THE PREDICTION OF GENERAL QUASI-GEOSTROPHIC MOTIONS

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ABSTRACT

Methods are presented for the numerical integration of several rather general approximations to the quasi-geostrophic equations of motion. The choice of the vertical grid interval involved in these approximations is justified by scale considerations. Three of the approximations involve the integration of non-linear partial differential equations of elliptic type, and the method of solution used is described in some detail because of its quite general applicability.

The results of the integrations for each version are presented and compared with those obtained earlier from simple versions. The principle conclusion is that the prediction error is not eliminated by a more refined treatment of the quasi-geostrophic equations, but is at least partly inherent in the geostrophic formulation itself. The retention of terms that are neglected in the first-order geostrophic approximation changes the aspect of the error but does not eliminate it.

1. Introduction

This article carries further an investigation of the quasi-geostrophic formalism as applied to the problem of short-range prediction. Earlier papers 2 have dealt with integrations of simplified versions of the quasi-geostrophic equations obtained by introducing certain linearizations. In the present treatment, the equations are integrated in more nearly their full generality; and where artificial constraints were imposed before to reduce the number of degrees of freedom in the vertical to three or less, the division of the vertical scale is now determined solely by the requirement that the vertical resolution of the fields of motion shall be equal to the horizontal resolution. The actual division is based on the following considerations.

It can be shown by means of scale arguments that the characteristic horizontal and vertical scales of the quasi-geostrophic motions, \( S \) and \( H \), respectively, satisfy the order of magnitude relationship

\[
\frac{S^2}{H^2} \sim \frac{g \partial \ln \Theta}{f^2 \partial z},
\]

in which \( g \) is the acceleration due to gravity, \( f \) the Coriolis parameter, \( z \) the vertical coordinate and \( \Theta(z) \) the characteristic potential temperature. 3 This relation may be given the following two interpretations. First, since \( (g \partial \ln \Theta/\partial z)^1 \) is the frequency of a vertical buoyancy oscillation and \( f \) the frequency of a horizontal inertial oscillation, the right-hand side of (1) is the ratio of a force per unit displacement in the vertical to a force per unit displacement in the horizontal. Thus, (1) states that the squares of the horizontal and vertical scales are in inverse ratio to the Coriolis and buoyancy forces per unit displacement. Second, if it is assumed that the large-scale transient disturbances of middle latitudes originate as dynamically unstable perturbations of a mean condition specified by the parameters \( f, H \) and \( \partial \ln \Theta/\partial z \), (1) may be interpreted as giving the order of the horizontal scale of the most unstable disturbance. Indeed, theory predicts a relationship of exactly this form between the wavelength of maximum instability and the parameters characterizing the mean flow (Eady, 1949).

In middle latitudes, the approximate numerical value of the right-hand side of (1) is 10^4. Hence, \( S \sim 100 \ H; \) and if the horizontal space increment in the finite-difference equations is the usual 300 km, the vertical increment should be 3 km. Taking 12 km as the scale height of the atmosphere, we find that four vertical grid intervals give the required vertical resolution. Empirically also, from inspection of vertical sections of the pressure anomaly or vertical velocity, one sees that four intervals, or five levels, in the \((fC V/gH)(\partial \ln \Theta/\partial z)\) with \( C \) denoting the characteristic phase speed. A similar evaluation of the terms in the quasi-geostrophic vorticity equation gives the expression \( CV/f S^2 \) for the characteristic horizontal divergence \( W/H \). Equation (1) is then obtained by eliminating \( W \).
vertical are equivalent to a horizontal resolution of about 300 km.

We proceed now to describe the methods of integration used and the results obtained from the solution of a number of different finite-difference approximations to the general quasi-geostrophic equations, all however with the vertical spacing based on the above reasoning. It should be noted that while we report here on several versions, they were developed independently and therefore do not form a definite hierarchy.

2. Derivation of basic equations

Since the quasi-geostrophic equations are but approximations to the equations of motion, one may exercise a certain choice in the selection of the appropriate formulations. A quite general system was originally proposed by Charney (1948). Taking pressure as the vertical coordinate and eliminating the horizontal divergence between the vorticity equation

$$\frac{D\eta}{Dt} + \omega \frac{\partial \eta}{\partial \rho} = - \eta \nabla \cdot V - k \cdot \nabla \omega \times \frac{\partial V}{\partial \rho},$$  

and the continuity equation

$$\nabla \cdot V + \frac{\partial \omega}{\partial \rho} = 0,$$

one obtains

$$\frac{D\eta}{Dt} + \omega \frac{\partial \eta}{\partial \rho} = \eta \frac{\partial \omega}{\partial \rho} - k \cdot \nabla \omega \times \frac{\partial V}{\partial \rho}. $$

These equations are then combined with the primitive Eulerian equations

$$\frac{DV}{Dt} + \omega \frac{\partial V}{\partial \rho} + f k \times V = - \nabla \phi,$$

and the adiabatic equation

$$\left( \frac{D}{Dt} + \omega \frac{\partial}{\partial \rho} \right) \theta = 0, \quad \theta = - \frac{1}{R} \frac{\partial}{\partial \rho} \rho \frac{\partial \phi}{\partial \rho},$$

to yield the equation of conservation of potential vorticity:

$$\left( \frac{D}{Dt} + \omega \frac{\partial}{\partial \rho} \right) q = 0,$$

$$q = - \eta \frac{\partial \theta}{\partial \rho} - k \cdot \frac{\partial V}{\partial \rho} \times \nabla \theta.$$  

The last two equations [or alternatively (4) and (6)], together with the hydrostatic and geostrophic equations,

$$\frac{\partial \phi}{\partial \rho} = - \frac{1}{\rho},$$

and

$$V = f^{-1} k \times \nabla \phi = V_0,$$  

constitute the required system. It is here assumed that the flow, in addition to being quasi-hydrostatic and quasi-geostrophic, is also non-viscous and adiabatic. It is, of course, recognized that eventually one will have to take into account both friction and heat sources and sinks. For the present, however, we are concerned only with determining the properties of a frictionless, thermally inactive atmosphere. In (2) to (9), the following notation is used: $\rho$ = pressure; $\rho$ = density; $V$ = horizontal velocity; $k$ = vertical unit vector; $\nabla$ = horizontal gradient operator; $D/ Dt$ = material derivative, $\partial/\partial t + V \cdot \nabla$; $\omega$ = horizontal component of motion; $\eta$ = vertical component of absolute vorticity, $f + k \cdot \nabla \times V$; $\omega$ = material derivative of pressure; $\phi$ = the geopotential of an isobaric surface; $R$ = gas constant; and $c_1$ and $c_\rho$ are the specific heats of air at constant volume and constant pressure, respectively.

The geostrophic approximation is based on the assumption that the intrinsic Rossby number $V/fS$, which measures the ratio of inertial to Coriolis force, is small compared with unity. This implies that the relative vorticity is small compared with $f$; hence, undifferentiated $\eta$'s should be replaced by $f$. Further, on the basis of scale arguments (Charney, 1948), it can be shown that the second terms on both sides of (4) are small in the same proportion as the intrinsic Rossby number is small. To be consistent, therefore, one should replace (4) by

$$\frac{D\eta}{Dt} = f \frac{\partial \omega}{\partial \rho}. $$

If, however, the second term is dropped from only the right-hand member of (4), (7) is replaced by the conservation equation

$$\left( \frac{D}{Dt} + \omega \frac{\partial}{\partial \rho} \right) q = 0,$$

and the adiabatic equation

$$\left( \frac{D}{Dt} + \omega \frac{\partial}{\partial \rho} \right) \theta = 0, \quad \frac{\partial}{\partial \rho} \rho \frac{\partial \phi}{\partial \rho},$$

Again, if the second terms in both members of (4) are dropped but $\eta$ is not replaced by $f$, one obtains the equation

$$\frac{D\eta}{Dt} = \frac{\partial \omega}{\partial \rho} = - \frac{1}{R} \frac{\partial}{\partial \rho} \rho \frac{\partial \phi}{\partial \rho}. $$

The various finite-difference systems to be derived are based on (6), (8) and (9), together with one of (7), (10), (11) or (12).

3. Non-linear approximation

This version is based on (6), (8), (9) and (11). Eliminating $\omega$ between (6) and (12), we obtain the

*Hereafter referred to as NLI. The term "non-linear" is used to distinguish the present approximation from others in which certain partial linearizations are made. In all versions, however, the non-linearity of the advective terms is retained.}

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equation
\[
\frac{\partial \theta}{\partial \rho} \frac{Dq}{Dt} - \frac{\partial q}{\partial \rho} \frac{D\theta}{Dt} = 0.
\] (13)

Using the finite-difference approximations for the vertical derivatives of \( \phi \), we have
\[
\frac{\partial \phi}{\partial \rho} = (2 \Delta \rho)^{-1}(\phi_{k+1} - \phi_{k}),
\]
\[
\frac{\partial^2 \phi}{\partial \rho^2} = (\Delta \rho)^{-2}(\phi_{k+1} + \phi_{k-1} - 2 \phi_{k}).
\]

The spherical earth is mapped conformally onto a plane with the Cartesian coordinates \( x \) and \( y \). The geostrophic absolute vorticity \( \eta \) then becomes, in view of (9),
\[
\eta = \nabla \cdot \left( \frac{1}{f} \nabla \phi \right) + f \approx \frac{m^2}{f} \nabla^2 \phi + f,
\] (14)

where the symbol \( \nabla^2 \) stands for the Laplace operator in the plane, and \( m(x, y) \) is the scale factor of the mapping. If the region of interest is between 30 and 60 deg lat, and a Lambert conformal projection, with standard parallels at 30 and 60 deg lat is used, the scale factor can be taken to be unity without appreciable error. Thus, when the terms in (13) are expanded, we obtain
\[
\nabla^2 \left( \frac{\partial \phi}{\partial t} \right)_k + \sum_{r=k+1}^{k+1} A_{kr} \left( \frac{\partial \phi}{\partial t} \right)_r + \mu_k = 0,
\] (15)

where the coefficients \( A_{kr} \) and \( \mu_k \) are functions of \( \phi \) and its spatial derivatives only.

In accordance with the reasoning given in the first section, the vertical scale for the integration to be described was divided into four intervals, and the five isobaric levels listed below were adopted as information levels:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(mb)</td>
<td>200</td>
<td>400</td>
<td>600</td>
<td>800</td>
<td>1000</td>
</tr>
</tbody>
</table>

Equation (15) holds only for the three interior levels, \( k = 1, 2 \) and \( 3 \); the \( \phi \)-tendencies on the boundaries, \( k = 0 \) and \( 4 \), must be determined by use of the boundary conditions. These were assumed to be that \( \omega \) is zero on the upper and lower boundaries, with the consequence [from (6)] that temperature is horizontally advected at these levels. If the 1000-mb temperature is expressed as the thickness between 800 and 1000 mb, and the 200-mb temperature as the thickness between 200 and 400 mb, we obtain from the boundary conditions expressions relating the 1000-mb tendency to the 800-mb tendency and the 200-mb tendency to the 400-mb tendency.

It further follows from the boundary conditions that \( q \) is advected horizontally at the upper and lower boundaries. This enables us to calculate \( q \) at these boundaries at successive time steps, a necessary prerequisite to the computation of the coefficients \( A_{kr} \) and \( \mu_k \) which involve \( \partial q/\partial \rho \).

Charney et al. (1956), using heuristic arguments, arrived at the conclusion that in barotropic non-divergent flow the conditions at a horizontal boundary should be the specification of \( \phi \) everywhere on the boundary and the absolute vertical vorticity, \( \eta \), on that part of the boundary at which fluid is entering the interior region. By similar reasoning, one arrives at the conclusion that the same conditions should apply in the baroclinic case if \( q \) is the potential vorticity. However, Platzman (1954) has shown from the simple advection equation,
\[
\frac{\partial q}{\partial t} + u \frac{\partial q}{\partial x} = 0,
\]
that a variation of this condition, namely to specify \( q \) everywhere on the boundary instead of just at inflow points, although physically incorrect, will nevertheless yield a mathematically stable forecasting system. We have found by numerical experimentation that this conclusion holds for both barotropic and three-level predictions and have consequently, in all the integrations of the quasi-geostrophic equations described in this article, assumed that both \( \phi \) and \( q \) remain constant and equal to their initial values on the lateral boundaries for all time. Furthermore, the initial values of \( q \) have been computed on the assumption that \( \eta = f \) on these boundaries.

The prediction proceeds in the following manner:

1. The coefficients \( A_{kr} \) and \( \mu_k \) are computed from the \( \phi \) and \( q \) fields.
2. The \( \phi \)-tendencies at the interior levels are obtained by integrating (15).
3. The \( \phi \)-tendencies at the upper and lower boundaries are obtained from those at the adjacent interior levels by application of the boundary conditions.
4. The \( \phi \)-tendencies are used to compute the \( \phi \)-values at the next time step.
5. The \( q \)-values at the next time step are computed at interior levels by using the definition of \( q \), and at the upper and lower boundaries by using the boundary condition that \( q \) is advected horizontally at 200 and 1000 mb.
6. This cycle is repeated as many times as is required to span the forecast interval.

The computations, which were made in ½-hr time steps, proceeded without mishap until a 7½-hr prediction was obtained. At this point, difficulties were encountered which made it impossible to continue the computations. These difficulties were associated with the solution of the differential equation (15). The method of solution used for this equation was a relaxation scheme analogous to the one described by Charney (1954) for a similar equation with constant coefficients. For a rectangular grid \( x = i \Delta s, y = j \Delta s (i = 0, 1, \ldots, n; j = 0, 1, \ldots, m) \), the finite-difference analog
of (15) is

\[
(\Delta s)^{-2} \left\{ b_{i+1,j,k} + b_{i-1,j,k} + b_{i,j+1,k} + b_{i,j-1,k} - 4b_{ijk} \right\} \\
+ \sum_{r=k-1}^{k+1} A_{ijk}b_{ijr} + \mu_{ijk} = 0, \quad (16)
\]

in which the subscripts \(i\) and \(j\) denote quantities at \((x = i \Delta s, y = j \Delta s)\), and \(b = \partial \phi / \partial t\). If the superscripts \(\nu\) and \(\nu + 1\) denote successive approximations, the solution of (16) is obtained by the iterative scheme

\[
b_{ijk}^{\nu+1} = b_{ijk}^{\nu} + \lambda \left[ 4(\Delta s)^{-2} - A_{ijk} \right]^{-1} \times \left\{ b_{i+1,j,k} + b_{i-1,j,k} + b_{i,j+1,k} + b_{i,j-1,k} - 4b_{ijk} \right\} - 4b_{ijk}^{\nu} + (\Delta s)^{2} \left( \sum_{r=k-1}^{k+1} A_{ijk}b_{ijr}^{\nu} + \mu_{ijk}^{\nu} \right),
\]

\((i = 1, \ldots, n - 1; j = 1, \ldots, m - 1; k = 1, 2, 3; \nu = 0, 1, 2, \ldots), \quad (17)\)

applied to the points \(ijk\) in lexicographic order with \(k\) first, \(j\) second and \(i\) last. The \(\lambda\) is chosen by experiment so as to optimize the rate of convergence of the iterative process.

Equation (16) actually represents a particular set of linear algebraic equations, the general form of which is

\[
\sum_{j=1}^{N} a_{ij} u_j + \alpha_i = 0, \quad i = 1, 2, \ldots, N.
\]

Collatz (1951) has shown that a sufficient condition for obtaining the \(N\) unknowns \(u_i\) by an iterative process of the type described above is that

\[
|a_{ij}| \geq \sum_{j=1, j \neq i}^{N} |a_{ij}|,
\]

where for some \(i\) the strict inequality holds. In the case of (16) this convergence condition becomes, for a division of the vertical scale into an arbitrary number of intervals,

\[
q_k > \frac{1}{2a^2} \left( \sigma - \frac{c_p}{c_p + c_k} \right)^{-1} (q_{k+1} - q_{k-1}),
\]

where \(\sigma = 4\) or 2, according as the \(k\)-level is adjacent to a boundary or not. We note that the terms on the right of the above inequalities have no physical significance, since they vanish in the limit as \(\Delta \rho\) approaches zero. It was the failure to satisfy these conditions which apparently led to the difficulties experienced in trying to extend the prediction past 7½ hr. Since the Collatz conditions can be shown to be only sufficient, the failure may have arisen from other causes. However, in another case, namely the integration of SL1 (see section 5, below), the conditions did appear, by experiment, to be necessary.

Unfortunately, in the present version of the quasi-geostrophic equations the history of the motion at any given time is carried by the \(\phi\)-field, and other quantities such as \(q\) are deduced therefrom. Hence it is not possible to adjust the \(q\)’s so as to satisfy the convergence conditions and then to make the corresponding \(\phi\) adjustments. By contrast, in the other versions to be described, the history of the motion is carried by the \(q\)-field and the \(\phi\)-field is derived from it as required. In these cases, if the \(q\)’s have to be modified to satisfy convergence criteria, the \(\phi\)’s can easily be adjusted accordingly. An attempt was made to adjust the coefficients \(A_{ijr}\) to comply with the Collatz convergence criteria before (16) was solved for the tendencies. This process, however, brought no improvement, for it became impossible to maintain correspondence between the \(\phi\) and \(q\)-fields, and the predicted \(\phi\)-field rapidly diverged.

4. Non-linear approximation 2

This version is based on (7), (8) and (9), together with the additional, rather artificial, assumption that we can neglect the vertical-advection terms in (7). We obtain the horizontal conservation equation

\[
Dq / Dt = 0, \quad (18)
\]

with \(q\) defined by

\[
q = -\eta \frac{\partial \ln \theta}{\partial \rho} - \rho \left( \nabla \frac{\partial \phi}{\partial \rho} \right)^2. \quad (19)
\]

If we wish to integrate these equations using \(q\) to determine the motion in a manner analogous to that described by Charney (1954) for the three-level model, we require a method of solving (19) for \(\phi\) when \(q\) is given. However, whereas in the three-level model the corresponding equation was linear, we now have a non-linear equation to solve. Since such non-linear equations occur in other meteorological contexts, for example in the theory of “balanced” non-geostrophic flow (Charney, 1955a; 1955b), we now describe the integration process in some detail.

It is convenient, before proceeding to the actual method used, to discuss the nature of (18) and (19) and an alternative method of solution. If the indicated time differentiations in (18) are carried out, we obtain

\[
\frac{\partial q}{\partial t} = F \left( \frac{\partial \phi}{\partial t} \right) = -\frac{1}{f} \frac{\partial \ln \theta}{\partial \rho} \frac{\partial \rho}{\partial t} \left( \frac{\partial \phi}{\partial \rho} \right)^2 + \rho \frac{\partial^2 \phi}{\partial t^2} + \rho \frac{c_p}{c_p + c_k} \frac{\partial \phi}{\partial t} + 2\rho \left[ \frac{\partial^2 \phi}{\partial \rho \partial x} \frac{\partial \phi}{\partial \rho} \frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial \rho \partial y} \frac{\partial \phi}{\partial \rho} \frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial t} \right] = -V \cdot \nabla q. \quad (20)
\]

\(^{4}\) Hereafter referred to as NL2.
Equation (20) is of second order in the tendency $\partial \psi / \partial t$. To determine its type, we have to consider the corresponding quadratic form

$$
\xi^2 + \mu^2 + 2\alpha \xi \mu + 2b \mu \alpha + c \alpha^2,
$$

where

$$
a = \frac{\rho}{\partial \ln \theta / \partial \rho} \frac{\partial^2 \psi}{\partial \rho \partial \alpha}, \quad b = \frac{\rho}{\partial \ln \theta / \partial \rho} \frac{\partial^2 \psi}{\partial \rho \partial \gamma},
$$

and

$$
c = \frac{\rho}{\partial \ln \theta / \partial \rho}.
$$

Since this form may be written

$$(\xi + \alpha \eta)^2 + (\mu + \beta \eta)^2 + (\eta - \alpha^2 - \beta^2) \eta^2,$$

it follows immediately that a necessary and sufficient condition for (20) to be elliptic in $\partial \psi / \partial t$ is that

$$
c - \alpha^2 - \beta^2 > 0.
$$

Substituting in this relationship the expressions for $a$, $b$ and $c$, we obtain the criterion

$$
\frac{f_p}{(\partial \ln \theta / \partial \rho)^2} \left[ - \frac{\partial \ln \theta}{\partial \rho} - \frac{\rho}{f} \left( \frac{\partial \psi}{\partial \rho} \right)^2 \right] = \frac{f_p q}{(\partial \ln \theta / \partial \rho)^2} > 0.
$$

Now, if the potential vorticity is everywhere positive initially, it must remain positive thereafter because of the conservative form of (18). Consequently, the condition for (20) to remain elliptic will always be satisfied as long as $q$ is initially positive everywhere. The method of solution of (20) may be patterned after the iterative method described for (15). The presence of mixed terms should cause no difficulty. Although no rigorous proof of convergence of the iterative process is known to us, it is likely that the process will converge since convergence has been obtained with a very similar system, namely the one to be described below.

It was seen in the case of NL1 that if the convergence criteria for the solution of (16) were not satisfied, it was not possible to correct the coefficients successfully owing to the impossibility of making the corresponding $\phi$-changes. The same difficulty could also arise in the present case. The property that the ellipticity of (20) remains unchanged, provided it is elliptic initially, holds only for the continuous case. Once finite differences are introduced, the possibility occurs that its type may be changed by truncation and round-off errors. To overcome this difficulty, the following integration procedure, which uses $q$ to determine the motion, was developed.

We extrapolate $q$ forward, using the formula

$$
\frac{\partial q}{\partial t} = - V \cdot \nabla q,
$$

and then solve the non-linear equation

$$
N(\psi) = \rho \left[ \left( \frac{1}{f} \nabla \psi + f \right) \left( - \frac{\partial \psi}{\partial \rho} - \frac{1}{c} \frac{\partial \psi}{\partial \mu} \right) \right. + \frac{1}{f} \left( \frac{\partial \psi}{\partial \mu} \right)^2] = q(x, y, \rho, t + \Delta t),
$$

(21)

to determine the new $\psi$ at $t + \Delta t$. To do this, we utilize the fact that the solution for a slightly different $q$, namely $q(x, y, \rho, t)$, is known. Thus, if the time step $\Delta t$ is sufficiently small, the change $\psi(x, y, \rho, t + \Delta t) - \psi(x, y, \rho, t)$ very nearly satisfies the equation obtained by multiplying (20) by $\Delta t$, namely

$$
F(\psi - \psi) = q(x, y, \rho, t + \Delta t) - q(x, y, \rho, t) = \hat{q} - q,
$$

(22)

where the circumflex distinguishes quantities at $t + \Delta t$ from quantities at $t$, and where it is understood that the coefficients involved in $F$ are evaluated from $\psi$.

When (22) is written in finite-difference form, it may be solved by an iterative process analogous to (17):

$$
(\hat{\psi} - \psi)_{ij,k} = (\psi - \hat{\psi})_{ij,k} + \frac{\lambda}{h_{ijk}} R_{ij,k}^{n+1},
$$

or, since $\psi$ is unchanged,

$$
\hat{\psi}_{ij,k} = \hat{\psi}_{ij,k} + \frac{\lambda}{h_{ijk}} R_{ij,k}^{n+1}.
$$

In these expressions, $R_{ij,k}^{n+1}$ is the difference between the value of $F$ at the current stage of the iteration and $\hat{q} - q$, the quantity $h_{ijk}$ is the coefficient of $\psi_{ijk}$ in the finite-difference equivalent of (22), $\lambda$ is an appropriate constant, and the double superscript $\nu, \nu + 1$ indicates that the superscript $\nu$ or $\nu + 1$ is to be used according as the subscripts of the $\hat{\psi}$'s are $(i, j, k)$ or $(i + 1, j, k)$ or $(i, j + 1, k)$ or $(i, j, k + 1)$ or $(i - 1, j, k)$ or $(i, j - 1, k)$ or $(i, j, k - 1)$. If, however, $\hat{q}$ differs very little from $q$, we have that $F(\hat{\psi} - \psi)$ is very nearly equal to $\hat{q} - q$, where $\hat{q}^{n+1}$ denotes the value of $q$ obtained from the current values of $\psi$ and $\psi^{n+1}$. Thus,

$$
R_{ij,k}^{n+1} = \hat{q}^{n+1} - q - (\hat{q} - q) = \hat{q}^{n+1} - \hat{q}.
$$

This suggests that the iterative process

$$
\hat{\psi}_{ij,k}^{n+1} = \hat{\psi}_{ij,k}^{n} + \frac{\lambda}{h_{ij,k}^{n+1}} [\hat{q}^{n+1} - \hat{q}],
$$

in which $h_{ijk}^{n+1}$ is the value of $h_{ijk}$ obtained by replacing $\phi$ by $\phi^{n+1}$, will converge to the solution of (21) provided that the time steps are chosen such that $\hat{q}(x, y, \rho, t + \Delta t) - q(x, y, \rho, t)$ is sufficiently small. This conclusion has been verified by the actual integrations.

It may be noted that it is necessary to specify more than the boundary values of $\psi$ to ensure that
the finite-difference form of (21) shall have a unique solution. This need arises from the second-degree character of this equation and is common to all non-linear finite-difference equations. The finite-difference form of (21) yields a system of $N$ simultaneous algebraic equations in $N$ unknowns, where $N$ is the number of interior points in the three-dimensional grid. Each of these algebraic equations is quadratic and there are, in general, $2^N$ independent solutions of the system. It can, however, be shown in the above case that one solution of each of these quadratic equations corresponds to a solution in which both the absolute vorticity and static stability are negative at the point in question. Since both these quantities are positive initially and the conservative form of the continuous equation (18) requires their product to remain so, we have a criterion for choosing the desired solution from the $2^N$ possible ones; namely, we choose that solution which satisfies the boundary condition on $\phi$ and yields positive values of the absolute vorticity and static stability at every point in the grid. Although in the continuous case this is the only possible solution, it is not always so in the finite-difference case because of truncation error. Thus, in the above case, it was found necessary sometimes to adjust $\tilde{\phi}$ at the point $ijk$ so as to keep $\tilde{\phi}^{+,+,+}$ at that point bounded positively away from zero.

The method outlined above for the integration of a non-linear elliptic equation is both powerful and general. One may, for instance, consider equations of the form

$$N \left( \frac{\partial^2 \phi}{\partial x^2}, \cdots; \frac{\partial \phi}{\partial x}, \cdots; \phi \right) = F(x, y, p),$$

which are elliptic when differentiated with respect to a parameter, provided $F$ satisfies certain inequalities. Suppose that a solution $\phi_0$ is known for

$$N \left( \frac{\partial \phi_0}{\partial x^2}, \cdots \right) = F_0(x, y, p),$$

where $F_0$ satisfies the given inequalities. We then transform the function $F_0$ into $F$ continuously in such a way that the inequalities remain satisfied. If this is done in sufficiently small steps, the successive changes in $\phi$ can be calculated by the foregoing procedure. In the present problem the continuously varying parameter is the time, and the conservative equation for $q$ provides the means for transforming the initial $q_0$ corresponding to $F_0$, into the final $q_k$ corresponding to $F$. The inequality to be satisfied is here $q \geq 0$.

5. Semi-linear approximation 1

The versions described in this and the following sections are termed semi-linear, because they each involve some linearization of the equations derived in section 2. The first such version is derived as follows:

We divide the vertical pressure scale of the atmosphere, $p = 0$ to $p = p_0$, into $n$ equal subintervals, $\Delta p$, and denote quantities at the mid-points of each interval by the subscripts $k$ ($k = 1, 2, \cdots, n$), and the lower and upper end points of each interval by $k - \frac{1}{2}$ and $k + \frac{1}{2}$, respectively. If we put $k = 1$ at $p = (\frac{1}{2})\Delta p$, and $k = n$ at $p = (\frac{3}{2})\Delta p$, the points $p = 0$ and $p = p_0$ correspond to $k = \frac{1}{2}$ and $k = n + \frac{1}{2}$, respectively.

If, using this notation, we express all vertical derivatives as centered finite differences, and approximate $\theta$ and its derivatives where they occur as coefficients by their mean values, (6) and (12) can be written for a layer centered at pressure $p_k$:

$$\omega_{k+1} = -\Delta p \left( \frac{\partial \theta}{\partial p} \right)_{k+\frac{1}{2}} \frac{D}{Dt} \left( \phi_k - \phi_{k+1} \right),$$

$$\omega_{k-1} = -\Delta p \left( \frac{\partial \theta}{\partial p} \right)_{k-\frac{1}{2}} \frac{D}{Dt} \left( \phi_{k-1} - \phi_k \right),$$

and

$$\frac{D}{Dt} \ln \eta_k = (\Delta p)^{-1} \omega_{k+1} - \omega_{k-1}. $$

The bar denotes a mean value. Eliminating $\omega_{k-1}$ and $\omega_{k+1}$ between these three equations, we obtain, for $k \neq 1$ or $n$, the equation

$$\frac{D}{Dt} q_k = 0, \quad q_k = \eta_k \left( \frac{\phi_k - \phi_{k+1}}{\phi_{k-1} - \phi_k} \right)^{S_{k+1}},$$

where

$$S_k = (\Delta p)^{-1} \left( \partial \ln \theta / \partial p \right)_{k-\frac{1}{2}}.$$
merely a sufficient condition for convergence, experience has proved that it is, practically, a necessary one. If any negative values of \( q \) appeared during the extrapolation of \( q \) by means of the formula

\[
\frac{\partial q}{\partial t} = -V \cdot \nabla q,
\]

they were replaced by zeros.

6. Semi-linear approximation 2

This version,\(^7\) which has been described previously by Charney (1954), has become generally known as the Princeton three-level model. A brief description of its derivation is given here for comparison purposes.

If, using the same notation as in section 5, we express the vertical derivatives in (5) and (9) by their finite-difference equivalents and eliminate \( \omega_{k-4} \) and \( \omega_{k+3} \), we obtain by approximating \( \rho \) and \( \theta \) where they occur as coefficients by their mean values:

\[
\frac{D}{Dt}\left\{ s_{k} + s_{k+1}(\varphi_{k} - \varphi_{k+1}) - s_{k-1}(\varphi_{k-1} - \varphi_{k}) \right\} = 0
\]

\[(k \neq 1, n), \quad (27)\]

where

\[ s_{k} = -\bar{\rho}_{k}(\Delta \rho)^{-1}(\partial \ln \bar{\theta} / \partial \rho)_{k}^{-1}. \]

As in the case of SL1, the boundary conditions \( \omega_{1} = 0 = \omega_{n+1} \) are used to obtain analogous equations for the levels \( k = 1 \) and \( n \). Equation (27) with possibly a transformation, \( \sigma = \sigma(k) \), of the vertical coordinate, is the basic equation for this version.

7. Further semi-linearized versions

By modifying the computational procedures for the integration of the equations for non-linear versions NL1 and NL2, it becomes possible to integrate two semi-linear approximations. While these versions do not differ in any essential respect from SL2, they are valuable for comparison purposes, for the information levels are placed differently, the coefficients \( s_{k} \) in (27) are computed slightly differently, and the upper and lower boundary conditions are applied in a different way.

The first of these versions was obtained directly from NL1 by neglecting the vertical advection of relative vorticity and replacing \( \eta \) by \( f \), and \( \theta \) and \( \theta / \partial \varphi \) by their mean values wherever they occurred as coefficients. Considerably simpler formulae for calculating the coefficients \( A_{r} \) and \( \mu \) were thereby obtained. The integrations were then carried out in exactly the same way as for NL1. In this version,\(^8\) however, the tendency equation has coefficients which always satisfy the Collatz convergence criteria, and therefore no difficulty was encountered in taking the prediction to 24 hr.

Version SL3 differed from SL2 in that the \( \phi \)-field rather than the \( q \)-field determined the motions at a given time, and the boundary conditions appeared explicitly at each time step rather than implicitly as in the basic equations for SL2.

The second version\(^9\) retained the explicit use of the boundary conditions at each time step but used the \( q \)-field to determine the motion. With the above-mentioned linearizations, (11) can be written

\[
\frac{D}{Dt} q_{k} = 0, \quad q_{k} = \zeta_{k} + \alpha_{k} \frac{\partial^{2} \phi}{\partial p^{2}} + \beta_{k} \frac{\partial \phi}{\partial p}, \quad (28)
\]

the second part of which, when \( \partial \varphi / \partial p \) and \( \partial^{2} \varphi / \partial p^{2} \) are expressed in terms of finite differences, becomes

\[
q_{k} = \eta_{k} + \alpha'_{k}(\varphi_{k+1} - \varphi_{k}) + \beta'_{k}(\varphi_{k} - \varphi_{k-1}),
\]

where \( \alpha' \) and \( \beta' \) are functions of \( k \) only. The \( k \)-levels were then chosen as follows:

\[
\begin{array}{cccccc}
\text{(mb):} & 0 & 291 & 600 & 800 & 1000 \\
\text{k:} & 0 & 1 & 2 & 3 & 4
\end{array}
\]

These levels were so chosen that, at \( k = 1 \), the coefficient of \( (\varphi_{1} - \varphi_{0}) \) becomes zero, and therefore the only condition necessary at the upper boundary is that \( (\varphi_{1} - \varphi_{0}) \) remain finite. The lower boundary condition was the same as for NL1, namely that the thickness between 800 and 1000 mb is advected horizontally with the 800-mb wind. Equation (28) is solved in the same way as NL2, in which the lower boundary condition is used explicitly at each time step. This boundary condition could have been incorporated in (28) and the integration performed in the same way as with SL2.

8. Results and conclusions

The successive application of the barotropic, two-level and three-level models to the prediction of the intense development of 24–25 November 1950 had led, in the case of the last model, to a moderately accurate forecast for the 400-, 700- and 900-mb levels. These results are shown in fig. 1. However, at least two major errors remained. First, the cyclogenesis was predicted to occur some 4 deg lat north of where it actually occurred; and the subsequent path of the new low center was predicted to be parallel to its observed path, with the result that the final displacement error was also 4 deg lat. Secondly, in the southwestern part of the forecast area, a spurious intensification of an anticyclone was predicted. This second error was at first thought to be associated with the boundary conditions, as it occurred quite close to the southern edge of the forecast area. Recalculation with the boundary moved much farther south did not, however, yield any improvement, and it was concluded that

\(^7\) Hereafter referred to as SL2.

\(^8\) Hereafter referred to as SL3.

\(^9\) Hereafter referred to as SL4.
the anticyclogenesis was not a boundary effect. Interest in these two errors was heightened by the occurrence of similar errors in later predictions for cyclogetic situations.

Charney (1954), in his discussion of the spurious anticyclogenesis, suggested that the replacement of \( \eta \) by \( f \) in the derivation of (10) destroys the mechanism existing in (7) which prevents the unlimited growth of anticyclonic vorticity. Version SL1 is therefore of considerable interest as it differs from SL2 — the three-level model — only in not having the replacement of \( \eta \) by \( f \).

During the first 6 hr, SL1 appeared to predict the development of the storm at 900 mb somewhat better than SL2. However, beyond this time, the deepening became excessive at all levels. This may be seen by contrasting the 12-hr prediction at 900 mb shown in fig. 2 with the corresponding prediction shown in fig. 1. We see, however, from the two 12-hr error maps given in fig. 3, that while SL1 does show considerably too much development, it does not show any sign of the spurious anticyclogenesis present in the prediction made using SL2. This result would appear to support Charney's suggestion as to the cause of the anticyclogenesis.

Since the geostrophic wind is greater than the observed wind in a cyclone and less than the observed wind in an anticyclone, we have the following relationships between the geostrophic vorticity, \( \eta_g \), and the
true vorticity $\eta$:

$$\eta_0 > \eta > f \quad (\text{cyclone}),$$

$$f > \eta_0 > \eta \quad (\text{anticyclone}).$$

Thus in the cyclonic case, the replacement of $\eta$ by $\eta_0$ in (13) results in an overestimation of $D\eta/Dt$, whereas the replacement of $\eta$ by $f$ has the opposite effect. In the anticyclonic case, the replacement of $\eta$ by $\eta_0$ results in considerably less exaggeration of $D\eta/Dt$ than does the replacement of $\eta$ by $f$. This, in a qualitative sense, is a possible explanation for the excess cyclogenesis occurring in SL1 and the spurious anticyclogenesis occurring in SL2.

The results obtained with NL2 were very similar to those obtained with SL1. In both, the deepening of the low was considerably over-predicted and its track was placed too far north. Outside of the low-pressure area, the predictions were reasonably good at all levels. Since NL2 was the only version to include the
term representing the conversion of horizontal to vertical vorticity — the second term on the right-hand side of (19) — it was of interest to examine the effect of excluding this term. It was found that at all levels the inclusion of the rotation of the vortex tubes gave rise to an additional erroneous deepening of the low center. To infer too much from this result would be unjustified, since in non-linear systems various effects can often strongly interact. Thus, it is quite possible that the effects of the turning of the vortex tubes would, in a completely accurate version, combine with other effects omitted in the present version to give quite different results.

Although the boundary conditions used for NL2 were the same as those used for NL1, it was found that, in the case of NL2, slight variations of the upper boundary condition produced quite large changes in the predictions for the 1000-mb level. This result, which is contrary to what is known about vertical propagation of influences (Charney, 1949), was not entirely unexpected in view of the severe artificial constraint imposed by the neglect of the vertical advection of potential vorticity. Experimental integrations were therefore made with SL4 in which the vertical advection of potential vorticity was at first ignored and then retained. The conclusion reached was that only when the vertical advection of potential vorticity was ignored did the vertical influence propagate become large.

It is difficult to draw valid conclusions from the results of the integration of NL1, since the integrations could only be carried to 7½ hr. The principal feature of the 6-hr prediction that was obtained was the accuracy with which the surface development was predicted. Fig. 4 shows this 6-hr prediction and also the 6-hr prediction made with SL3 which, as stated in section 7, is the same as the three-level model which predicted the development to occur about 4 deg lat north of its actual position. It is apparent that NL1 predicted the development with more accuracy.

When, however, the results from NL1 were compared at all levels with those obtained from SL3, it was found that NL1 predicted more deepening over the whole forecast region. The amount of the excess deepening was comparable to that found at 6 hr with SL1 as compared with SL2. The conclusion may therefore be drawn that even if it had not been possible to continue the integration of NL1 past 7½ hr, it is very likely that the excess deepening would have continued and would have given results as erroneous as those found for longer-period predictions with SL1.

In section 7 it was shown how several versions of the three-level model could be derived. Our experience with these versions was that the main features of the predictions were essentially independent of the version used or from which level the initial information was taken, provided that these levels reasonably spanned the vertical extent of the atmosphere. Fig. 5 shows the comparison between the 900-mb 24-hr prediction obtained with SL2 and the corresponding 900-mb prediction obtained by linear interpolation with respect to pressure between the 800- and 1000-mb predictions made with SL4. We see that these two predictions are very similar. In fact, two predictions made with the same model but with two independently prepared initial analyses might have been expected to yield differences as great.

The general conclusion reached as a result of the integrations reported here is that the constraints imposed by the quasi-geostrophic assumption are such that any significant improvement over the three-level model must be obtained by the use of non-geostrophic methods. Our conclusions must, however, be tempered by the fact that all the versions described suffered from one of two classes of defects: either the geostrophic approximation was used inconsistently, to first order in some terms and to second order in others, or else artificial constraints other than the geostrophic constraint itself were imposed. As illustration of the first defect, we remark that it is inconsistent in (4) not to replace η by f where it appears as a coefficient, or to retain the term involving the turning of the vortex tubes, or to include the vertical advection of vorticity term; for all these are second-order terms.

To expand on the nature of the first class of defects, we construct a possible hierarchy of geostrophic approximation equations according to the scheme:

\[ V^{(o+1)} = V_o + \frac{k}{f} \times \left( \frac{\partial}{\partial t} + V^{(o)} \cdot \nabla + \omega^{(o)} \frac{\partial}{\partial p} \right) V^{(o)} \],

\[ \frac{\partial \omega^{(o+1)}}{\partial p} = - \nabla \cdot V^{(o+1)} \],

and

\[ \left( \frac{\partial}{\partial t} + V^{(o)} \cdot \nabla + \omega^{(o+1)} \frac{\partial}{\partial p} \right) \theta = 0 \],

where \( \nu = 0, 1, 2, \ldots \), and \( V^{(-1)} = \omega^{(-1)} = 0 \). For simplicity, we neglect the variability of \( f \) since this will not alter our essential argument.

The zero-order approximation is horizontal geostrophic flow, i.e., \( V^{(0)} = V_o \), \( \omega^{(o)} = 0 \). The first-order approximation is the quasi-geostrophic system in the
form proposed by Eliassen (1949):

\[ V^{(1)} = V_{\varphi} + \frac{k}{f} \times \left( \frac{\partial}{\partial t} + V_{\varphi} \cdot \nabla \right) V_{\varphi}, \tag{32} \]

\[ \partial \omega^{(1)}/\partial \varphi = -\nabla \cdot V^{(1)}, \tag{33} \]

and

\[ \left( \frac{\partial}{\partial t} + V_{\varphi} \cdot \nabla + \omega^{(1)} \frac{\partial}{\partial \varphi} \right) \theta = 0. \tag{34} \]

Taking the horizontal divergence of (32) and substituting (33), we get the geostrophic vorticity equation,

\[ \left( \frac{\partial}{\partial t} + V_{\varphi} \cdot \nabla \right) \eta_{\varphi} = \frac{f}{\partial \varphi} \frac{\partial \omega^{(1)}}{\partial \varphi}, \tag{35} \]

\[ \eta_{\varphi} = k \cdot \nabla \times V_{\varphi} + f, \]

which, in view of the way in which \( \omega^{(1)} \) occurs in (34), is identical to the system (9) and (10).

In a similar manner, we find the second-order vorticity equation to be

\[ \left( \frac{\partial}{\partial t} + V^{(1)} \cdot \nabla + \omega^{(1)} \frac{\partial}{\partial \varphi} \right) \eta^{(1)} = \frac{f}{\partial \varphi} \frac{\partial \omega^{(2)}}{\partial \varphi} \]

\[ + \eta^{(1)} \frac{\partial \omega^{(1)}}{\partial \varphi} - k \cdot \nabla \omega^{(1)} \times \frac{\partial V^{(1)}}{\partial \varphi}, \tag{36} \]

where \( \eta^{(1)} = k \cdot \nabla \times V^{(1)} + f \) is given by the curl of (32). Thus,

\[ \eta^{(1)} = \eta_{\varphi} - \frac{2m^2}{f^3} \left[ \frac{\partial^2 \varphi}{\partial x^2} \frac{\partial^2 \varphi}{\partial y^2} - \left( \frac{\partial^2 \varphi}{\partial x \partial y} \right)^2 \right], \tag{37} \]

where \( m \) is the scale factor of the conformal mapping.

In comparing the first-order vorticity equation, (35), with the second, (36), we see that to obtain a consistent second-order approximation it is necessary not only to add terms but to insure that the terms themselves are approximated to the right order. For example, we must take \( \eta^{(1)} \), not \( \eta_{\varphi} \), on the left of (2); and we must advect \( \theta \) with \( V^{(1)} \), not \( V_{\varphi} \) as in (6).

The second class of defects is present in the non-linear version, NL2, in which the vertical-advection-of-potential-vorticity term was ignored, and in all the semi-linear versions where the static stability was replaced by a constant where it occurred as a coefficient.

Despite the presence of the various inconsistencies and artificial constraints, it is difficult for the writers, on the basis of the actual forecast results, to imagine how their elimination could combine to overcome to any large extent the most typical errors of the numerical forecasts.

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