta \alpha/\theta_3 = 5 \)). For most meteorological radars on the market, however, a Gaussian approximation, which will be derived below, remains sufficient.

The general formulation Eq. (5) shows that because of the normalization that is done there, the absolute values of \( \tilde{f}^4 \) are not important; using a function that is proportional to that is sufficient. Therefore, the ansatz for the approximation formula is

\[
\tilde{f}^4 = \exp \left( -8 \ln2 \left\{ \left( \frac{\alpha - \alpha_0}{\phi_3,\text{eff}} \right) \cos \theta_3 + \left( \frac{\epsilon_3,\text{eff} - \epsilon_0}{\theta_3,\text{eff}} \right) \right\}^2 \right),
\]

(13)

where \( \phi_3,\text{eff} \) and \( \theta_3,\text{eff} \) are the quarter-widths of \( \tilde{f}^4 \) in the beam coordinate system. Note that for use in Eq. (4), whose detailed derivation is presented in appendix A, Eq. (13) would additionally need proper normalization to ensure that (4) is an unbiased estimate of \( Z_{\text{RR}} \) from the single \( Z_{\text{R}}(\theta) \) measurements. This is due to the approximation to the coordinate transformation in (13) and the beam function integral in the prefactor of Eq. (4). As mentioned earlier, to circumvent these issues the more general formulation Eq. (5) should be preferred.

If we use the numerically derived radar system values of \( \alpha_{3,\text{eff}} \) and \( \epsilon_{3,\text{eff}} \) in (13), valid for certain \( \Delta \alpha/\theta_3 \) and \( \epsilon_0 \), then it follows from the coordinate transformation (9) and (10) that

\[
\phi_{3,\text{eff}} = \alpha_{3,\text{eff}} \cos \epsilon_0 \quad \text{and} \quad \theta_{3,\text{eff}} = \epsilon_{3,\text{eff}}.
\]

This form has the disadvantage that \( \alpha_{3,\text{eff}} \) and \( \epsilon_{3,\text{eff}} \) have to be available as tabulated values.

Note, however, that in our numerical calculations \( \epsilon_{3,\text{eff}} \) remains nearly the same if \( \Delta \alpha/\theta_3 \) is not larger than 1.5, as previously mentioned (and as is the case in Fig. 4). It is therefore justifiable to assume that

\[
\theta_{3,\text{eff}} = \phi_3.
\]

Now, the following reasoning leads to an approximation for \( \phi_{3,\text{eff}} \). Because \( \tilde{f}^4 \) is an azimuthal average over \( \Delta \alpha \), the width in the \( \phi \) direction of this function is composed of a contribution due to \( \theta_3 \) plus a contribution \( \Delta \phi \) caused by the antenna rotation \( \Delta \alpha \). After transforming to the radar system, \( \Delta \phi \) becomes \( \Delta \alpha \cos \epsilon_0 \) and, to a first approximation (denoted by the tilde),

\[
\tilde{\phi}_{3,\text{eff}} = \theta_3 + \Delta \alpha \cos \epsilon_0.
\]

(14)

At an elevation of \( \epsilon_0 = 0^\circ \), this is

\[
\tilde{\phi}_{3,\text{eff},0} = \theta_3 + \Delta \alpha.
\]
Eliminating \( \theta_3 \) from Eq. (14) leads to

\[
\hat{\phi}_{3,\text{eff}} = \hat{\phi}_{3,\text{eff},0} + (\cos \epsilon_0 - 1) \Delta \alpha. \tag{15}
\]

For \( \hat{\phi}_{3,\text{eff},0} \), it is reasonable to take the value \( \alpha_{3,\text{eff},0} \) at \( \epsilon_0 = 0^\circ \) from the universal function \( \alpha_{3,\text{eff}}/\theta_3 \) over \( \Delta \alpha/\theta_3 \) (see Fig. 5 and Table 1), because in this case \( \alpha_{3,\text{eff},0} = \hat{\phi}_{3,\text{eff},0} \) holds true.

Equation (15) is a first crude approximation that correctly describes the overall dependence on elevation. A better approximation can be achieved by inventing a correctional term in the last summand on the right side of Eq. (15), which is derived by comparing the numerically calculated \( \hat{\phi}_{3,\text{eff}} = \alpha_{3,\text{eff}} \cos \epsilon_0 \) from \( f^4 \) with the predicted values of Eq. (15) and fitting an analytic function dependent on \( \epsilon_0 \) and \( \Delta \alpha/\theta_3 \). In doing so, it has become clear that the dependence on \( \epsilon_0 \) is very weak and can safely be neglected. The result is

\[
\hat{\phi}_{3,\text{eff}} = \alpha_{3,\text{eff},0} + (\cos \epsilon_0 - 1) \Delta \alpha \left[ 1 - \exp \left( -1.5 \frac{\Delta \alpha}{\theta_3} \right) \right]. \tag{16}
\]

The correctional term \( [1 - \exp(-1.5\Delta \alpha/\theta_3)] \) on the
right-hand side only depends on the normalized azimuthal averaging interval. Figure 5 also shows the approximated $\tilde{\beta}_{3,\text{eff}}/\theta_3 = \tilde{\phi}_{3,\text{eff}}/\theta_3 \cos \epsilon_0$ as a function of $\epsilon_0$ (dashed) for $\Delta \alpha/\theta_3 = 0.1$ (center) and for $\Delta \alpha/\theta_3 = 1$ (right). For comparison, the numerically derived values of $\alpha_{3,\text{eff}}$ are shown as solid lines. The agreement is excellent.

With this, Eq. (13) becomes

$$\tilde{\beta}^4 = \exp \left\{ -8 \ln2 \left[ \frac{(\alpha - \alpha_0) \cos \epsilon}{\alpha_{3,\text{eff},0} + (\cos \epsilon_0 - 1)\Delta\alpha[1 - \exp(-1.5\Delta\alpha/\theta_3)]} + \left( \frac{\epsilon - \epsilon_0}{\theta_3} \right)^2 \right]^2 \right\}. \quad (17)$$

Here, only the two radar-specific parameters $\alpha_{3,\text{eff},0}$ and $\theta_3$ and the antenna positioning angles $\alpha_0$ and $\epsilon_0$ have to be specified. Following the above-mentioned facts, $\alpha_{3,\text{eff},0}$ has to be proportional to $\theta_3$ where the proportionality constant can be taken from Fig. 5 or Table 1 at $\epsilon_0 = 0^\circ$ and the radar-specific value of $\Delta \alpha/\theta_3$. In Fig. 6, this Gaussian function is shown as a gray dashed line for the exemplary radar for an elevation of 40°, normalized to the integral value of $\tilde{\beta}^4$. Again, there is no significant difference from the numerically derived function $\tilde{\beta}^4$ (in fact, the gray dashed line is totally hidden by the original function).

Following above-mentioned arguments, the parameterization (17) is valid if $\Delta \alpha/\theta_3$ is not larger than approximately 1.5. Moreover, $\epsilon_0$ has to be smaller than approximately 50° because otherwise the coordinate transformations (9) and (10) are no longer valid.

To evaluate the overall quality of the approximation $\tilde{\beta}^4$ compared to the numerically calculated $\tilde{\beta}^4$, their differences are analyzed as follows, considering Eq. (4) without range weighting. With the operator

$$S[\ldots] = \int_{\pi + \alpha_0}^{\pi + \alpha_0 + \pi/2 + \epsilon_0} \int_{-\pi + \epsilon_0}^{-\pi + \epsilon_0 + \pi/2 + \epsilon_0} (\ldots) \cos \theta \, d\alpha, \quad (18)$$

a suitable measure of overall error $E$ may be defined as

$$E = \left[ \frac{\tilde{\beta}^4(\alpha - \alpha_0, \epsilon - \epsilon_0)}{S[\tilde{\beta}^4(\alpha - \alpha_0, \epsilon - \epsilon_0)]} - \frac{\tilde{\beta}^4(\alpha - \alpha_0, \epsilon - \epsilon_0)}{S[\tilde{\beta}^4(\alpha - \alpha_0, \epsilon - \epsilon_0)]} \right]. \quad (19)$$

Here, $E$ is the integrated absolute difference of the properly scaled weighting functions $\tilde{\beta}^4 \cos \epsilon$ and $\tilde{\beta}^4 \cos \epsilon$ relative to the integral value of the scaled $\tilde{\beta}^4 \cos \epsilon$ [the denominator of (19)], which equals 1; $E$ may be interpreted as the relative importance of differences compared to the overall weight. Scaling is necessary because the integral over a weighting function has to be 1, ensuring a bias-free reflectivity averaging in Eq. (4).

Here $E$ is favored over other common error measures (e.g., the integrated relative absolute or relative quadratic deviation) because those might overweight the relative differences in the outer tails of $\tilde{\beta}^4$ and $\tilde{\beta}^4$ (which do not contribute much to reflectivity weighting) and make $\tilde{\beta}^4$ look worse than it really is. Moreover, these tails are inaccurate because of the neglect of sidelobes, which might be present in reality. Because the focus of this paper is on the more important main lobes (see, e.g., the comments at the end of section 3), the tails of $\tilde{\beta}^4$ and $\tilde{\beta}^4$ should not receive too much attention in error analysis.

Because $\tilde{\beta}^4$ and $\tilde{\beta}^4$ are explicit functions of $\Delta \alpha$, $E$ is a function of $\Delta \alpha/\theta_3$ and $\epsilon_0$ (i.e., there is no dependence on $\alpha_0$ because of azimuthal scanning, which is assumed here). Figure 8 shows $E$ as a function of $\Delta \alpha/\theta_3$ and $\epsilon_0$. It is evident that the error measure $E$ is less than 10% if $\Delta \alpha/\theta_3 < 1.5$. For $\Delta \alpha/\theta_3 \geq 1.5$, the error increases continually. This is a manifestation of the fact that the shape of $\tilde{\beta}^4$ becomes more and more cuboidal with growing values of $\Delta \alpha/\theta_3$, as was previously shown in Fig. 7, and can no longer be described by a simple Gaussian.

Often, for practical applications like numerical radar operators for mesoscale atmospheric models or specialized data analyses, the question arises as to which part of the weight is concentrated within certain angular limits around the beam center (or vice versa). For example, in a numerical radar operator, considerable computational cost can be saved if calculations of radar observables are limited to the relevant domain. To this end, additional numerical computations have been performed. Because $\tilde{\beta}^4$ does not depend on $r$, a two-
dimensional analysis with respect to the angles $\alpha$ and $\epsilon$ is sufficient.

Figure 9 shows the results for the numerically calculated $\tilde{f}^4$ of the above-mentioned exemplary radar at an elevation of $\epsilon_0 = 0^\circ$. Because $\tilde{f}^4$ is nearly Gaussian in this case, the relevant results apply also to a two-dimensional Gaussian in general and to most meteorological radars with $\Delta\alpha/\theta_3 < 1.5$. The left ordinate (black solid line) denotes the fraction of the integral over the reflectivity weighting function $\tilde{f}^4(\alpha - \alpha_\epsilon, \epsilon - \epsilon_0) \cos \phi$ that is concentrated within a certain contour line of $\tilde{f}^4$ (approximately elliptical; see, e.g., Fig. 4) as a function of the corresponding weighting function value in decibels relative to its maximum; thus, the quantity

$$G_{f4}(\zeta) = \int_{\zeta}^{\infty} \int_{C(\zeta)} f^4(\alpha - \alpha_\epsilon, \epsilon - \epsilon_0) \cos \phi \, d\alpha \, d\epsilon$$

with

$$\zeta = 10 \log_{\text{dB}} \left[ \frac{\text{contour level of } \tilde{f}^4}{\text{max}(\tilde{f}^4)} \right]$$

The integration limit $C(\zeta)$ represents the closed $\tilde{f}^4$ iso-line of the corresponding relative level $\zeta$ in the $(\alpha_\epsilon, \epsilon_\epsilon)$ plane. For example, the relative level $\zeta = -6$ dB is the quarter-value width of $\tilde{f}^4$ (the half-width of the one-way beamwidth $\tilde{f}^2$, comparable to $\theta_3$).

The integrations in Eq. (20) were calculated by means of the trapezoidal rule using very small discrete angular steps. Within the $-6$-dB level, 75% of the weight of $\tilde{f}^4$ is concentrated. A value of 90% is reached at the $-10$-dB level. To show which angular coordinate range is covered by certain contour lines, the corresponding lengths of the main axes of $C(\zeta)$ (approximately ellipses) in azimuthal and elevational direction ($\alpha_{3,\text{eff}}$ and $\epsilon_{3,\text{eff}}$, respectively) are given in Fig. 9 as functions of $\zeta$. Note that $\alpha_{3,\text{eff}}$ and $\epsilon_{3,\text{eff}}$ have been normalized to $\theta_3$, which makes them depend only on $\Delta\alpha/\theta_3$ and $\epsilon_0$. For example, if the main lobe should contain 90% of the weight ($-10$-dB level), the corresponding azimuthal range $\alpha_{3,\text{eff}}$ has to be at least 1.8 $\theta_3$ and the elevational range $\epsilon_{3,\text{eff}}$ at least 1.3 $\theta_3$ [for consistency with the notation previously used in this paper, $\alpha_{3,\text{eff}}(\zeta = -10$ dB) is written as $\alpha_{3,\text{eff}}$ just as for $\epsilon_{3,\text{eff}}$ and other values of $\zeta$].

Besides, the values for $\alpha_{3,\text{eff}}/\theta_3$ and $\epsilon_{3,\text{eff}}/\theta_3$ can be
read off at the level −6 dB (j₁² is a two-way formulation) because Fig. 9 is valid for ε₀ = 0°. The former has exactly the value for the exemplary radar (see Table 1 for Δα/θ₃ = 1) and the latter has the value 1, as discussed earlier.

In the Gaussian approximation [see Eq. (17)], its −10-dB widths \( \tilde{\alpha}_{10, \text{eff}} \) and \( \tilde{\epsilon}_{10, \text{eff}} \) (the double central angle on the principal axes in the radar system, analogous to \( \theta_3 \)) are given by

\[
\tilde{\alpha}_{10, \text{eff}} = \alpha_{3, \text{eff}} \sqrt{\frac{\ln 10}{2 \ln 2}} = \frac{\alpha_{3, \text{eff}, 0} + (\cos \epsilon_0 - 1) \Delta \alpha [1 - \exp(-1.5 \Delta \alpha / \theta_3)]}{\cos \epsilon_0} \sqrt{\frac{\ln 10}{2 \ln 2}},
\]

(21)

\[
\tilde{\epsilon}_{10, \text{eff}} = \theta_3 \sqrt{\frac{\ln 10}{2 \ln 2}}.
\]

(22)

From Eq. (21) one readily obtains the same value for \( \alpha_{10, \text{eff}} / \theta_3 \) that can be read from Fig. 9 at a level of −10 dB because \( \tilde{f}^4 \) is approximately Gaussian.

To summarize, Eq. (17) constitutes an easy-to-use parameterization of the effective beam weighting function of an azimuthally scanning radar, together with the azimuthal and elevational range containing 90% of the total weight. Formulas for scan geometries other than constant-elevation scans and for applications using the beam coordinate system are given in appendix B.

In contrast to the present derivation, DZ93 and Zrnić and Doviak (1976) chose a one-dimensional presentation of \( f^4 \) and substituted the summation in Eq. (4) by a continuous integration, which enabled an analytical solution based on the Gaussian error function. However, the dependence on the elevation \( \epsilon_0 \) is not treated, and they focus mainly on the effective quarter-widths of \( \tilde{f}^4 \).

As already mentioned in the first half of section 2, all presented results concerning \( Z^\text{ref} \), \( \tilde{f}^4 \) [Eqs. (4) and (5)] and \( \tilde{\alpha}^4 \) remain valid if a more realistic range weighting function is considered instead of a simple step function as in this paper. For example, when using Eq. (5) as a radar simulator in an NWP model, one merely has to multiply the range weighting function to \( \tilde{f}^4 \) or \( \tilde{\alpha}^4 \) and extend the limit of integration with respect to \( r \) to cover the relevant range of \( W(r) \).

5. Results and conclusions

Scanning meteorological radar systems that average over several consecutive radar pulses to obtain a statistically stable measurement exhibit an effective beam weighting function \( \tilde{f}^4 \) that is different from the corresponding beam pattern of a single ray. Even for systems with axisymmetrical antennas, \( \tilde{\alpha}^4 (\alpha - \alpha_s, \epsilon - \epsilon_0) \) is no longer symmetrical about (\( \alpha_s, \epsilon_0 \)) but is “elongated” in the direction of antenna rotation. This broadening effect depends on the ratio of the angular averaging interval to the one-way single-beam half-width \( \Delta \alpha / \theta_3 \) as well as the constant elevation angle in cases of azimuthal scanning. The latter (artificial) effect is present only when the radar coordinate system is used and stems from the coordinate transformation from beam system to radar system. For example, if \( \Delta \alpha / \theta_3 = 1 \) (a common value for meteorological radars) and \( \epsilon_0 = 20^\circ \), then the effective beamwidth is larger by a factor of about 1.5 in the along-scan direction as compared to the single beamwidth (see Table 1).

Although the effect has been described previously in literature (Zrnić and Doviak 1976; DZ93), it seems to be seldom recognized. Therefore, the present paper analyzes the effective beam weighting function in detail, extending the work of the previous literature to a two-dimensional description.

Additionally, and as a main result, with Eq. (17) and the parameters in Table 1 an easy-to-use and relatively simple parameterization for azimuthal scanning radar has been presented in section 4. This may, together with Eqs. (4) or (5), be used, for example, as a “radar operator” to derive radar observables from the output of numerical weather models. It may also affect the results of specialized radar data analyses in which the beam weighting function is of importance. In appendix B it is demonstrated how to modify or simplify this parameterization to apply it also to vertical sweeps (RHI scans) as well as relative to the beam coordinate system. The given parameterization is quite general and is only limited to systems with an axisymmetrical antenna, which is the case for almost all meteorological radars.

The applicable range of the parameterized effective beam weighting function was explored by comparison with the numerically calculated effective beam weighting function. Figures 4 and 6 depict the effective beam patterns in the case of \( \Delta \alpha / \theta_3 = 1 \) and for elevations of 0° and 40° and show an excellent agreement between numerically calculated and parameterized patterns, aside from insignificant differences in the along-scan direction. In contrast, Fig. 7 shows the more cuboidal
effective pattern for a larger value of $\Delta \alpha/\theta_3 = 5$, which cannot be approximated by a simple Gaussian function.

The overall error estimate $E$ given by Eq. (19), depicted in Fig. 8, suggests that the parameterized effective beam pattern is applicable in the case of $\Delta \alpha/\theta_3 \lesssim 1.5$. In this case $E$—the integrated absolute difference between the numerically calculated and parameterized pattern, scaled with the overall weight—remains smaller than approximately 10%. For the elevation range, an upper limit of $\varepsilon_0 < 50^\circ$ is imposed by the validity of the underlying approximated coordinate transformation between the beam system and the radar system [see formulas (9) and (10)]. Again, both constraints should meet the operating conditions of most scanning meteorological radar systems.

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APPENDIX A

General Form of the Effective Beam Weighting Function

In this appendix, Eq. (4) is derived using the notation from section 2.

The reflectivity $\eta$ is the local backscattering cross section density and is given by

$$\eta(r, \phi, \theta) = \int_0^\infty \sigma_b(D) N(D, r, \phi, \theta) \, dD,$$  \hfill (A1)

where $N(D)$ denotes the size distribution of the scattering particles; $\sigma_b(D)$ is the backscattering cross section that is a function of some size parameter (here, the volume equivalent spherical particle diameter $D$); and $\eta$ is the primary radar measurable quantity, which can be obtained from the received power and the radar parameters contained in the radar constant $C$.

For spherical scatterers in the Rayleigh regime, this leads to the well-known relation

$$\eta = \frac{\pi^5}{\lambda_0^4} |K|^2 \int_0^\infty N(D)D^6 \, dD = \frac{\pi^5}{\lambda_0^4} |K|^2 Z, \quad (A2)$$

defining the radar reflectivity factor $Z$, where $\lambda_0$ denotes radar wavelength, $|K|^2 = \left(\frac{m^2 - 1}{(m^2 + 2)}\right)^2$, and $m$ is a complex refractive index of the particle's material.

It is common practice in radar data processing to store not the reflectivity $\eta$ but a value representative for spherical raindrops in the Rayleigh regime, such that $\eta$ is normalized by the Rayleigh prefactor to obtain the so-called equivalent radar reflectivity factor $Z_e$, defined as

$$Z_e = \frac{\lambda_0^4}{\pi^5} |K_{w,0}|^2, \quad (A3)$$

where $|K_{w,0}|^2 = 0.93$ is a representative value for water in the microwave spectral band. In the case where all scatterers are Rayleigh scattering raindrops, $Z_e$ is nearly independent of wavelength because $|K_{w,0}|^2$ (the value for water, which depends on wavelength and temperature) does not vary much in the microwave spectral band and at atmospheric temperatures.

Moreover, by applying the mean value theorem to Eq. (1),

$$P_r(r_0, \alpha_0, \varepsilon_0) = \frac{\pi}{\varepsilon_0} \left[ \overline{\eta(r, \phi, \theta)} \right] \int_{r_0-c/4}^{r_0+c/4} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \frac{f_4(\phi, \theta)}{r^2} \cos \theta \, d\theta \, d\phi \, dr, \quad (A4)$$

with the double overbar denoting integral average,

$$\left\langle \cdot \cdot \cdot \right\rangle = \frac{\int_{r_0-c/4}^{r_0+c/4} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} f_4(\phi, \theta)}{r^2} \cos \theta \, d\theta \, d\phi \, dr, \quad (A5)$$
and inserting Eq. (A1), approximating $r^2$ with $r_0^2$—valid for $r_0 \gg c\tau/2$—and extending the angular integration limits to infinity, one can readily execute the integration to obtain to a very good degree of approximation (Probert-Jones 1962)

$$P(r, \alpha, \epsilon) = C \frac{c\tau \pi \phi_3 \theta_3}{2 \pi} \int \frac{\eta(r, \alpha, \epsilon)}{\pi^2} d\alpha d\epsilon. \quad (A6)$$

As already mentioned in section 2, attenuation by hydrometeors is disregarded (i.e., it is assumed that $\lambda_0 = 1$).

To get an estimate of $Z_e$, the radar software performs the following normalization to an instantaneous power measurement $P_i^{(R)}$,

$$\overline{Z_e^{(R)}}(r_0, \alpha_0, \epsilon_0) = \frac{1}{N} \sum_{i=0}^{N-1} P_i^{(R)}(r_0, \alpha_0^{(i)}, \epsilon_0), \quad (A8)$$

with

$$\alpha_0^{(i)} = \alpha_0 + \left( \frac{i}{N-1} - \frac{1}{2} \right) \Delta\alpha, \quad i = 0, 1, \ldots, N-1. \quad (A9)$$

Again, $\alpha_0^{(i)}$ denotes the azimuthal angle of the beam center of each single measured $Z_i^{(R)}$, where an equidistant distribution of $\alpha_0^{(i)}$ over the averaging interval $\Delta\alpha$ is assumed. The angle $\alpha_0$ henceforth denotes the center of the averaging interval (see Fig. 2).

Assuming that the average over $P_i^{(R)}$ equals its ensemble average, by substituting Eq. (1) together with (A3) for the single $P_i^{(R)}(r_0, \alpha_0^{(i)}, \epsilon_0)$ (neglecting attenuation) after transforming to the radar system $(r, \alpha, \epsilon)$ according to section 3, one readily obtains Eq. (4).

**APPENDIX B**

**Other Formulas for Effective Beam Parameterization**

For an elevational scan (RHI scan) at a constant azimuth $\alpha_0$, the parameterization of the effective beam weighting function is also applicable after setting $\Delta\alpha = 0$ in the $\alpha$ term of Eq. (17) and replacing $\theta_3$ in the $\epsilon$ term by $\alpha_3\text{,eff,0}$; thus,

$$f_3^\ast = \exp \left\{ -8 \ln2 \left[ \frac{(\alpha - \alpha_0) \cos \theta_3}{\theta_3} \right]^2 + \left( \frac{\epsilon - \epsilon_0}{\alpha_3\text{,eff,0}} \right)^2 \right\}. \quad (B1)$$

Again, $\alpha_3\text{,eff,0}$ can be taken from Table 1, where $\Delta\alpha$ now denotes the elevational scanning interval of the beam center. As with $\alpha_0$, $\epsilon_0$ is the center of $\Delta\alpha$. For application in the beam coordinate system, just assume $\epsilon_0 = 0^\circ$, respectively, $\epsilon_0 = 0^\circ$, and backsubstitute the transformation formulas (9) and (10) in Eq. (17) for azimuthal scans or in (B1) for RHI scans.

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