ON THE MECHANISM OF LARGE-SCALE VERTICAL MOTION

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(Original manuscript received 8 November 1957; revised manuscript received 4 February 1958)

ABSTRACT

An equation is derived from the linearized equations of planetary aerodynamics which expresses the vertical component of velocity as a function of baroclinity, static stability, geostrophic vorticity, and geostrophic wind speed toward the north. The properties of this equation are compared with those of other equations and it is concluded that the linear equation provides an insight into the mechanism of large-scale vertical motion and that it is simple enough for easy application to real data.

Calculations of vertical velocity using the linear theory are compared with calculations using the adiabatic method, with calculations using two nonlinear methods, and with cloudiness and precipitations. The linear theory yields results which are relatively insensitive to errors in observation and analysis, and which compare favorably with results calculated from other methods. On the other hand, it is not notably more accurate than are other methods; all introduce errors which are comparable to the calculated values.

1. Introduction

Observational and theoretical studies during the past decade have emphasized the importance of large-scale vertical motion both in its direct causal relation to cloudiness and precipitation and in its more subtle relation to pressure change and development of cyclones and anticyclones. In spite of this recognition, vertical motion remains difficult to determine accurately and its relation to basic physical principles remains obscure and difficult to understand.

One may distinguish here between manipulations of basic equations which offer a complete description of vertical motion in terms of more or less measurable quantities and integrals of the governing equations which may offer an insight into the mechanism of vertical motion. The former are illustrated by the kinematic, adiabatic and vorticity methods of calculation which are thoroughly discussed in current texts. They will not be considered in what follows, although the accuracy of the final equation will be evaluated, in part, by comparison with adiabatic computations.

Equations which offer insight into the mechanism of vertical motion have made frequent appearances in meteorological literature, usually as by-products of investigations directed at other objectives. Some of these mechanisms will be discussed in this paper. Before considering these ideas, however, it should be recognized that the mechanism which governs the vertical component of large-scale velocity can only be an inseparable part of the mechanism of the generation, maintenance and decay of atmospheric systems, that is, of the general circulation. Considerable progress has been made in recent years toward understanding these processes, so that it may now be possible to consolidate somewhat our understanding of the dependence of large scale vertical motion on the physical state of the atmosphere.

The purpose of this paper is to explore one mechanism of large-scale vertical motion, the transformation of potential energy of mass distribution into kinetic energy. Recent work, summarized by Fleagle (1957), suggests that this process determines the major features of middle and high latitude motions. An equation will be derived which expresses the vertical velocity as a simple function of its determining parameters. The extent to which this mechanism accounts for vertical motion occurring in the atmosphere and the accuracy of the equation will be assessed very roughly by comparing calculated vertical velocities with corresponding results using other methods.

2. Mechanisms of vertical motion

The most elementary mechanism which has been suggested as an explanation of large-scale vertical motions is based on the concept of the horizontal divergence (hereafter, divergence) of the geostrophic wind. This concept yields convergence and, through the equation of continuity, upward motion in northward moving air (in the northern hemisphere), divergence and sinking in southward moving air. Often this result agrees qualitatively with observations, although the magnitudes computed from geostrophic divergence are nearly always too small. However, the concept is unsatisfactory on a deeper and more important level; namely, the air hardly can know what its geostrophic velocity is, so that the assumption of
complete adjustment to geostrophic velocity is quite arbitrary and unfounded. The assumption, if made at all, should be justified by demonstrating the smallness of other terms.

A more sophisticated mechanism of divergence of the gradient wind has been suggested by Bjerknes and Holmboe (1944). The results of their analysis give a vertical as well as a horizontal distribution of divergence in wave-shape streamline systems which is in rough qualitative agreement with observations. However, as in the geostrophic case, the magnitudes often are too small and there are no obvious reasons why the wind should be gradient or why the disturbances should occur at all.

The distribution of divergence may be determined from the potential vorticity equation by postulating certain arbitrary values of wave speed, vertical shear, wavelength, and amplitude. But this analysis, like those described above, leaves unexplained the selection of the arbitrary values. Until the physics of vertical motion is contained in the method, one must make computations from real data only with great caution and skepticism.

A mechanism based on the principle of conservation of energy and transformation of potential energy to kinetic energy has been suggested by Eady (1949) following discussion of the same problem by Holland (1941). A vertical cross section is imagined in which the potential isotherms make the angle $\mu$ with the horizontal as shown in fig. 1. Associated with this temperature field is a certain potential energy of mass distribution which may be released and converted into kinetic energy; this conversion occurs through vertical displacement, so that a vertical velocity relation is implied in the process. There follows a brief derivation of this equation which will be useful in later discussions.

The symbols used throughout the paper are listed below in the order of their introduction.

<table>
<thead>
<tr>
<th>Eq. No.</th>
<th>Symbol</th>
<th>Definition</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>$P$</td>
<td>potential energy per unit volume</td>
</tr>
<tr>
<td>2</td>
<td>$\rho$</td>
<td>density</td>
</tr>
<tr>
<td>3</td>
<td>$g$</td>
<td>acceleration of gravity</td>
</tr>
<tr>
<td>4</td>
<td>$z$</td>
<td>vertical coordinate</td>
</tr>
<tr>
<td>5</td>
<td>$\theta$</td>
<td>angle between direction of motion and the horizontal plane</td>
</tr>
<tr>
<td>6</td>
<td>$\delta$</td>
<td>angle between potential temperature surface and the horizontal plane</td>
</tr>
<tr>
<td>7</td>
<td>$\omega$</td>
<td>virtual component of velocity</td>
</tr>
<tr>
<td>8</td>
<td>$v$</td>
<td>component of velocity in $y$ direction</td>
</tr>
<tr>
<td>9</td>
<td>$u$</td>
<td>component of velocity in $x$ direction</td>
</tr>
<tr>
<td>10</td>
<td>$\psi$</td>
<td>amplitude of perturbation of the pressure surface</td>
</tr>
<tr>
<td>11</td>
<td>$\phi$</td>
<td>potential temperature</td>
</tr>
<tr>
<td>12</td>
<td>$\alpha$</td>
<td>specific heat at constant pressure</td>
</tr>
<tr>
<td>13</td>
<td>$\beta$</td>
<td>vertical component of velocity</td>
</tr>
<tr>
<td>14</td>
<td>$\phi$</td>
<td>wave number in $x$ direction</td>
</tr>
<tr>
<td>15</td>
<td>$\psi$</td>
<td>virtual shear of undisturbed zonal wind</td>
</tr>
<tr>
<td>16</td>
<td>$\mu$</td>
<td>coefficient of piezotropy [$c_{\mu}/c_{\mu}p$ for adiabatic change of state]</td>
</tr>
<tr>
<td>17</td>
<td>$\lambda$</td>
<td>latitude</td>
</tr>
<tr>
<td>18</td>
<td>$\lambda$</td>
<td>longitude</td>
</tr>
<tr>
<td>19</td>
<td>$\pi$</td>
<td>meridional (latitudinal) wave number</td>
</tr>
<tr>
<td>20</td>
<td>$K$</td>
<td>angular velocity of air about earth's axis relative to earth's surface</td>
</tr>
<tr>
<td>21</td>
<td>$\sigma$</td>
<td>angular velocity of the earth's surface about a vertical axis</td>
</tr>
<tr>
<td>22</td>
<td>$\sigma^2$</td>
<td>angular velocity of the undisturbed current about a vertical axis</td>
</tr>
<tr>
<td>23</td>
<td>$\Omega$</td>
<td>angular velocity of earth</td>
</tr>
<tr>
<td>24</td>
<td>$\Omega$</td>
<td>zonal (longitudinal) wave number</td>
</tr>
<tr>
<td>25</td>
<td>$\Omega$</td>
<td>coefficient of piezotropy [$c_{\mu}/c_{\mu}p$ for adiabatic change of state]</td>
</tr>
<tr>
<td>26</td>
<td>$\Omega$</td>
<td>radius of the earth</td>
</tr>
<tr>
<td>27</td>
<td>$\Omega$</td>
<td>complex angular frequency relative to undisturbed zonal current</td>
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</table>

Under rotation the sloping potential isotherms shown in fig. 1 may be in equilibrium; for instance, under certain conditions of zonal flow in a north-south temperature gradient field. Now, if an air parcel is imagined to move in this field, continuity requires that another parcel must replace the first. One may imagine a simple interchange of the positions of two parcels within the cross section shown in fig. 1, and one may calculate the potential energy change per unit volume resulting from this interchange. Notice that there is a net downward transport of mass; this is the essential feature in the release of potential energy. Of course, there must be a region of the atmosphere in which there is at some time a compensating net upward transport of mass, but this is irrelevant to the problem discussed here. Rotation has the effect of deviating each parcel (toward the right in the northern hemisphere) so that the two parcels shown in fig. 1 do not actually replace one another. Instead, a chain of parcel displacements is required to complete the interchange. However, the net potential energy change is unaffected by the lateral deviation, so that the interchange shown in fig. 1 properly may be called a virtual interchange. The process will be assumed not to influence the surrounding pressure field, and the change of state will be assumed to be adiabatic. Extension to polytropic change of state is obvious.
The initial potential energy per unit volume of the two parcels shown in Fig. 1 may be expressed

$$P = \rho_1 g z_1 + \rho_2 g z_2.$$  \hspace{1cm} (1)

No loss of generality occurs if $z_1$ is set equal to zero. The parcels then interchange positions along a surface of slope $\delta$. The new potential energy is

$$P' = \rho_1' g z_1 + \rho_2' g z_2,$$  \hspace{1cm} (2)

and the change in potential energy is, upon setting $z_1$ equal to zero

$$P' - P \equiv \Delta P = g z_2 (\rho_1' - \rho_2).$$  \hspace{1cm} (3)

Upon expressing $\rho_1 - \rho_1$ as $(\Delta \rho)$ and $\rho_1' - \rho_1$ as $\Delta \rho_1$, (3) may be written

$$\Delta P = g \Delta z [\rho_1 + \Delta \rho_1 - \rho_1 - (\Delta \rho)],$$  \hspace{1cm} (4)

where the subscript $t$ represents change with time held constant and $z_2$ has been replaced by $\Delta z$. From the definition of potential temperature and the equation of state, the density increment holding $t$ constant may be expressed by

$$\Delta \rho_1 = \frac{\rho}{\rho} \left( 1 - \frac{R}{c_p} \right) \Delta \rho - \frac{\rho}{\rho} \Delta \theta.$$  \hspace{1cm} (5)

The corresponding change of density following the motion is expressed for adiabatic change of state by setting $\Delta \theta$ equal to zero. Thus

$$\Delta \rho_1 = \frac{\rho}{\rho} \left( 1 - \frac{R}{c_p} \right) \frac{\Delta \rho}{\theta}$$  \hspace{1cm} (6)

and

$$\Delta P = \frac{\rho}{\theta} \Delta z \Delta \theta.$$  \hspace{1cm} (7)

The increment of potential temperature may be expressed by

$$\Delta \theta = \frac{\partial \theta}{\partial y} \Delta y + \frac{\partial \theta}{\partial z} \Delta z$$  \hspace{1cm} (8)

or, upon replacing $\partial \theta/\partial y$ by $-\tan \mu \partial \theta/\partial z$ and replac-

![Fig. 1. Vertical cross section showing slope ($\mu$) of the potential temperature surfaces ($\theta$), virtual interchange of two unit volumes of density $\rho_1$ and $\rho_2$ along a slope ($\delta$) in the $y, z$ plane. North (pos. $y$) is directed to the left.](image)

$$\Delta \rho = -\frac{g \rho}{\rho} \Delta \theta$$

ing $(1/\theta) \partial \theta/\partial z$ by $s$, (7) becomes

$$\Delta P = -gs \Delta z (\tan \mu \Delta y - \Delta z).$$  \hspace{1cm} (9)

Finally, upon replacing $\Delta z$ by $\Delta y \tan \delta$, (9) becomes

$$\frac{\Delta P}{\rho} = -gs (\Delta y)^2 \tan \delta (\tan \mu - \tan \delta).$$  \hspace{1cm} (10)

Equation (10) shows that for positive $s$ and positive $\mu$ the net potential energy decreases if $0 < \delta < \mu$. The total potential and kinetic energy is considered to remain constant, so that kinetic energy must increase during interchange if $0 < \delta < \mu$. Hoiland (1941) has concluded from an equivalent equation that motions occurring along a slope lying between the slope of the isentropic surface and the horizontal will be amplified, whereas motions occurring along other slopes will be damped. Eady (1949) has suggested that motions along all slopes may occur, but those occurring along the slope of greatest instability will be amplified most rapidly and will dominate. The slope of greatest instability ($\delta_m$) may be found by setting the derivative of (10) with respect to $\delta$ equal to zero. This gives

$$\tan \delta_m = \frac{1}{2} \tan \mu.$$  \hspace{1cm} (11)

and, upon eliminating $\delta$ from (10)

$$\frac{\Delta P}{\rho} = -\frac{gs}{4} (\Delta y)^2 \tan^2 \mu.$$  \hspace{1cm} (12)

From (11) it follows that the vertical velocity component is given by

$$w = -\frac{v \partial \theta/\partial y}{2 \partial \theta/\partial z} = -\frac{\partial \theta}{\partial z} \frac{y}{2gs} \frac{\partial^2 U}{\partial y^2}.$$  \hspace{1cm} (13)

Equation (13) predicts (a) upward motions with flow toward cold air in a statically stable atmosphere, (b) an inverse relation of static stability and vertical velocity. It also may be inferred that isotherms on pressure surfaces should move with half the speed of the normal horizontal velocity component. These conclusions are in rough agreement with observations in the middle troposphere, and the magnitudes are roughly correct. However, (13) fails to account for vertical motion in many specific situations; and this failure is consistent with the fact that the equations of horizontal motion and of continuity and the boundary conditions have not been used in its derivation. To state the limitation differently, the complete set of equations may limit the angle $\delta$ in ways other than the selection of the slope of greatest instability.

Several investigators (notably, Charney, 1947; and Kuo, 1952) have successfully integrated linear equations which describe certain atmospheric models and have been able to express the vertical component of velocity by closed equations. However, in each of
these cases the final equation has been too complex for easy comprehension, and it has been poorly suited to application with real data because it has included a "wavelength." The wavelength is difficult to define in a real situation and, in any case, is an arbitrary factor. In an equation which explained vertical motion in terms of fundamental atmospheric parameters, the dimensions of the disturbance should be a result of calculation, not an observed parameter. However, these studies represent a great step forward in providing a sound physical basis for understanding large-scale atmospheric behavior. Other investigators (e.g., Charney and Phillips, 1953, and Gates, Pocinki, and Jenkins, 1955) have derived non-linear differential equations in the vertical velocity which can be solved by numerical or graphical methods. The solutions are, of course, in numerical form so that they do not reveal the part played by separate factors and, therefore, do not provide explanations of vertical motion in the sense used here.

It is also possible to derive differential equations in the vertical velocity, then to specify the differential terms arbitrarily, thereby obtaining a vertical velocity equation. Kuo (1953), for example, has developed the following equation

\[ f \frac{\partial \omega}{\partial z} - \frac{s_0 \omega}{k^2 \partial p^2} - \frac{s_0 \omega}{fg \delta} = -2 \frac{\partial u}{\partial \varphi} + \frac{1}{k^2} \frac{\partial v}{\partial \varphi}. \]  

(14)

Then if \( \partial^2 \omega / \partial p^2 \) is replaced by \( 2 \omega / \varphi (p - p_0) \), equation (14) becomes

\[ \omega = \frac{k^2 \varphi (p - p_0)}{2f + \frac{k^2 \varphi_s (p - p_0)}{2f \varphi_s \delta}} \left[ \frac{2v}{\partial \varphi} + \frac{\partial v}{k^2 \partial \varphi} \right]. \]  

(15)

or

\[ w = \frac{k^2 \varphi (p - p_0)}{2f \varphi_s (p - p_0)} \left[ \frac{2v}{\partial \varphi} - \frac{\partial v}{k^2 \partial \varphi} \right]. \]  

(16)

The selection of the arbitrary function for \( \partial^2 \omega / \partial p^2 \) is based on observations which show that \( \omega \) increases upward to a maximum in the neighborhood of 500 mb and decreases to zero at the top of the atmosphere. For short wavelengths, \( k \) becomes very large and (16) yields a slope of the streamsurface given by

\[ \frac{w}{v} = \frac{2f}{2f \varphi_s \delta} \]  

(17)

which is four times the slope given by (13) and is twice the slope of the isentropic surface. The effect of increasing wavelength is to decrease the vertical velocity given by (17). The presence of the wavenumber in (15) makes it difficult to use in calculating vertical velocities from actual data.

Several equations of large-scale vertical motion have been reviewed in the preceding pages. Each is deficient in certain respects. Apparently, the basis for a comprehensive but simple equation suitable for calculation of vertical motion and incorporating an intuitive understanding of the phenomenon is still incomplete.

3. The method

Recently, Fleagle (1957) has explained certain outstanding features of the general circulation as consequences of the baroclinity and static stability of the atmosphere. Small disturbances poleward of a critical latitude were found to increase in amplitude and to form vortices roughly 4000 km in horizontal extent, whereas disturbances in the tropics were found to develop only in areas of neutral static stability and were limited to small horizontal extent. "Jet streams" were found to form just poleward of the critical latitude and, under certain conditions, at higher latitudes. These results arose from the linearized equations of planetary aerodynamics applied to a streamsurface of greatest slope which was considered to be in the middle troposphere. Because real disturbances may be represented by a superposition of solutions of the linear equations each having a different dimension, the theoretical results are applicable, in principle, to actual data. However, to perform a superposition would be very tedious, so that the conclusion was reached that "specific predictions of development of cyclones, anticyclones, jet streams and the like probably cannot be made."

However, contrary to the conclusion just restated, if the dimensions of the disturbance could be eliminated from the solution, the problem of superposition would become trivial, and the theory might easily be applied to actual data. Omission of the nonlinear terms, of course, may introduce serious errors. Now, the linear theory shows that disturbances of a certain dimension grow more rapidly than do disturbances of greater or lesser dimensions. One may, therefore, postulate that the atmosphere practices selectivity in that only those dimensions of greatest rate of growth are chosen, and one may then eliminate the dimension of the disturbance from the solution. Thus, the selection of a slope of greatest instability, which was suggested by Eady (1949), is replaced by selection of a dimension of greatest instability. The latter is more plausible a priori because interaction of the boundary conditions with the horizontal dimension of the disturbance is surely less direct than is interaction with the slope of the streamsurface. This elimination may be applied to the equation of vertical velocity yielding an equation which expresses vertical velocity as a function of static stability, baroclinity, latitude,
and the pressure distribution in the vicinity of the point.

4. The vertical velocity equation

The ratio of the vertical and south-to-north perturbation velocity components at the streamsurface of greatest slope has been shown by Fleagle (1957) (equation 27) to be given by

$$\frac{C}{B} = \frac{[mU_s + g a X \sin \theta]}{g m s_z} + \frac{i a X \sin^2 \theta}{m^2 s_z}. \quad (18)$$

Equation (18) shows that a phase difference between the vertical and meridional velocity components occurs if the disturbance is amplified (X complex) or if it is limited in meridional extent (v different from zero). In either case the vertical velocity is retarded so that upward motion extends westward from a trough in the streamlines. The second term is usually an order of magnitude smaller than the first so that it will be neglected in what follows.

The complex angular frequency relative to the undisturbed zonal current (X) has also been shown (equations 29 and 30 from Fleagle, 1957) to be given by

$$\sigma - m \kappa = -\frac{m \Gamma [U_s + \beta s_z / l \gamma]}{a \sin \theta [2 s_z a^2 + g a^4 \gamma]} \pm i \sigma,$$  

where

$$\sigma = \frac{m \Gamma [U_s + \beta s_z / l \gamma]}{a \sin \theta [2 s_z a^2 + g a^4 \gamma]} \times \left[ \frac{s_z + \frac{g \gamma l}{a^2}}{l \gamma} \right]^4.$$

Substitution of (19) in the first term of equation (18) gives

$$\frac{C}{B} = \frac{2 i l U_s a^2 - g \gamma l}{2 g s_z a^2 + g a^4 \gamma} \pm \frac{m \gamma i \sin \theta}{m s_z}.$$

Only the positive sign is used because it corresponds to the amplifying disturbance which must always dominate the damped disturbance. Neutral disturbances, for which \( \sigma \), is imaginary, cannot be treated by this method.

Equation (20) may be differentiated with respect to \( \alpha^2 \) to determine the dimension of greatest instability. However, in order to reach a tractable result, it is convenient first to specialize \( \alpha^2 \) by considering only meridional extent large enough so that \( \alpha^2 \) may be approximated by \( m^2 / a^2 \sin^2 \theta \). This is a reasonable assumption because \( m \) appears as a linear multiplier only. Kuo's (1953) \( k \) and the \( \alpha \) used here are identical under these conditions. It then follows that the maximum amplification occurs for

$$\alpha_{m}^2 = \frac{g \gamma l [U_s + 3 \beta s_z / l \gamma]}{2 s_z [U_s - \beta s_z / l \gamma]}.$$

The dimensions of the most unstable disturbance which may be calculated from equation (22) for a wide range of observed magnitudes of \( U_s, s_z \) and latitude are several thousands of kilometers. This specifies the horizontal acceleration as small compared to the Coriolis acceleration, thereby justifying the use of the geostrophic approximation in succeeding substitutions. Under near critical conditions, \( l U_s \sim \beta s_z / l \gamma \), the dimensions become small and the geostrophic assumption fails. However, the preference of the atmosphere for the most unstable disturbance, which has been assumed in the development of (22), can be realistic only under conditions of \( l U_s > \beta s_z / l \gamma \). Under these conditions (22) may be substituted into (20) and (21) giving

$$\frac{C}{B} = \frac{1}{2 g s_z} \left[ l U_s + \frac{\beta s_z}{l \gamma} + \frac{i (l U_s - \beta s_z / l \gamma)}{l (U_s + 3 \beta s_z / l \gamma)} \right] \times \left[ \frac{s_z + \frac{g \gamma l}{a^2}}{l \gamma} \right]^4. \quad (24)$$

The left side of (24) may be replaced by \( w / v \). Then, it is convenient, although not essential, to represent \( iv \) in the second term by geostrophic approximation. On a surface of constant pressure the relation between wind speed and slope of the pressure surface is

$$v = \frac{g}{a l \sin \theta} \frac{\partial \sigma_p}{\partial \lambda} = \frac{g}{l} \frac{\partial \sigma_p}{\partial \lambda}.$$

The difference between the spherical and Cartesian coordinate systems is neglected in what follows. The perturbation height of the pressure surface may be expressed by

$$z_p = F(z) e^{i (m \lambda + v \theta - \nu t)} \quad (26)$$

from which it follows with good approximation that

$$\frac{i}{a l \sin \theta} \frac{\partial z_p}{\partial \lambda} = \frac{m}{a l a \sin \theta} \nabla z_p.$$

Upon setting \( m / a \sin \theta \) equal to \( \alpha \), substitution of (25) and (27) in the second term of (24) gives

$$w = \frac{1}{2 g} \left( \frac{U_s}{s_z} + \frac{\beta}{\gamma l^2} \right) v \quad (28)$$

For numerical calculations it is convenient to introduce the slope of the pressure surface from (25) in the first term of (28). This gives in symbolic form

$$w = W_{1} \frac{\partial z_p}{\partial x} + W_{2} \nabla z_p. \quad (29)$$

Each component disturbance has associated with
it an equation of the form of (29). The vertical component of velocity associated with a real disturbance may be expressed by the summation of a series of such component disturbances.

Now, the equations used here are derived from linear differential equations with constant coefficients. Therefore, $W_1$ and $W_2$ may be treated as independent of the summation so that

$$ w = \sum_{i=1}^{\infty} w_i = W_1 \sum_{i=1}^{\infty} \left( \frac{\partial s_p}{\partial x} \right)_i + W_1 \sum_{i=1}^{\infty} (\nabla^2 s_p)_i. $$  (30)

The right side expresses the product of the observable functions $W_1$ and $W_2$ and summations of component disturbances; these summations represent the observable derivatives of the pressure distribution. $W_1$ and $W_2$ are presented graphically in fig. 2.

Contrary to the assumption of constancy, the parameters which constitute $W_1$ and $W_2$: $U_s$, $s_s$, $\beta$, $l$, $\gamma$, all vary in space so that $W_1$ and $W_2$ also vary in space. Nevertheless, it is plausible that the vertical velocity at a point is primarily influenced by the values of the parameters in the neighborhood of the point and only secondarily by values at distant points. By way of illustration, this is the assumption which justifies the use of a mean latitude in applications of the barotropic trough formula or other related equations.

If one, then, considers $W_1$ and $W_2$ to vary in space, a separate superposition may be invoked at each point. There appears to be no reasonable way to

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**Fig. 2.** Nomograms plotted from (28) and (29) showing the functions $W_1$ (solid lines) and $W_2$ (dashed lines) for 30, 40, 50 and 60 deg lat. The labelled values of $W_1$ and $W_2$ are multiplied respectively by $10^4$ and $10^5$ C.G.S. units.
compare the error which results from this assumption and the errors which result, say, from linearization or others of the simplifying assumptions already introduced.

The first term of (28) usually is considerably larger than the second term and therefore accounts for the larger part of the vertical velocity. The first part of the first term, \((U, \nu/2gs)\), may be recognized as identical with the right side of (13) except for the slight difference between \(l\) and \(f\). It will be called the baroclinic term. This identity justifies the conclusion that the mechanism developed by Eady (1949) from particle concepts represents a major part of the mechanism of vertical motion in the middle troposphere. The baroclinic term, therefore, may be visualized as indicating that cold air sinks and warm air rises along a slope equal to one-half the slope of the isentropic surface. This is the condition required for the greatest possible decrease of potential energy, and it is what one expects intuitively on the basis of every-day experience with nonrotating fluids.

The second part of the first term, \(3\nu/2g\gamma\), represents the divergence of the geostrophic wind multiplied by \(7/10\) the height of the homogeneous atmosphere, that is, about 5.5 km. It adds to the vertical motion expressed by the baroclinic term. The baroclinic and geostrophic divergence terms are comparable in magnitude in middle latitudes, although in limited regions the baroclinic term may be very much the larger. Evidently, the mechanism of geostrophic divergence accounts for an important part of the vertical motion occurring in the middle troposphere. The baroclinic and geostrophic divergence terms together account for the strong correspondence between upward motion and poleward flow.

The second term in (28), although usually smaller than the first term, may be comparable to or even larger than the first term in limited regions in ridges and troughs or where east-west shear is particularly strong. It will be referred to as the vorticity term. There is no elementary mechanism which describes this term, but the qualitative behavior may be made clear by substituting again from (22) and (23), giving for this term

\[
\frac{g\gamma \sigma_i}{4\pi^2 \gamma \left( \frac{1}{L_x^2} + \frac{1}{L_y^2} \right)} \nabla \cdot \nabla \sigma_i.
\]

The term is directly proportional to the rate of growth \((\sigma_i)\) of unstable disturbances and to the geostrophic vorticity and depends also on the static stability and the dimensions of the disturbance. Therefore this term contributes upward motion in developing troughs, downward motion in developing ridges.

5. Comparison with Kuo's equation

The linear theory presented here may be compared with Kuo's equation by applying (16) to the 500 mb surface. This gives

\[
\frac{w}{v} = \frac{2fU, k^2 - f\beta y}{g_s k^2 + g^2 \gamma^2 (\sigma v/c_s)^2}
\]

which may be compared with (21) in the form

\[
\frac{w}{v} = \frac{2fU, \alpha^2 - g\beta y}{2g_s \alpha^2 + g^2 \gamma^2} + \frac{i\gamma a \sin \theta}{ms} \sigma_i.
\]

If \(\alpha\) is equated with \(k\) and \(l\) with \(f\), (31) and (32) may be compared easily. For short waves (31) gives values twice as large as the first term of (32), that is, the slope of the streamline is twice the slope of the isentropic surface. This results in a sharp cut-off to baroclinic instability on the short wavelength side of the region of maximum instability. Observations of large-scale motions and more complete theories (e.g., Kuo, 1952) suggest that (32) has somewhat the better of the argument to this point. With positive values of the various parameters and increasing wavelength, the first term of (32) diminishes to zero and may become negative at very great wavelengths; (31) behaves similarly, but an additional parameter, \(v_s/\sigma\), is required. It is not completely clear whether this is an advantage of (31) or a disadvantage, but it is clear that the additional parameter adds one step to calculation. Finally, (31) and (32) differ in that \(\alpha\) may be eliminated from (32) as shown in the preceding pages, whereas the corresponding elimination has not been made for (31).

6. Calculations of vertical velocity

In what follows calculations using the linear equations are compared with precipitation and with calculations using other methods. Comparisons with a variety of methods are shown.

Equation (30) describes the vertical velocity at the stream-surface of greatest slope. There is assurance, therefore, that the computed vertical velocity represents, at least approximately, the greatest vertical velocity occurring in the column. The height of this largest value could be estimated from (37), given by Fleagle (1957), but it is much more convenient to compare computed values with values measured on a particular surface. The 500-mb surface has been chosen as being representative of a rather thick layer in the troposphere and all calculations are made from and for that surface. The static stability was averaged between 400 and 700 mb and the vertical shear of the undisturbed wind was determined from the north-south temperature gradient using the thermal wind equation. Calculations were made at intervals equiva-
lent to \(2\frac{1}{2}\) deg lat. Error of unknown amount is introduced through using data in (30) observed on a surface other than the surface at which (30) applies. An additional error occurs in the use of the nomograms which were constructed by treating \(l\) as determined by latitude alone. In cases of north-south shear of 1 m sec\(^{-1}\) (100 km\(^{-1}\)) this will lead to an error in effective Coriolis parameter (\(l\)) of roughly 10 percent in middle latitudes. Nevertheless, the vertical uniformity of the temperature and velocity fields is such that comparisons made in this way are of interest.

Fig. 3 shows the vertical velocity fields for 2100 GMT 10 April 1950, computed from (30) and measured by the adiabatic method from 12-hr trajectories centered at 2100 GMT. Over most of the chart there is agreement in sign. In the central Mississippi Valley the linear equation indicates upward motion where the adiabatic method gives downward motion; the rain area suggests that the linear method has the best of the argument in this case. The resolution of this test probably is insufficient to conclude that other regions of disagreement in sign represent errors in the values computed by either method.

Fig. 4 shows the vertical velocity field for 0900 of 11 April 1950. The vertical velocity field compared with the corresponding field determined from 12-hr trajectories indicates agreement somewhat closer than does fig. 3. In both cases rain was occurring predominantly in areas of upward motion.

Fig. 5 shows the results of calculations for 0300 GMT of 14 December 1953. This case exhibited unusually large horizontal gradients of wind speed and temperature. Numerical values of vertical velocity calculated from (30) are compared with values determined by numerical integration of a modification of the Sawyer-Bushby (1953) nonlinear baroclinic model. The latter represents mean values for the layer between 1000 mb and 500 mb. The Sawyer-Bushby equation is similar to (14) [Kuo, 1953] except that Kuo has assumed periodic disturbances in \(x\) and \(y\). The two methods [(30) and Sawyer-Bushby nonlinear] show reasonable agreement except in the base.
of the trough where the linear method greatly underestimated the subsidence calculated by the nonlinear method. This probably represents a real failure of the linear method; the failure may result from neglecting the variation of the zonal shear with latitude in calculation of \( \lambda \); for, in passing from the north to the south of the jet, the shear varied roughly from the Coriolis parameter to the negative of the Coriolis parameter.

In fig. 6 calculations using the linear method are compared with calculations using the thermotropic model (Gates, Pocinki, Jenkins, 1955) for 8 January 1953. The linear equation yielded strong upward motion over the eastern U. S., whereas the thermotropic calculation yielded only weak upward motion. The rain area was less extensive than the upward motion indicated by the linear equation, but this may be resolved by the fact that the rain observations were made two and one-half hours before the time of the vertical velocity chart.

Within the region of upward motion in eastern U. S. the static stability was unusually small. The static stability is used as observed in (30), whereas in the nonlinear relaxation process an average static stability was used over the entire chart. This difference accounts for the larger positive values calculated by the linear method. This case emphasizes the importance of static stability in determining the vertical velocity; nonlinear methods probably must provide for variable static stability if they are to describe accurately large scale vertical motions.

7. Evaluation

Equation (28) expresses the maximum vertical component of velocity occurring in a vertical column by: a) the sum of two terms which have simple intuitive meanings and b) a smaller more complex term. The first or baroclinic term shows that, without regard to curvature of the earth, air tends to flow

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**Fig. 5.** Charts showing (1) 500-mb contours in hundreds of feet and isotherms (left), (2) vertical velocity component computed from nonlinear equation by modified Sawyer-Bushby method and from equation (29) (right) for 0300 GMT of 14 December 1953. Rainfall is indicated for 0630 GMT.

**Fig. 6.** Charts showing (1) 500-mb contours in hundreds of feet and isotherms (left), (2) vertical velocity component computed by thermotropic method and by equation (29) (right) for 1500 GMT of 8 January 1953. Rainfall is indicated for 1230 GMT.
along a slope equal to one-half the slope of the isentropic surface projected onto a north-south vertical cross section. This effect is in all essential respects analogous to the vertical motion occurring in a laboratory tank when cold water is allowed to underrun warm water. The second or geostrophic divergence term shows that, without regard to baroclinic effects, air currents directed toward the poles should experience convergence and upward motion. This convergence has often been suggested and as often scorned because there appeared to be no clear reason why the velocity should adhere so closely to the geostrophic velocity. Confirmation of the existence of geostrophic divergence suggests that in a theory which permitted finite amplitude disturbances the effect of trajectory curvature, on which gradient wind divergence depends, should also appear.

The final small term has no simple interpretation. It is directly proportional to the baroclinic instability or rate of development and to the geostrophic vorticity; upward motion accompanies positive vorticity and downward motion accompanies negative vorticity.

These terms may be calculated easily and rapidly from standard analyze charts, in this way providing an estimate of the vertical component of velocity in the middle troposphere. The accuracy of the calculation is relatively insensitive to errors in the basic data, so that the calculated vertical velocity field can be determined with the same resolution as can derivatives of the fields of pressure or temperature. Of course, there remain errors, largely unknown, which arise from the approximations made in the linear theory.

The assumption of selection of the dimension of greatest baroclinic instability probably is most reliable for large scale of the isentropic surface. It should not be expected to hold in the absence of baroclinic instability. To the extent that theory and observations agree, one may conclude that this assumption is reasonable. However, it should also be recognized that in many cases baroclinic instability is difficult to determine quantitatively, that is, $U_\alpha/s_\alpha - \beta/\gamma^2$ tends to be small compared to either term separately. This causes no difficulty in measurement of the baroclinic and geostrophic divergence terms, but does introduce considerable uncertainty in measurement of the vorticity term. It is attractive, in view of the agreement of theory and observations demonstrated here, to visualize the atmosphere in middle latitudes as being in a chronic state of baroclinic instability, small numerically but sufficient to determine the major modes of atmospheric motion.

Comparison with results calculated by the adiabatic method and by numerical integration indicate that

the linear equation yields values which are reasonable in magnitude and which agree in sign, for the most part, with these other more elaborate methods. This appears to be true even where baroclinic instability is small or entirely absent. An exception must be noted in regions of very large north-south shear where the absolute vorticity of the zonal wind departs substantially from the Coriolis parameter. The linear method has the advantage of great simplicity, but it must be recognized that other methods are less dependent on arbitrary simplification. However, the theoretical advantage does not lie entirely on the side of nonlinear methods, for nonlinear methods usually neglect the vortex tube terms and the vertical transport of vorticity, whereas in this treatment the first order terms are retained. The linear method as well as the numerical methods provide 'instantaneous' values, whereas the adiabatic method provides average values.

Equation (30) provides both intuitive understanding of large-scale vertical motion and a simple method for calculation of large-scale vertical motion which is comparable in accuracy to the adiabatic method or to current nonlinear numerical methods.

Acknowledgment.—The work reported here has been supported in part by the Air Force Cambridge Research Center and in part by the National Science Foundation. The writer is indebted to R. J. Reed for providing the vertical velocity calculations by the Sawyer-Bushby method for 14 December 1953.

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