Inferring Propagation Direction of Nonlinear Internal Waves in a Vertically Sheared Background Flow

RAMZI MIRSHAK AND DANIEL E. KELLEY
Department of Oceanography, Dalhousie University, Halifax, Nova Scotia, Canada

(Manuscript received 13 February 2008, in final form 3 July 2008)

ABSTRACT
Internal waves heave the background flow through which they propagate. If the background flow is vertically sheared, the high-pass-filtered velocity field will thus contain signals of both the wave velocity and the heaved flow. Under conditions of large wave amplitude and large background shear—a common situation for nonlinear internal waves in coastal waters—the velocity fluctuations caused by wave heaving of the background flow can be comparable to the wave velocity itself. This complicates the inference of wave properties such as energy flux and propagation direction. The present study deals with methods to infer propagation direction in such situations. Attention is given to three methods that may be applied to acoustic Doppler current profiler measurements: a “filtering” method that estimates wave signals from high-pass-filtered time series, a “beamwise” method that infers wave direction from lagged correlations of echo intensity between the spatially separated acoustic beams of the profiler, and a “modal” method that separates background and wave signals by regressing the high-pass-filtered velocity field onto a normal-mode wave model. The methods are tested using synthetic datasets. The results suggest that the filtering method is biased by wave heaving of the background shear, while the beamwise and modal methods are resistant to heaving. The beamwise method provides accurate predictions of wave propagation angle for cases in which the measurements have high temporal resolution and the environment exhibits no depth-averaged background flow. The limitation on depth-averaged flow is relaxed for the modal method, but it requires the measurement of stratification. These issues are illustrated, and the applicability of these methods is explored with a series of sensitivity tests, and it is found that the different methods perform well under different circumstances.

1. Introduction
Nonlinear internal waves (NIWs) are common features in coastal zones and marginal seas (Helfrich and Melville 2006) where they are thought to play a significant role in transport and nonlocal mixing (Klymak and Moum 2003; Scotti and Pineda 2004; Bourgault et al. 2007). Such waves are “typically the most energetic high-frequency events in the coastal ocean” (Nash and Moum 2005). Determining the wave propagation direction, so that one may in turn identify potential locations of dissipation and mixing, can be an important part of estimating the energetics of a given region. In cases where three or more moorings are deployed in a nonlinear array (e.g., a triangular array), the wave propagation direction may be estimated by comparing the arrival times of the waves at each mooring. For moorings deployed in a linear fashion, it is necessary to determine the direction of travel by examining the currents themselves. This task, however, can be challenging. Currents in coastal waters often vary greatly in the vertical, especially in estuaries. As a result, velocity fluctuations due to wave heaving of the background flow can be comparable to wave velocities. This heaving can mask the wave velocity field and hinder the determination of wave propagation direction.

Here, we examine the capability of three methods to reveal wave propagation direction using a series of synthetic experiments. In these experiments, velocity and echo-intensity fields due to a solitary internal wave propagating across a sheared background flow are recorded by an upward-looking ADCP, and then the wave propagation direction is estimated from the synthetic ADCP data.

Corresponding author address: Ramzi Mirshak, Department of Oceanography, Dalhousie University, 1355 Oxford St., Halifax, NS B3H 4J1, Canada.
E-mail: ramzi@dal.ca

DOI: 10.1175/2008JTECHO632.1

© 2009 American Meteorological Society
The three methods are presented in order of increasing complexity. First, we present a “filtering method” that is intuitive and simple to implement. It involves the direct use of high-pass-filtered velocity data to estimate the propagation direction $\theta$. Second, we consider a “beam-wise method” that was introduced by Scotti et al. (2005). It makes use of temporal offsets in echo intensity between individual ADCP beams to infer wave propagation direction. Last, we introduce a “modal method” that fits the observations to internal wave data. First, we present a “filtering method” that was introduced by Scotti et al. (2005). It makes use of temporal offsets in echo intensity between individual ADCP beams to infer wave propagation direction and celerity. Last, we introduce a “modal method” that was introduced by Scotti et al. (2005). It makes use of temporal offsets in echo intensity between individual ADCP beams to infer wave propagation direction and celerity. Last, we introduce a “modal method” that was introduced by Scotti et al. (2005). It makes use of temporal offsets in echo intensity between individual ADCP beams to infer wave propagation direction and celerity.

This paper is organized as follows. First, the ADCP field for the synthetic experiments is developed and the parameters used in sensitivity analyses are introduced (section 2). The three methods are then outlined and subjected to the sensitivity analysis (section 3). Possible extensions and limitations of the methods are discussed in section 4, and the work is summarized in section 5.

2. Synthetic experimental design

a. Defining the domain

Consider a horizontally unbounded domain of constant depth $H$ with the background state described by the density $\rho_o = \rho_o(z)$ and the unidirectional flow $\mathbf{u}_o = [u_o(z), 0, 0]$. (All variables are listed in Table 1.) To represent the effects of wave-induced vertical displacements, the heaved density and velocity are written as $\hat{\rho} = \rho_o(z - \eta)$ and $\hat{\mathbf{u}} = \mathbf{u}_o(z - \eta)$, respectively, where $\eta = \eta(x, y, z, t)$ is vertical displacement due to wave heaving. Within this flow, a solitary internal wave propagates across the background flow at angle $\theta$. If the wave velocity field is denoted $\hat{\mathbf{u}} = \hat{\mathbf{u}}(x, y, z, t)$, then the total velocity field can be written as

$$\mathbf{u} = \hat{\mathbf{u}} + \hat{\mathbf{u}},$$

where it is assumed that there is no interaction between the wave field and the background flow apart from the heaving effect.

To represent a simplified estuarine environment, the stratification is set as

$$\rho_o = \hat{\rho} + \frac{\Delta \rho}{2} \left[ 1 - \tanh \left( \frac{z - Z_p}{D} \right) \right],$$

where $\hat{\rho}$ is a reference density, $\Delta \rho$ is the density difference across a pycnocline with thickness, and $D$ is centered at a depth $Z_p$ (Fig. 1). (Here, $\hat{\rho}$, $\Delta \rho$, $D$, and $Z_p$ are all taken to be constants throughout the synthetic experiments.) The background flow is prescribed similarly by

$$u_o = U \tanh \left( \frac{z - Z_p}{D} \right),$$

where $U$ is varied between the synthetic experiments.

Solitary internal wave properties are determined numerically, using the fully nonlinear theory used by Stastna and Lamb (2002). The theory applies the nonlinear elliptic eigenvalue problem

$$\nabla^2 \eta = -\frac{\hat{u}'}{c - \hat{u}'} \left[ \eta'' + \eta_z^2 \right] - \frac{N^2 \eta}{(c - \hat{u})^2},$$

where $N$ is the heaved background buoyancy frequency calculated using $\hat{\rho}$, and $c$ is the wave celerity. The eigenvalue problem is subject to the boundary conditions

$$\eta(x, -H) = \eta(x, 0) = 0$$

and

$$\lim_{x \to \pm \infty} \eta = 0.$$
b. Generating synthetic ADCP time series

Consider an ADCP with four beams inclined at $\phi = 20^\circ$ to the vertical and separated by $90^\circ$ in the horizontal. If the ADCP is mounted on a level, planar bottom, the velocities measured by the ADCP along its beams are the real components of

$$b_k = \begin{cases} 
(-1)^k(u_k - i w_k)e^{i\alpha} \sin \phi - w_k \cos \phi + \epsilon_k & \text{for } k = 1, 2 \\
(-1)^k(u_k + w_k)e^{i\alpha} \sin \phi - w_k \cos \phi + \epsilon_k & \text{for } k = 3, 4
\end{cases}$$

(6)

where $k$ is the beam number (labeled 1, 3, 2, and 4 counterclockwise from above), $\alpha$ is the angle of orientation of beam 1 with respect to the background flow, $i = \sqrt{-1}$, and we take $\epsilon$ to be a normally distributed noise field with zero mean and a standard deviation of $\sigma_\epsilon$. (In cases where the bottom is not level, corrections for pitch and roll become necessary.) The velocities used for the synthetic experiments are the continuous velocities averaged over the space and time domains of each bin. While ADCPs often use triangular time gates that will cause bins to overlap, we note that the difference in binning did not strongly affect the velocity profiles generated in this work. Here, $\mathbf{u}_k = (u_k, v_k, w_k)$ is defined as

$$\mathbf{u}_k = \begin{cases} 
\mathbf{u}[(-1)^{k-1}(z + H) \tan \phi \cos \alpha, (-1)^{k-1}(z + H) \tan \phi \sin \alpha, z, t] & \text{for } k = 1, 2 \\
\mathbf{u}[(z + H) \tan \phi \sin \alpha, (-1)^{k-1}(z + H) \tan \phi \cos \alpha, z, t] & \text{for } k = 3, 4.
\end{cases}$$

(7)
To recover velocities in a Cartesian frame from the beam velocities recorded by the ADCP, it is necessary to perform the following operation:

\[
\begin{align*}
  u &= \frac{b_2 - b_1}{2 \sin \phi} \cos \alpha + \frac{b_4 - b_3}{2 \sin \phi} \sin \alpha, \\
  v &= \frac{b_2 - b_1}{2 \sin \phi} \sin \alpha + \frac{b_4 - b_3}{2 \sin \phi} \cos \alpha, \\
  w &= -\frac{b_1 + b_2 + b_3 + b_4}{4 \cos \phi},
\end{align*}
\]

(8)

at discrete times and depths that are separated by the intervals \(\Delta t\) and \(\Delta z\), respectively. Scotti et al. (2005) suggest lagging the beam velocities before making this calculation in order to improve velocity measurements. We shall return to this point later in the text.

The beamwise method requires an echo intensity field \(I\), which we take to have an averaged state \(\bar{I} = I(z - \eta)\). For this study, we set \(\bar{I}\) by taking a 90-s time average of echo intensity data from a towed Biosonics 400-kHz narrowbeam echosounder at a location and time where there was little internal wave energy (Fig. 2). To reproduce noise in the echo-intensity field, we use

\[
I = \bar{I} + \gamma,
\]

(9)

where \(\gamma\) is a normally distributed noise field with zero mean and a standard deviation \(\sigma_I\). For simplicity, \(\sigma_I\) is taken to be a constant in a given synthetic experiment.

The echo-intensity data are mapped onto beam co-ordinates using the same technique used for the velocity field in (6), namely,

\[
I_k = \begin{cases} 
I((-1)^{k-1}(z + H) \tan \phi \cos \alpha, (-1)^{k-1}(z + H) \tan \phi \sin \alpha, z, t) & \text{for } k = 1, 2 \\
I((-1)^k(z + H) \tan \phi \sin \alpha, (-1)^{k-1}(z + H) \tan \phi \cos \alpha, z, t) & \text{for } k = 3, 4.
\end{cases}
\]

(10)

c. Trials

The background density field was defined using (2), with pycnocline depth \(Z_p = -H/3\), pycnocline thickness \(D = H/10\), and density step \(\Delta \rho = 10^{-2} \rho\) (Fig. 1). The values for \(Z_p\) and \(D\) were also used to determine the background velocity described in (3), and \(U\) was varied to explore different shear levels. The propagation direction relative to the mean flow \(\theta\) was varied from 0° to

Fig. 1. (a) Background flow \(\bar{u}(z)\) in the absence of a passing wave. (b) The \(x\) component of the wave velocity field of a solitary internal wave propagating across the background flow at an angle of \(\theta = 45°\). Contours are spaced at \(U/20\) intervals. The filled gray region shows areas where the velocity is greater than zero. As \(\theta = 45°\), this panel also illustrates \(\bar{v} = \bar{u} \mu \sin \theta\). (c) Heaved background flow \(\bar{u}\) during wave passage. Contours are as in (b), but intervals are \(U/10\). Here \(u' = \bar{u} - \bar{u}(\eta, z)\). Contours are as in (c). (e) Background density field \(\rho(z)\) (solid line), and the gravest vertical internal mode, \(\Pi(z)\) (dashed line). The black dots represent the depths at which the filtering method is tested.
90° across 31 equally spaced angles. To build the solitary waves using the Stastna and Lamb (2002) model, the prescribed available potential energy was set to give a wave amplitude of $h_o' = Z_p/2$. The exact amplitude changed between trials because changes in $U$ and $u$ altered the results extracted from (5). The full velocity field was defined using (5) subject to (3). For each value of $u$, the ADCP orientation $a$ was varied from 0° to 45° across 16 equally spaced angles. This series of trials was repeated for various choices of $U$, $s_b$, $s_I$, and $D_t$ with the noise fields $\epsilon$ and $\gamma$ re-calculated for each trial. Four nondimensional parameters were used to test the various methods:

1) A nondimensional parameter describing the strength of the background shear relative to the wave shear,

$$\mathcal{U} = \frac{\eta_o \frac{\partial u}{\partial z}}{\sqrt{U}},$$

where $\tilde{U}$ is the scale of the horizontal wave velocity. For the filtering method, the background shear $\partial u/\partial z$ is calculated at the depth where it is applied, while for the modal method, we use its maximum value at the pycnocline. Typical oceanic values for $\mathcal{U}$ will range from 0 to 1.

2) An ADCP velocity signal-to-noise ratio,

$$S = \frac{\tilde{W}}{\sigma_b \cos \phi},$$

where $\tilde{W}$ is the scale of the vertical (wave) velocity. Normal ADCP configurations will have $S$ greater than 1.

3) A signal-to-noise ratio for the echo-intensity gradient:

$$I = \frac{\eta_o}{\rho \frac{\partial I}{\partial z}} \frac{1}{\sigma_I}.$$  

In field applications, this parameter generally has values between 1 and 5.

4) A ratio of the temporal bin size to the time required for a wave to travel the distance separating opposing beams,

$$T = \frac{c\Delta t}{X_{1,2}},$$

where $X_{1,2}$ is the distance between opposing beams 1 and 2 at $z = -0.45H$ (Fig. 2). The distance $X_{1,2}$ will depend on the depth at which the beamwise method is applied. Typically, this time discretization parameter will range between 0.3 and 1.

Not all parameters are used to explore a given method. We expect that any effect that backscatter intensity might have on velocity measurements is incorporated into $S$, so $I$ is neglected for the filtering and modal methods. In addition, it was found that averaging in time did not greatly affect the results presented here for the filtering and modal methods, so only the parameters $U$ and $S$ are examined for those two methods. As the beamwise method does not use velocity measures, it depends on only $I$ and $T$.

The trials used $S = (0.1, 0.3, 1, 3, 10)$, $I = (0.1, 0.3, 1, 3, 10)$, and $T = (0.25, 0.5, 1)$. Except when testing the effects of temporal discretization on the beamwise method, $T$ was set to 0.5. The Stastna and Lamb technique used to determine the wave field relative to the background flow hindered a similar systematic examination of $U$, but values spanned the range $0 < U \leq 1$. A fifth parameter, $Z = |\Delta z/\eta_o|$, was also examined, but over the range $0.1 \leq Z \leq 1$ it was found to have little effect compared with the other parameters, and so $Z$ was set to 0.3 for the results presented here.

3. Methods for inferring direction of travel

a. The filtering method

A straightforward method for estimating the direction of wave travel is to examine only the horizontal component of the anomaly velocity field:

$$\mathbf{u}' = \mathbf{u} - \langle \mathbf{u} \rangle,$$
with \( \langle u \rangle = \langle u \rangle(z) \) being defined as a time-averaged flow prior to wave arrival or a low-pass-filtered velocity field. An inferred (or predicted) propagation direction \( \theta_p \) may be found by using

\[
\theta_p = \arctan \left( \frac{v'}{u'} \right) \tag{16}
\]

at some depth above the pycnocline where wave velocities are in the direction of wave propagation.

The sensitivity of the method was tested for \( S \) and \( U \). Figure 3 shows the skill of the model at three positions in the water column: 0.13\( H \) (near the middle of the mixed layer where \( U = 10^{-2} \)), 0.27\( H \) (in the mid-pycnocline where \( U = 0.2 \)), and 0.40\( H \) (near the pycnocline base where \( U = 1 \)). The standard deviation of \( \theta_p \) is roughly inversely proportional to \( S \), and is equal to 4\(^\circ\) when \( S = 3 \). Consistently, the results are most accurate at shallower depth, where isopycnal displacements are small and, in this case, the background shear is also small. At the depth 0.13\( H \), \( \theta_p \) showed negligible offset from \( \theta \). At the depth 0.27\( H \), \( \theta_p \) is biased by the background flow, causing a 10% reduction in the inferred propagation angle. At the depth 0.4\( H \), where shear in the mean flow is strongest, the bias reduces \( \theta_p \) by 50%.

The filtering method produces this bias because the velocities it uses include a component of the heaved background flow. Note that unlike \( u \), the low-pass-filtered velocity \( u' \) neglects heaving effects. Substituting (1) into (15) yields

\[
u' = \bar{u} + \langle \bar{u} - \langle u \rangle \rangle. \tag{17}
\]

Hence, if the background flow contains no vertical shear, then the high-pass-filtered velocity \( u' \) will resemble the wave velocity \( \bar{u} \); otherwise, \( u' \) will also include \( \bar{u} - \langle u \rangle \), which will influence the inferred propagation direction. The effect is demonstrated in Fig. 1 for a case where \( \theta = 45^\circ \). Each of the depths where the filtering method was tested is shown as a black dot in Fig. 1e. The increased heaving effect with depth is visible by comparing Figs. 1b,d. The velocity field \( u' = \bar{u} + \langle \bar{u} - \langle u \rangle \rangle \) is shown in Fig. 1d while \( v' = \bar{v} \) is shown in Fig. 1b. In this particular case, \( \bar{u} = \bar{v} \), so the difference between the two panels represents \( \bar{u} - \langle u \rangle \).

The heaving effect can be calculated with a scaling analysis of (17):

\[
\begin{align*}
\left\{ u' \right\} & \approx \left\{ \bar{U} \cos \theta + \eta_o \frac{\partial \bar{u}}{\partial z} \right\}, \\
\left\{ v' \right\} & \approx \bar{U} \sin \theta,
\end{align*}
\]

where \( \bar{U} \) is a horizontal wave velocity scale. With (16), this yields

\[
\frac{\theta_p}{\theta} = \arctan \left( \frac{\sin \theta}{\cos \theta + U} \right). \tag{19}
\]

indicating that in the absence of background shear (i.e., when \( U = 0 \)), the filtering method will be successful in determining direction of wave propagation, as demonstrated in Fig. 3. In the presence of background shear, however, the value of \( \theta_p \) will be shifted toward the axis of the background flow field.

The bias due to the heaved flow, shown in Fig. 3, suggests a linear relationship between \( \theta \) and \( \theta_p \). This linearity is consistent for \( U \leq 1 \), implying that one may predict—and take into account—the bias due to wave heaving. Using \( \theta = 90^\circ \) in (19), and assuming a linear relationship as seen in Fig. 3, suggests that for \( U \leq 1 \),

\[
\frac{\theta_p}{\theta} = \frac{\arctan(U^{-1})}{90^\circ}. \tag{20}
\]

This finding is supported by the synthetic experiments. The result of (20) for \( U = 1 \) is plotted on Fig. 3, where its prediction coincides with \( \theta_p \) calculated at 0.4\( H \).

If one could estimate \( U \), the bias could be corrected with (20). Without a priori knowledge of \( \theta \), however, the relationship between \( \bar{v} \) and \( \bar{U} \) will be unknown, so determination of \( U \) may not be possible. While characterization of \( \eta_o \) may be extracted from backscatter images, and \( \bar{U} \) may be estimated with an appropriate wave theory, a 10% error in each may lead to a 14% error in the bias correction term.

In the synthetic examples used here, there is almost no shear in the upper quarter of the water column, so the filtering method is adequate when applied in this region. In field settings where shear persists throughout the water column, however, this method will produce biased...
predictions like those seen near the pycnocline at 0.4H in Fig. 3. In such situations, other techniques may be more successful at determining propagation direction.

**b. The beamwise method**

The four ADCP beams sample the water column at different locations, so changes in echo intensity due to wave heaving will occur at different times in the acoustic beams (Scotti et al. 2005). The difference in wave arrival between beams $i$ and $j$ may be inferred from the lagged cross-covariance function

$$R_{i,j}(z,s) = \text{cov}[I_i(z,t), I_j(z,t - s)],$$

with the lag $s$ that maximizes $R_{i,j}$ providing an estimate of wave travel time $\tau_{i,j}$ from beam $i$ to beam $j$. This lag is modeled by Scotti et al. (2005) as

$$\tau_{i,j} = \frac{X_{i,j} \cos(\theta - \alpha - \beta_{i,j})}{c},$$

where $X_{i,j} = X_{i,j}(z, \phi)$ and $\beta_{i,j}$ are the horizontal distances and angles between beams $i$ and $j$. The inferred propagation direction $\theta_p$ and celerity $c_p$ may be deduced by minimizing the misfit:

$$J = \sum_{i=1}^{4} \sum_{j=i+1}^{4} \left[ \frac{c_p \tau_{i,j}}{X_{i,j}} - \cos(\theta_p - \alpha - \beta_{i,j}) \right]^2.$$  

In the time-discretized case, $\tau_{i,j} = n_{i,j}\Delta t$, where $n_{i,j}$ is an integer and $\Delta t$ is the temporal bin size, so (22) yields

$$n_{i,j}\Delta t = \frac{X_{i,j} \cos(\theta - \alpha - \beta_{i,j})}{c},$$

or, using (14),

$$\theta_p - \alpha - \beta_{i,j} = \cos(n_{i,j}\Delta t).$$

Hence possible values of $n_{i,j}$ are constrained to

$$|n_{i,j}| \leq \frac{1}{T}.$$  

This method has limitations due to time discretization (Fig. 4). Results for a sampling period with $T = 0.25$ suggest that nine discrete angles may be inferred in the 90° sweep. Doubling $T$ halves the number of angles, consistent with (26).

In low noise conditions ($I = 10$), deviation in $\theta_p$ from its expected value spans a narrow range, but the spread increases with increasing noise levels. Figure 4 includes a scatterplot of $\theta_p$ against $\theta$ for $T = 0.5$ and $I = 3$, as well as bands that exhibit the distribution of binned $\theta_p$ values. These bins were determined by assuming that $\theta_p$ would tend toward the result for $I = 10$ and $T = 0.5$ (solid line) with decreasing noise. In the low noise condition, $\theta_p$ shows discrete steps, so the data were grouped accordingly for $I = 3$. At some values of $\theta_p$, it is difficult to discern the propagation angle to a precision that would likely be of use in applications. For example,
if $\theta_p = 40^\circ$ for $I = 3$ and $T = 0.5$, the actual propagation angle could lie between $14^\circ$ and $76^\circ$.

For the intensity field tested here, this method exhibited limited success at inferring the propagation direction of single solitary internal waves (SIWs). When $I \approx 10$ and $T \leq 1/4$, $\theta_p$ was within about $10^\circ$ of $\theta$ (Fig. 4).

c. Modal regression

This method assumes that the general shape of solitary internal waves will resemble that predicted by normal-mode theory (Apel 2003; Helfrich and Melville 2006), and models the wave field by fitting it to a normal-mode shape. To estimate $\tilde{w}$, it is assumed that the vertical velocities are attributed only to wave motions, so that

$$\tilde{w} = w' = k\Pi + \epsilon_w,$$  \hspace{1cm} (27)

where $\kappa = \kappa(t)$ is a time-dependent velocity scale that is fit by regression, $\epsilon_w = \epsilon_w(z, t)$ is a combination of measurement and model error, and $\Pi = \Pi(z)$ is the first normal mode, which is determined by solving the eigenvalue problem (Gill 1982):

$$\frac{d^2\Pi}{dz^2} + \frac{N_o^2}{c_p^2}\Pi = 0,$$ \hspace{1cm} (28)

where $c_p$ is a wave speed and $N_o = N_o(z)$ is a background buoyancy frequency calculated with $\rho_o$. This solution is subject to the boundary conditions $\Pi(0) = 0$ and $\Pi(-H) = 0$. The linear problem described here assumes $\eta_o$ is small, so the buoyancy frequency $N_o$ and the normal mode $\Pi$ do not depend on wave amplitude or wave heaving. This eigenvalue problem is a simplified version of the Taylor–Goldstein equation and does not include the effects of a background current, because wave direction relative to the background flow is not known a priori.

Substituting an inferred celerity $c_p$, derived from internal shallow-water wave theory and applying (27) to the continuity equation while assuming a frozen field yields an inferred value the horizontal wave velocity for $\tilde{u}_H$,

$$\tilde{u}_H = -c_p \frac{d\Pi}{dz} \int_{t_o}^{t'} \kappa dt',$$ \hspace{1cm} (29)

where $t_o$ is some time before internal wave arrival. With an estimate of the wave velocity, the direction of wave travel may then be estimated by regression of $\tilde{u}_H$ onto $u'$ and $v'$, that is,

$$u' = a_u \tilde{u}_H + \epsilon_u,$$ \hspace{1cm} (30)

and

$$v' = a_o \tilde{u}_H + \epsilon_o,$$ \hspace{1cm} (31)

where $\epsilon_{uo} = \epsilon_{uo}(z, t)$ contains model uncertainty, measurement uncertainty, and the vertically advected mean velocity field. The values $a_u$ and $a_o$ resemble expectations of $(c/c_p) \cos \theta$ and $(c/c_p) \sin \theta$, respectively, so the ratio $a_o/a_u$ leads to a celerity-independent estimate of the propagation direction:

$$\theta_p = \arctan \left( \frac{a_o}{a_u} \right).$$ \hspace{1cm} (32)

Regressing the $w'$ velocity field onto a $\Pi$ to infer $\kappa$, rather than regressing $u'$ and $v'$ onto $d\Pi/dz$, is done in order to avoid having the heaved background flow alter the inferred time dependence on wave structure. In situations where $w'$ might not be reliably resolved (e.g., ADCP measurements from a ship) one might attempt to regress the modal shape directly onto the horizontal field.

Figure 5 compares $\theta_p$ to $\theta$ for values of $S = 0.3, 1, and 3$, while holding $U = 1$ constant. At $U = 1$, this method exhibits skill comparable to the filtering method in the absence of shear. For $S \ll 3$, outliers in the data suggest the need for repeated observations or higher values of $S$. At $65^\circ \leq \theta \leq 80^\circ$, the propagation direction is over-predicted for intermediate values of $S$. The reason for this is unclear, but it does not appear at larger $S$ (Fig. 5c).

The model skill for the modal method is mapped in parameter space in Fig. 6. In general, the median value of $(\theta - \theta_p)$ is weakly sensitive to $U$, but most of the sensitivity is attributed to $S$. For $S > 5$, the predicted angle is expected to be within $3^\circ \pm 5^\circ$ (67% confidence interval) of the actual propagation direction. Here, waves had relatively large amplitudes of $\eta_o = Z_p/2$. Because the modal method is based on a linear eigenvalue problem, its precision is expected to increase with decreasing wave amplitude.

4. Discussion

The three techniques each exhibit strengths and weaknesses, so the choice of method will depend on the oceanographic scenario and the instrument setup. The filtering technique is simple, but heaving effects cause a bias in the direction of the background flow. The beamwise technique is resistant to wave heaving, but it is limited by the temporal discretization of the measurements. The modal method is resistant to heaving, but is only half as accurate as the filtering method. Suggestions for when to use each of the techniques is given in Table 2, with the reasoning outlined below.
Figures 3 and 6 suggest that in situations where $U \lesssim 0.25$, the filtering method should be used as it will outperform the modal method. While $U$ is generally an unknown in systems with shear throughout the water column, it will be small in locations where $\frac{\partial u}{\partial z} \approx \frac{\partial \bar{u}}{\partial z}$ is small. For example, if an internal wave has amplitude $\eta_0 \approx 10\, \text{m}$ and velocity $\bar{U} \approx 0.5\, \text{m}\,\text{s}^{-1}$, then in order to achieve $U < 0.25$, it would be necessary that the background shear was $|\partial \bar{u}/\partial z| \approx 10^{-2}\,\text{s}^{-1}$ over a vertical range $\eta_0$. While the bias effect in the filtering method for large $U$ seems to follow a clear pattern, measuring $U$ adequately may become challenging in field settings. If $U$ is unknown or is anticipated to be large, then the modal method should be used rather than the filtering method.

The choice to use the beamwise method, as opposed to the filtering and modal methods, is more complicated. Consider a wave propagating across a uniform flow at angle $\theta$. The velocity anomalies will not include the uniform background flow, so the filtering and modal methods will infer the direction of wave travel relative to the moving fluid. By contrast, disturbances to the echo-intensity field will be advected by the background flow, albeit in a manner that depends on $\theta$. In the case where $\theta = 0^\circ$, the projected celerity will include the background flow, but in the case where $\theta = 90^\circ$, it will not. Now, consider the case where $u_o$ is constant and $\theta = 45^\circ$. In that scenario, only the background transport component parallel to the wave propagation will be registered in the interbeam lags. An effect similar to the bias found for the filtering method can be expected where, for constant $u_o$,

$$\theta_p = \arctan \left[ \frac{c \sin \theta}{(c + u_o) \cos \theta} \right].$$

We therefore expect that the lag in echo intensity between beams, while providing the correct shift to minimize uncertainty in velocity measurements (Scotti et al. 2005), will result in an estimate of $\theta_p$ that is shifted by an “advection effect” of the depth-averaged background flow. This advection effect was not encountered for the beamwise method here because of the technique used to infer time series from the spatial fields (see the end of section 2a). In field settings, however, the shift would require consideration.

A factor not corrected here is the effect of lags between beams on the velocity measurements themselves. This effect severely distorted the measured wave velocities examined by Scotti et al. (2005) and motivated them to develop (22) and (23). The interbeam lags affected their velocity values considerably more than those in the wave fields examined here. The key difference between the two situations was the ratio of the distance separating the beams to the horizontal wavelength scales. The solitary waves studied here were propagating in depths of $H \approx 30\, \text{m}$. The ratio of wave half-widths to beam separations at the surface ranged from 4 to 5. By contrast, the wave groups discussed by Scotti et al. (2005) were in deeper water ($H \approx 80\, \text{m}$). As a result, beam separations led to beam velocity time series that were nearly in quadrature. Based on this
difference, it is anticipated that the need to correct for lags will increase as the water depth increases.

After making the velocity corrections, the filtering or modal techniques will still have use in deep water. In fact, while the velocity correction becomes increasingly important, the angles inferred by the beamwise method can become less reliable in deep water. This is because in deep water deployments, ADCPs are often set to record a greater number of depth bins and the deployments themselves often last for longer periods. The combined limitations of memory and battery power may therefore result in an increased value of $D_t$, leading to an increased value of $T$. In deep waters where shears will usually be small, one should therefore apply the filtering method after velocity measurements are corrected for interbeam lags (Scotti et al. 2005).

Another potential use for the filtering or modal methods may arise in situations where interbeam lags are not reliable enough to correct velocity measurements (e.g., due to low values of the echo-intensity gradient $I$). In such a case, wave direction can be inferred by the filtering or modal method and an approximation of wave celerity can be determined from wave theory. Combined with measurements of background transport and applied to (22) these values might provide an improved estimate of the lags $\tau_{ij}$ thereby improving the correction to velocity measurements.

5. Conclusions

The skill of three methods for inferring the direction of SIW’s travel through a vertically sheared background flow has been tested with synthetic experiments. A filtering method, involving analysis of the high-pass-filtered velocity field, is biased by shear in the heaved background flow. A beamwise method, using cross covariance of the echo-intensity field, is resistant to heaving but its predictions will be altered in the presence of a nonzero background transport. Its precision and accuracy are limited by noise and time discretization, respectively. A modal method, applying a regression of the velocity field onto a normal-mode wave model, is resistant to the effects of background vertical shear, but provides only half the accuracy of the filtering method. Recommendations for method selection are provided in Table 2. In general, the filtering method is a powerful tool when background shears are known to be weak. The beamwise method may prove useful when

Table 2. Summary of situations in which each of the three techniques will be useful. The beamwise method should only be used if the depth-averaged background current is small compared to wave celerity. In all cases, ADCP data should be used to correct for lags, as in Scotti et al. (2005), if possible.

<table>
<thead>
<tr>
<th>Distance to pycnocline (m)</th>
<th>$T$</th>
<th>$U$</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>30–50 m (estuary or shelf)</td>
<td>0.2</td>
<td>&lt; 0.25</td>
<td>Beamwise or filtering</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
<td>1</td>
<td>Beamwise or modal</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
<td>?</td>
<td>Beamwise or modal</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>&lt; 0.25</td>
<td>Filtering</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>Modal</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>?</td>
<td>Modal</td>
</tr>
<tr>
<td>100 m (shelf)</td>
<td>1</td>
<td>&lt; 0.25</td>
<td>Filtering</td>
</tr>
<tr>
<td>200 m (shelf break)</td>
<td>0.8–5</td>
<td>&lt; 0.25</td>
<td>Filtering</td>
</tr>
</tbody>
</table>

Fig. 6. (a) Contours of $\theta - \theta_p (\cdot)$ in $U - S$ space for the modal method. (b) Contours of the median value of the 67% spread of $\theta_p$.
there is no background transport and temporal discretization is small compared to the time required for a wave to travel between ADCP beams. In other situations, the modal method is recommended.

Acknowledgments. The authors wish to thank the anonymous reviewers whose comments greatly improved the manuscript, particularly by suggesting the use of a nonlinear wave model to test the methodology. Kevin Lamb kindly provided the numerical code used to generate the waves. Discussions with Doug Schillinger and Clark Richards were also beneficial. This work was supported by the Natural Sciences and Engineering Research Council of Canada and the Canadian Foundation for Climate and Atmospheric Sciences.

REFERENCES


