Analytical Expressions for Doppler Spectra of Scatter from Hydrometeors Observed with a Vertically Directed Radar Beam

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(Manuscript received 10 January 2011, in final form 4 August 2011)

ABSTRACT

A generalized expression is derived for the correlation function of signals backscattered from hydrometeors observed with a vertically pointed radar beam in which particle size distribution, turbulence, and mean wind are not homogeneous. This study extends the previous work of Fang and Doviak by including the effects of particle size distribution on the measured Doppler spectrum. It shows the measured Doppler spectrum to be the volumetric mean of the weighted convolution of the normalized Doppler spectra associated with turbulence, mean wind, particle oscillation/wobble, and terminal velocity. Without particle oscillation/wobble, mean wind, and turbulence, the Doppler spectrum is the mirror image of the terminal velocity spectrum under the condition that the second-order effect of finite beamwidth can be ignored. This generalized Doppler spectrum reduces further to a previously derived expression if the particle size distribution, or equivalently reflectivity, is uniform. Provided there is a unique relationship between the particle’s terminal velocity and its effective diameter, the derived equations can be applied to scatterers consisting of ice particles as well as water droplets. This study derives the analytical expression for the Doppler spectrum of mean wind and also shows that if stationary homogeneous turbulence is the only contributor to spectral broadening, then the average of a large number of radar-measured Doppler spectra will be equal to the velocity probability density function of turbulence independent of the angular, range, reflectivity, and weighting functions.

1. Introduction

The Doppler radar with a vertically or nearly vertically pointed beam forms a big family. It comprises most ground-based cloud radar, airborne and space-based radar, some research radar, and wind profilers. For Doppler radar, the Doppler spectrum is essential in calculating the widely used radial velocity and spectrum width. In addition, the Doppler spectrum measured by the radar with a vertically pointed beam is often used to retrieve particle size distribution (PSD) and air motions in clouds (Gossard 1994; Kollias et al. 1999; Giangrande et al. 2010). Although the Doppler spectrum is extensively applied, there is neither a generally applicable analytical expression including all of the contributors, nor an expression for a single contributor, like turbulence or shear, based upon a robust derivation for a vertically directed radar beam.

The usually used analytical expression of the normalized Doppler spectrum of echoes from a resolution volume along a vertically pointed radar beam is expressed as [i.e., Eq. (8.77) of Doviak and Zrnić 2006]

\[ S_{np}(v) = S_{np}(-\nu_f) = \frac{\sigma(D)N(D)\frac{dD}{dv_p}}{\eta}, \]  

where \( S(-\nu_f) \) is the Doppler spectrum; \( v, \nu_f = v_p - w \), and \( v_p \) are the Doppler, fall (positive downward), and
terminal (≥0) velocities of a particle with effective diameter $D$; $w$ is the updraft velocity; $\sigma(D)$ is the backscattering cross section of the hydrometer of effective diameter $D$, one possessed by all particles falling at the same terminal velocity; $N(D)$ is the PSD; and $\eta$ is the reflectivity. The PSD and the reflectivity are assumed to be spatially uniform, and turbulence is assumed to be negligible. In Eq. (1), the subscript “np” is appended to $S$ to indicate that the spectrum is a normalized spectrum associated with the dispersion of particle terminal velocities. Because Eq. (1) is presented without a derivation based on scattering theory, one of the purposes of this paper is to fill this gap and show the conditions under which it is applicable (section 4).

The analytical expression for the estimated normalized Doppler spectrum, one that ignores the spectral broadening resulting from the dispersion of particle terminal velocities, but makes no assumption about the uniformity of drop oscillation, mean wind, and the homogeneity of turbulence, is (Fang and Doviak 2008, hereafter FD08)

$$S_n(v, r_0, t_n) = \frac{S_n(v, r, t_n)}{T_n},$$

(2a)

where $r_0$ locates the center of the resolution volume $V_0$ (Doviak and Zrnic 2006, their section 4.4.4). This estimated spectrum is obtained from averages over ensembles of particles’ cross sections and various configurations of scatterers, but is for one member of an ensemble of turbulent wind fields. The circumflex (ˆ) emphasizes that of scatterers, but is for one member of an ensemble of members by all particles falling at the same terminal ($\leq 0$) velocities not included in the work of FD08.

The approach given by FD08 will be applied in the next section to derive the autocorrelation function and a generalized analytical equation for the Doppler spectrum, including terminal velocity, mean wind, turbulence, and particle oscillation/wobble. Section 3 studies the spectrum resulting from shear and turbulence and investigates the
relationship between the estimated Doppler spectrum and the probability density of turbulence. Section 4 discusses the simplified equations obtained from the generalized equation under certain limiting conditions. Conclusions are in section 5.

2. Autocorrelation function

For a scanning beam, the single sample autocorrelation function $R_{\xi\alpha}(mT_s, r_{00}, r_{0}, t_n)$ at lag $mT_s$ for weather signals from a $V_s$ at $r_0$ and collected during a dwell time $MT_s$ ending at time $t_n$ can be expressed as

$$
\bar{R}_{\xi\alpha}(mT_s, r_{00}, r_{0}, t_n) = \sum_i \sum_l F_i^* (r_{00}, r_0) A_l^s (0, t_n) F_i (r_0, r_l) A_l (mT_s, t_n)
$$

$$
\exp\{j[\phi_l(0, t_n) - \phi_l(mT_s, t_n) - 2k\nu_l(0, t_n)mT_s]\}, \quad (3)
$$

where $T_s$ is the pulse repetition time and $n = 0, 1, 2, \ldots; M; i$ and $l$ denote the $i$th and $l$th MTs; $\phi_l$ is the phase of the $l$th scatterers’ echo wave, whereas $A_l$ is the echo amplitude; $\nu_l$ is the scatterers’ mean radial velocity over $mT_s$; $k = 2\pi/\lambda$ is the radar’s wavenumber; $*$ denotes the complex conjugate; $r_{00} = (r_{00}, \theta_{00}, \phi_{00})$ locates the center of $V_s$ at zero lag; and, because $V_s$ can be displaced $d_0$ in $mT_s$ by the scanning beam, $r_0 = r_{00} + d_0$. In this study, the beam is stationary (i.e., $d_0 = 0$). The subscripts $\sigma, \xi, \nu$ in Eq. (3) are appended to the correlation sample to emphasize that $R_{\xi\alpha}(mT_s, t_n)$ is dependent on three random variables: the backscatter cross section of the particles $\sigma$, the configuration of particle locations $\xi$, and the radial component of the wind field $\nu$.

To simplify notation, we will drop the arguments $r_{00}$ and $r_0$ in $R_{\xi\alpha}$, and the arguments $mT_s$ and $t_n$ in other functions, unless they are necessary for clarity. This will be the convention throughout and one that keeps the notation consistent with that of FD08. It should be pointed out that the noise also contributes to the correlation function, but it is neglected.

Up to this point we have focused on the estimate $R_{\xi\alpha}(mT_s, t_n)$ for a single sample pair drawn from an $M$ sample sequence ending at $t_n$. To reduce estimate variance, it is natural to average the $M$--$m$ sample pairs that are contained in the block of $M$ samples. In this case the $l$th scatterer velocity is averaged over the dwell time $MT_s$ and the argument of $\nu_l$ will change henceforth from $(0, t_n)$ to $(t_n, t_n)$.

Here, we consider the hydrometeors as incoherent scatterers; that is, hydrometeors do not contribute coherent scatter, as discussed by Jameson and Kostinski (2010) and Gossard and Strauch (1983, section 2.4.1). For incoherent scatter, an ensemble average $E_{\xi} \{[x]\}$ over $\xi$ [$x$ is the element of the matrix in Eq. (3)] eliminates the off-diagonal elements of the matrix in Eq. (3), and thus

$$
\bar{R}_{\xi\alpha}(mT_s, t_n) \rightarrow \bar{R}_{\xi\alpha}(mT_s, t_n)
$$

$$
= \sum_l A_l^s (0) A_l (mT_s) \sum_j \{F_l (r_0, r_l)\}^2 \exp[-j2k\nu_l(t_n)mT_s]. \quad (4a)
$$

Function $F_l (r_0, r_l)$ is a complex weighting function that depends on the radiation and reception patterns of the antenna, and the transmitted pulse width and receiver bandwidth (Doviak and Zrnić 2006); $F_l (r_0, r_l)$, the square root of Eq. (2c), weights the backscatter cross section $\sigma_l$ (i.e., proportional to $A_l$) of the $l$th scatterer at $r_l$.

For a vertically directed beam,

$$
\nu_l(t_n) = \langle v_{\xi l}(t) \rangle_{MT, r} - v_p(D_l) \cos(\theta_l)
$$

$$
= \langle v_{\xi l}(t_n) - v_p(D_l) \cos(\theta_l) \rangle, \quad (4b)
$$

where $\nu_l(t_n)$ is the time-dependent radial component of the wind at the $l$th particle location and $\langle \rangle_{MT, r}$ denotes the average over $MT_s$; $v_p(D_l)$ is the terminal velocity of the $l$th particle of diameter $D_l$ and $\theta_l$ is the zenith angle of the $l$th particle from the beam axis (assumed to be circular). To retrieve PSD from the Doppler spectrum, a unique relation between $D_l$ and $v_p(D_l)$ is required, and the impact resulting from finite beamwidth [i.e., the change of $\cos(\theta_l)$ over the radar beam in Eq. (4b)] should be negligible. For a 20° half-power beamwidth, the change of $\cos(\theta_l)$ from the beam axis to the edge of the beam is less than 1.5%, whereas the cross-beam wind causes radial velocity variations, resulting from $\sin(\theta_l)$, of about 17.4%. Thus, it seems safe for us to assume that $\cos(\theta_l) \approx 1$ holds everywhere where particles contribute significant scatter amplitude. Although a unique relation between $D_l$ and $v_p(D_l)$ is assumed, Montero-Martinez et al. (2009) show that this might not always be true.

To simplify the derivation, let us for now ignore particle oscillation/wobble. Because FD08 focused attention on weather radar applications at low elevation angles, the spectral broadening factor resulting from terminal velocity dispersion was negligible, but the spectrum $S_0(r, t_n)$ resulting from particle wobble/oscillation was included. After examining the final expression it will be apparent that we can simply insert $S_0(r, t_n)$ into the final expression to have a composite spectrum resulting from many, if not all, of the most significant spectral broadening mechanisms. Under the condition that we can ignore drop oscillation/wobble, the correlation coefficient $\rho_\sigma(\tau)$ due to particle oscillation and/or wobble (FD08) results in $\rho_\sigma(\tau) \approx 1$ and $A_l^s (0) A_l (mT_s) = \sigma(D_l)$. Thus, Eq. (4a) reduces to

$$
\bar{R}_{\xi\alpha}(r_0, r_l) = \sum_l |F_l (r_0, r_l)|^2 \exp[-j2k\nu_l(t_n)] \sigma(D_l) \exp[j2k\nu_p(D_l)r_l], \quad (5)
$$

where $\sigma(D_l) = \beta_l(D_l) \Phi_l(D_l)$, and $\beta_l(D_l)$ and $\Phi_l(D_l)$ denote the Doppler spectral broadening factor resulting from terminal velocity dispersion and orientation, respectively. Solved for $\beta_l(D_l)$, we have $\beta_l(D_l) = \sigma(D_l) / \Phi_l(D_l)$, and $\Phi_l(D_l)$ is the beam power beamwidth, the change of $\cos(\theta_l)$ over the radar beam in Eq. (4b) should be negligible. For a 20° half-power beamwidth, the change of $\cos(\theta_l)$ from the beam axis to the edge of the beam is less than 1.5%, whereas the cross-beam wind causes radial velocity variations, resulting from $\sin(\theta_l)$, of about 17.4%. Thus, it seems safe for us to assume that $\cos(\theta_l) \approx 1$ holds everywhere where particles contribute significant scatter amplitude. Although a unique relation between $D_l$ and $v_p(D_l)$ is assumed, Montero-Martinez et al. (2009) show that this might not always be true.

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$$
\bar{R}_{\xi\alpha}(r_0, r_l) = \sum_l |F_l (r_0, r_l)|^2 \exp[-j2k\nu_l(t_n)] \sigma(D_l) \exp[j2k\nu_p(D_l)r_l]. \quad (5)
$$
where $\tau = mT_0$. Because $\overline{R_v(r,0,t_n)}$ is the mean power [i.e., the average of $R_v(0,t_n)$ over the ensemble of scatter configurations $\xi$], and because $|F_I(r_0,0)|^2 = I(r_0,0)$, it can be shown that Eq. (5) reduces to Eq. (3.24) of Dovia and Zrnić (2006); Eq. (5) also has a form that is equivalent to Eq. (2d) of FD08, but Eq. (5) contains the terminal velocity $v_t(D_t)$, whereas FD08’s Eq. (2d) does not.

At this point the approach used by FD08 is modified to consider an elemental volume $dV$ at $r$. Because $dV$ is small compared to $V_0$, the weighting function and the exponential function containing the radial component of wind can be moved outside the summation. Inside $dV$, the number of particles whose diameter falls between $D$ and $D + dD$ is $N(D,r) dV dD$. Thus, the autocorrelation function $R_v(\tau,r,t_n)$ for signals backscattered from all particles in $dV$ can be written as

$$dR_v(\tau,r,t_n) = I(r_0,r) \exp[-j2kv_y(r,t_n)\tau] \left\{ \int_0^\infty \sigma(D)N(D,r) \exp[2kv_y(D)r] dD \right\} dV. \quad (6)$$

As stated earlier, the diacritical "^" emphasizes that the assigned parameter is an estimate obtained from observations over a dwell time ending at $t_n$ of a wind field that changes with time. Equation (7) has the same form as Eq. (7) of FD08, and the correlation coefficients for the steady wind and turbulence [i.e., $\rho_v(\tau,r)$ and $\hat{\rho}_v(\tau,r)$] are exactly the same, but we have the added correlation factor $P(\tau,r)$.

It is noteworthy that $v_r(r)$ in Eq. (7c) is the radial component of wind. It inherently contains the contributions from both vertical and horizontal components of the mean wind. Because this study utilizes the spherical coordinate system, the results from Eq. (7) are applicable not only to the narrow beam but also to relatively broad beams.

The correlation coefficient associated with signals from $dV$ is defined as

$$\hat{\rho}_v(\tau,r,t_n) = \frac{\tilde{R}_v(\tau,r,t_n)}{\tilde{R}_v(0,r,t_n)} = \frac{\tilde{R}_v(\tau,r,t_n)}{\int_V I(r_0,r) \eta(r) dV} \quad (8a)$$

Then, the measured correlation coefficient is the spatial integral of Eq. (8a), that is,

$$\hat{\rho}_v(\tau,r_0,t_n) = \hat{\rho}_v(\tau,r,t_n) = \frac{\tilde{R}_v(\tau,r,t_n)}{\tilde{R}_v(0,r,t_n)} = \frac{\int_V I(r_0,r) \rho_v(\tau,r,t_n) \eta(r) dV}{\int_V I(r_0,r) \eta(r) dV}. \quad (8b)$$

Fourier transforming Eq. (8a) to the velocity domain, noting that the Fourier transform and volumetric integration are commutative, and using $\tilde{F}_v(g_1,g_2,g_3) = \tilde{F}_v(g_1) \star \tilde{F}_v(g_2) \star \tilde{F}_v(g_3)$, where $\tilde{F}_v$ denotes the Fourier transform in the velocity domain, one obtains the normalized Doppler spectrum of signals backscattered from particles in $dV$ (see the appendix).
\[ \hat{S}_n(\upsilon, \mathbf{r}, t_n) = \hat{F}_v[\hat{p}(\mathbf{r}, t_n)] \]

\[ = \frac{\eta(\mathbf{r})I(\mathbf{r}_0, \mathbf{r})}{I(\mathbf{r}_0, \mathbf{r})\eta(\mathbf{r})} \hat{F}_v[\hat{p}_s(\mathbf{r}, \mathbf{r}, t_n)] \hat{F}_v[\hat{p}_t(\mathbf{r}, \mathbf{r}, t_n)] \hat{F}_v \left[ \frac{P(\mathbf{r}, \mathbf{r})}{\eta(\mathbf{r})} \right] dV \] (9a)

in which we have multiplied the numerator and denominator by \( \eta(\mathbf{r}) \). Thus using Eq. (2b) this equation reduces to

\[ \hat{S}_n(\upsilon, \mathbf{r}, t_n) = H_n(\mathbf{r}_0, \mathbf{r}) \hat{F}_v[\hat{p}_s(\mathbf{r}, \mathbf{r}, t_n)] \hat{F}_v[\hat{p}_t(\mathbf{r}, \mathbf{r}, t_n)] \hat{F}_v[\hat{p}_t(\mathbf{r}, \mathbf{r}, t_n)] dV \] (9b)

where each of the factors of Eq. (9a) has the form

\[ \hat{F}_v[\hat{p}_s(\mathbf{r}, \mathbf{r})] = \frac{2}{\lambda} \int_{-\infty}^{\infty} \hat{\rho}_s(\mathbf{r}, \mathbf{r}) \exp[j2k\upsilon] d\tau, \] (9c)

and

\[ \rho_p(\mathbf{r}, \mathbf{r}) = \frac{P(\mathbf{r}, \mathbf{r})}{\eta(\mathbf{r})} \] (9d)

is defined as the correlation coefficient associated with terminal velocities \( \upsilon_p \) within \( dV \).

The spatial integration of Eq. (9b) is the weighted normalized Doppler spectrum measured by radar.

Following the procedure of FD08, this measured Doppler spectrum can be expressed as

\[ \hat{S}_n(\upsilon, \mathbf{r}, t_n) = \int_V H_n(\mathbf{r}_0, \mathbf{r}) S_{np}(\upsilon, \mathbf{r}) \hat{S}_{nt}(\upsilon, \mathbf{r}, t_n) S_{np}(\upsilon, \mathbf{r}) dV, \] (10a)

in which we have defined

\[ S_{np}(\upsilon, \mathbf{r}) = \hat{F}_v[\hat{p}_t(\mathbf{r}, \mathbf{r})], \] (10b)

and we have incorporated into Eq. (10a) the normalized spectrum \( S_{np}(\upsilon, \mathbf{r}) \) resulting from the particle oscillation/wobble (FD08). It can be shown that [appendix, Eq. (A9)]

\[ S_{np}(\upsilon, \mathbf{r}) = \frac{1}{\eta(\mathbf{r})} \sigma(D) N(D, \mathbf{r}) dD \frac{dD}{\upsilon_p} \] (10c)

and

\[ S_{nt}(\upsilon, \mathbf{r}) = \hat{F}_v[\hat{p}_t(\mathbf{r}, \mathbf{r})] \] (10d)

Substituting Eq. (10c) into Eq. (10a), one obtains

\[ \hat{S}_n(\upsilon, \mathbf{r}_0, t_n) = \int_V H_n(\mathbf{r}_0, \mathbf{r}) S_{nt}(\upsilon, \mathbf{r}) \hat{S}_{nt}(\upsilon, \mathbf{r}, t_n) \hat{S}_{nt}(\upsilon, \mathbf{r}, t_n) \frac{N(D, \mathbf{r}) \sigma(D) dD}{\eta(\mathbf{r}) \upsilon_p} dV, \] (11a)

which can also be expressed as

\[ \hat{S}_n(\upsilon, \mathbf{r}_0, t_n) = \int_V \frac{I(\mathbf{r}_0, \mathbf{r}) S_{nt}(\upsilon, \mathbf{r}) \hat{S}_{nt}(\upsilon, \mathbf{r}, t_n) \hat{S}_{nt}(\upsilon, \mathbf{r}, t_n) \frac{N(D, \mathbf{r}) \sigma(D) dD}{\eta(\mathbf{r}) \upsilon_p} dV}{\int_V I(\mathbf{r}_0, \mathbf{r}) \eta(\mathbf{r}) dV} \] (11b)
Equations (11a) and (11b) are the generalized equations for the normalized Doppler spectrum observed with a vertically directed beam, and for which turbulence, mean wind, particle oscillation/wobble, and PSD are not necessarily homogeneous.

3. Analytical expressions for spectra related to shear and turbulence

a. Shear

The measured Doppler spectrum is usually the convolution of several component spectra. If the exact analytical expression of each component is known, one should in principle be able to separate each component spectrum from the measured spectrum. This separation will help to retrieve PSD and turbulence intensity. The goal of this subsection is to derive the analytical expression for the spectrum due to the shear of the mean wind, and in the next section 3b, the spectrum due to turbulence.

By substituting $v_s = -\lambda f_s(r)/2$ into Eq. (7c) and then substituting the result into Eq. (A5a) of the appendix, Eq. (10d) can be written as

$$S_{ns}(v, r) = \frac{2}{\lambda} \hat{F}[\rho_s(\tau, r)]$$

$$= \frac{2}{\lambda} \int_{-\infty}^{\infty} \exp\{-j2\pi[f - f_s(r)]\} \, d\tau. \quad (12a)$$

The integral in Eq. (12a) is the delta function $\delta[f - f_s(r)]$ having a zero value everywhere except at $f = f_s(r)$. Thus,

$$S_{ns}(v, r) = \frac{2}{\lambda} \delta[f - f_s(r)] = \delta[v - v_s(r)], \quad (12b)$$

where $v$ is the Doppler velocity and $v_s(r)$ is the mean of the radial component of wind. To obtain the last equality, relation (A4) has been used. It is emphasized that Eq. (12b) is the spectrum of the radial component $v_s$ of the mean wind in $dV$ at $r$, and is not the radar-observed spectrum. Furthermore, because $v_s$ is a deterministic function of $r$, the radar-observed spectrum, resulting from the spatially dependent $v_s$, is

$$S_{ns}(v, r_0) = \int_V H_n(r_0, r) \delta[v - v_s(r)] \, dV. \quad (13a)$$

Noting Eq. (2b) and that $\delta[v - v_s(r)]$ is infinity at $v = v_s$ and equal to zero elsewhere, Eq. (13a) shows that $S_{ns}(v, r_0)$ at $v$ is the ratio between the power scattered by particles with $v_s(r) = v$ and the total power scattered by all of the particles. Furthermore, assuming only horizontal wind $v_h$ changing with height $h$, $v_s$ can be written as

$$v_s(r) = v_h(h) \sin(\theta) \sin(\phi). \quad (13b)$$

Without loss of generality, it has been assumed that $v_h$ is eastward and $\phi$ is the azimuth of the point at $r$. From Eq. (13b), it can be clearly seen that $v_s(r)$ is the radial component of cross-beam wind, a function of position within the beam, and the cause of spectral broadening resulting from cross-beam wind. Thus, Eq. (13a) is applicable not only to the narrow beams, but also to relatively broad beams.

b. Turbulence

Using Doppler spectra to investigate turbulence can be dated back to the 1950s (Hitschfeld and Dennis 1956; Rogers 1957). This application implicitly assumes that the Doppler spectrum can be related to the probability density of the turbulence. This assumption was not theoretically proved until the work done by Doviak and Zrnić (2006). For statistically homogeneous turbulence with zero mean velocity, Doviak and Zrnić (2006, their section 5.2.3) show that the normalized expected Doppler spectrum $S_m(v)$, purely resulting from homogeneous turbulence, is equal to the probability density function $p(v)$ of the turbulent velocity $v$. The purpose of this section is to investigate the relationship between the estimated Doppler spectrum and the probability density of inhomogeneous turbulence.

The correlation coefficient related to statistically homogeneous and stationary turbulence is (Doviak and Zrnić 2006, their section 5.2.3)

$$\rho_\tau = E_v[\hat{\rho}_I(\tau)] = \int_{-\infty}^{\infty} p(v)(v) \exp(-j2k v \tau) \, dv. \quad (14a)$$

where $p(v)$ is the probability density of turbulent velocities $v$. It is deduced that the expression for inhomogeneous but stationary turbulence is

$$\rho_\tau(\tau, r) = E_{v, r}[\hat{\rho}_I(\tau, r, t_n)] = \int_{-\infty}^{\infty} p(v, r) \exp(-j2k v \tau) \, dv. \quad (14b)$$

Practically, we only have estimates of $\rho_\tau(\tau, r)$ corresponding to estimates of $p(v, r)$. An estimate of $\rho_\tau(\tau, r)$ is obtained from data collected over a relatively short dwell time compared to the overturning time of the largest scales of turbulence. In this case, Eq. (14b) has the form

$$\hat{\rho}_I(\tau, r, t_n) = \int_{-\infty}^{\infty} \hat{p}_I(v, r, t_n) \exp(-j2k v \tau) \, dv. \quad (15)$$
where \( \hat{p}_i(v, r, t_n) \) is an estimate of \( p_i(v, r) \). Equation (15) is similar to Eq. (7d), but Eq. (7d) assumes that turbulence is frozen [i.e., the dwell time \( MT_\lambda \) is sufficiently short that the turbulent field \( v_i(r, t_n) \) is fixed during the dwell time at \( t_n \)], whereas \( v_i \) in Eq. (15) has all of the values determined by \( \hat{p}_i(v, r, t_n) \). Although stationary turbulence is assumed, the estimates \( \hat{p}_i(v, r, t_n) \) change with time as the turbulent field changes from \( t_n \) to \( t_{n+1} \).

If no other spectral broadening mechanisms are active, the radial component of turbulence \( v_r = v \) the Doppler velocity. Thus, Eq. (10e) can be rewritten as

\[
\hat{S}_{nt}(v, r, t_n) = \hat{F}_v[\hat{p}_i(v, r, t_n)] = \hat{F}_v \left[ \int_{-\infty}^{\infty} \hat{p}_i(v, r, t_n) \exp(-j2kv) dv \right]. \tag{16a}
\]

The integral in Eq. (16a) is the inverse Fourier transform of \( \hat{p}_i(v, r, t_n) \) (see the appendix). Thus, Eq. (16a) reduces to

\[
\hat{S}_{nt}(v, r, t_n) = \hat{F}_v \hat{F}_v^{-1}[\hat{p}_i(v, r, t_n)] = \hat{p}_i(v, r, t_n), \tag{16b}
\]

and the radar-measured Doppler spectrum resulting from inhomogeneous but stationary turbulence is

\[
\hat{S}_{nt}(v, r, t_n) = \hat{F}_v \int_{-\infty}^{\infty} \hat{p}_i(v, r, t_n) dV.
\]

Equation (16c) states that the radar-measured Doppler spectrum is equal to the \( H_n(r_0, r) \)-weighted volumetric integral of the spatial distribution of the estimated probability density of statistically stationary turbulence. Equation (16c) even holds for the more general flow. This can be seen by replacing \( v = v \), with \( v = v_r + v_t \) and repeating the above procedure (Fang 2008, chapter 4). It is noteworthy that even if turbulence is stationary and homogeneous, \( \hat{p}_i(v, r, t_n) \) and \( \hat{S}_{nt}(v, r, t_n) \) remain functions of time and space because \( \hat{p}_i(v, r, t_n) \) and \( \hat{S}_{nt}(v, r, t_n) \) are estimated during the short dwell time during which the continuously changing turbulent field is considered frozen.

If a large number \( N \) of spectra \( \hat{S}_{nt}(v, r, t_n) \) are temporally averaged so that \( N_{\lambda_{nt}} \) is longer than the overturning time of the largest scales of turbulence, then Eq. (16c) can be expressed as

\[
\langle \hat{S}(v, r_0, t_n) \rangle_i = \int V H_n(r_0, r) \hat{p}_i(v, r, t_n) dV \simeq S(v, r_0) = \int V H_n(r_0, r) p_i(v, r) dV. \tag{16d}
\]

In this case temporal averaging approximates an average over the ensemble of turbulent fields. If turbulence is homogeneous [i.e., \( p_i(v, r) \sim p_i(v) \)], then \( p_i(v, r) = S(v, r_0) \), independent of the reflectivity, angular, and range weighting functions. Moreover, if turbulence is stationary and homogeneous in a layer of stratiform precipitation observed with a radar resolution volume that is small compared to the layer thickness, and temporal averages are taken over times that are long compared to the overturning time of the largest turbulence scale (about the thickness of the turbulent layer), then the probability density of turbulence in that layer can be derived.

4. The equation for Doppler spectra related to terminal velocity only

Equation (1) is a widely applied equation. This section will show the conditions under which Eq. (1) applies. To do so, assume \( v_s = 0 \), \( v_t = 0 \), and \( S_{nt}(v, r) = 1 \). Without mean and turbulent motion and particle’s oscillation/wobble, Eq. (11b) reduces to

\[
S_n(v, r_0) = S_{np}(-v_p, r_0) = \int V I(v_0, r) \sigma(D) N(D, r) \left( \frac{dD}{dt} \right) dV
\]

and if the integrand of Eq. (17a) is expressed in the particle velocity domain, then Eq. (17a) becomes

\[
S_n(v, r_0) = S_{np}(-v_p, r_0) = \int V I(v_0, r) \sigma(v_p) N(v_p, r) dV
\]

Because the Doppler spectrum is the mirror image of the terminal velocity spectrum, the Doppler velocity \( v \) is replaced with \(-v_p\); the argument \( t_n \) is not present because PSD is assumed to be stationary. Equation (17) is the integral form of the analytical expression for normalized Doppler spectrum associated with terminal velocity if the PSD is not homogeneous.

If the PSD is homogeneous, then \( \eta(r) \) is independent of \( r \) and Eq. (17b) reduces to

\[
S_{np}(-v_p, r_0) = \frac{\sigma(v_p) N(v_p)}{\eta}. \tag{18a}
\]

or, in terms of \( D \),
This is exactly Eq. (1). If we further assume particles are spherical and satisfy the Rayleigh condition (i.e., $D \ll \lambda$), then the nonnormalized version of Eq. (18b) (i.e., without $\eta$) is identical to Eq. (11) of WG09. However, if the PSD is not uniform, then the observed weighted normalized Doppler spectrum is Eq. (17).

Therefore, with the work presented in this section we have derived a general analytical Doppler spectrum expression for terminal velocity, and we show that the previously derived equation is a result of Eq. (17) when the PSD is uniform and wind is absent. Researchers have noted that either inhomogeneous reflectivity or a PSD inside $V_0$ can impact radar measurements (Campos et al. 2008; Johnston et al. 2002; Sarchilli et al. 1990; Atlas et al. 1969; Sloss and Atlas 1968). Equation (11), which includes nonuniform turbulence and mean wind shear, is the analytical expression for such kinds of studies.

The derivation of Eqs. (11), (17), and (18) neither requires that scatterers are spherical water drops nor that they satisfy Rayleigh scatter conditions. Ice particles can have different shapes, but we can define an effective diameter. The effective diameter is one possessed by all particles falling at the same terminal velocity. As long as the relation between the particle’s terminal velocity and its effective diameter is unique, Eqs. (11), (17), and (18) also apply to ice particles.

5. Conclusions

This study obtains a generalized analytical expression for the Doppler spectra of scatter from hydrometeors observed with a vertically pointed beam. This expression includes the spectral broadening mechanisms of mean wind, turbulence, particle oscillation/wobble, and the dispersion of particle terminal velocity, all of which are not necessarily spatially uniform. Without mechanisms other than the dispersion of terminal velocities, this equation can be related to the nonuniform particle size distribution that may find its application in studying the effect on the radar measurements of inhomogeneous reflectivity or PSD inside the radar resolution volume. If particle size distribution is uniform, then this generalized equation reduces to a previously derived expression. In this derivation we are able to relax the restriction of a “narrow beam” to take mean radial wind shear into account. That is, beam broadening resulting from mean cross-beam wind is included in the expressions for the Doppler spectrum associated with vertically pointed beams.

Moreover, the spectrum equations derived in this study are applicable to ice particles as well as water droplets if the relation between the effective diameter of an ice particle and its terminal velocity is unique.

This study also derives an analytical expression [i.e., Eq. (13a)] for the normalized Doppler spectrum of mean wind. It reveals that this expression is the ratio between the power scattered by particles with $v_p = v$ and the total power scattered by all particles. It is expected that these will find application in studying spectrum width broadening resulting from cross-beam wind and the horizontal shear of updraft.

It has been shown that without spectral broadening mechanisms other than turbulence the radar-measured Doppler spectrum is a volumetric mean of the estimated probability density of the turbulence weighted by $H_n(r_p, r)$. This holds even for the general flow with nonzero mean velocity. Moreover, if turbulence is statistically stationary and homogeneous in a layer of thickness that is large compared to range resolution, and temporal averages are taken over times that are long compared to the overturning time of the largest turbulence scale (about the thickness of the turbulent layer), then the probability density of turbulence can be derived from the spectra.

Acknowledgments. This work was supported by NSF Grant ATM0715235.

APPENDIX

The Expression for Doppler Spectra Associated with Particles’ Terminal Velocities

This appendix will derive the analytical expression for the expected Doppler spectrum associated with particles’ terminal velocities in $dV$ at $r$ in detail. In the frequency domain the normalized Doppler spectrum is the Fourier transform of the correlation coefficient and is defined as

$$S_n(f, r) = \tilde{F}[\rho(\tau, r)] = \int_{-\infty}^{\infty} \rho(\tau, r) \exp(-j2\pi f \tau) \, d\tau,$$  \hspace{1cm} (A1a)

and

$$\rho(\tau, r) = \tilde{F}^{-1}[S_n(f, r)] = \int_{-\infty}^{\infty} S_n(f, r) \exp(j2\pi f \tau) \, df$$  \hspace{1cm} (A1b)
is the inverse Fourier transform. Substituting \( f = -2\nu/\lambda \),
where \( \nu \) is the Doppler velocity, a dummy variable in Fourier transform that should not be confused with the radial physical velocity \( u(r, t) \) of wind, one obtains

\[
S_n(f, r) = \int_{-\infty}^{\infty} \rho(\tau, r) \exp\left(\frac{j4\pi\nu\tau}{\lambda}\right) d\tau
= \int_{-\infty}^{\infty} \rho(\tau, r) \exp(j2k\nu r) d\tau. \tag{A2}
\]

If \( S_n(\nu, r) d\nu \) represents the fraction of the power returned from scatterers whose radial velocity lies in \( d\nu \), then we have

\[
S_n(\nu, r) d\nu = -S_n(f, r) df, \tag{A3}
\]

where the negative sign is needed because increasing \( d\nu \) corresponds to decreasing \( df \). Using the relation \( f = -2\nu/\lambda \) and the above equation, one has

\[
S_n(\nu, r) = \frac{2}{\lambda} S_n(f, r), \tag{A4}
\]

which states that, given a power spectrum in frequency domain, one needs to substitute \(-2\nu/\lambda \) for \( f \) and multiply \( S_n(f) \) by \( 2/\lambda \) to obtain the power spectrum \( S_n(\nu) \) represented in the Doppler velocity domain. Thus, the Fourier transform \( \tilde{F}_v \) in velocity domain is defined as

\[
\tilde{F}_v[\rho(\tau, r)] = S_n(\nu, r) = \frac{2}{\lambda} \int_{-\infty}^{\infty} \rho(\tau, r) \exp(j2k\nu r) d\tau
= \frac{2}{\lambda} \tilde{F}(\rho(\tau, r)), \tag{A5a}
\]

and the inverse Fourier transform in velocity domain is

\[
\tilde{F}^{-1}_v[S_n(\nu, r)] = \rho(\tau, r) = \int_{-\infty}^{\infty} S_n(\nu, r) \exp(-j2k\nu r) d\nu
= \tilde{F}^{-1}[S_n(f, r)]. \tag{A5b}
\]

By substituting Eq. (7b) into Eq. (9d) and the result into Eq. (A5a), \( S_{np}(\nu, r) \) can be expressed as

\[
S_{np}(\nu, r) = \tilde{F}_v[p^p(\nu, \tau)]
= \frac{1}{\eta(\nu)} \tilde{F}_v\left[ \int_{0}^{\infty} \sigma(D) \exp[j2k\nu p^p(D)\tau] N(D, r) dD \right]. \tag{A6}
\]

The number of particles per unit volume per unit diameter interval must equal the number of particles per unit volume per unit terminal velocity interval, and because \( \nu_p = -\nu \),

\[
N(D, r) dD = N[\nu_p(D), r] d\nu_p = -N(\nu, r) d\nu. \tag{A7a}
\]

Thus, Eq. (A6) can be written as

\[
S_{np}(\nu, r) = \frac{1}{\eta(\nu)} \tilde{F}_v\left[ \int_{-\infty}^{\infty} \sigma(\nu) \exp(-j2k\nu r) N(\nu, r) d\nu \right], \tag{A7b}
\]

where \( N(\nu, r) = 0 \) when \( \nu > 0 \), that is, \( \nu_p < 0 \), has been used to change the upper integral limit to infinity. The integral in Eq. (A7b) has the same form as the integral in Eq. (A5b); thus by replacing \( S_n(\nu, r) \) in Eq. (A5b) with \( \sigma(\nu)N(\nu, r) \), one can rewrite Eq. (A7b) as

\[
S_{np}(\nu, r) = \frac{1}{\eta(\nu)} \tilde{F}_v\left[ \sigma(\nu)N(\nu, r) \right] = \frac{\sigma(\nu)N(\nu, r)}{\eta(\nu)}. \tag{A8}
\]

Substituting \( \nu \) with \( -\nu_p \) and using Eq. (A7a), we have

\[
S_{np}(\nu_p, r) = \frac{\sigma(\nu_p)N(\nu_p, r)}{\eta(\nu)} = \frac{\sigma(D)N(D, r) dD}{\eta(\nu)} d\nu_p. \tag{A9}
\]

REFERENCES


