A THEORETICAL EXPRESSION FOR THE ROOT MEAN SQUARE VERTICAL EDDY FLUCTUATION \((\bar{w}^2)^{1/2}\)

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ABSTRACT

A theoretical expression is derived for the root-mean-square vertical-eddy fluctuation \((\bar{w}^2)^{1/2}\) in terms of the friction velocity, the height and Monin and Obukhov’s function \(\phi\). Steady-state conditions and constant shearing stress are assumed.

It is shown that \((\bar{w}^2)^{1/2}\) increases with height under superadiabatic conditions, is constant with height when the lapse rate is adiabatic and decreases with height during inversions. Indirect evidence suggests that the height variation of \((\bar{w}^2)^{1/2}\) can be approximated by a power law.

1. Introduction

Consider the turbulent boundary layer with constant shearing stress and steady-state conditions. It seems intuitively reasonable to expect that at least some characteristics of the vertical fluctuations \(w\) might be determinate, given the roughness length \(z_0\), the friction velocity \(u^*\) and the mean wind profile. Such a theoretical relation would be very useful; e.g., \((\bar{w}^2)^{1/2}\) is required in the calculation of Sutton’s \(C_{\tau}\).

2. An expression for \((\bar{w}^2)^{1/2}\)

The physical model is specified in the usual orthogonal axes \(OX, OY\) and \(OZ\), with \(X\) measured in the direction of the mean wind \(\bar{u}\), \(Y\) measured across wind and \(Z\) measured vertically. Constant shearing stress and steady-state conditions are assumed. Hence,

\[ \tau/\rho = u^* = -\bar{u}\bar{w} \]

where \(\tau\) is the shearing stress, \(\rho\) is the density and \(u^*\) is the friction velocity. In order to advance the argument, a mixing length is usually introduced at this point. There are a number of definitions of mixing length, three of which will be considered here.

(a) The Prandtl definition: It is assumed that

\[ \bar{w} = l_X \frac{\partial \bar{u}}{\partial Z}, \quad \bar{w}' = l_Z \frac{\partial \bar{u}}{\partial Z} \]

and that \(l_X = l_Z = l\).

Thus,

\[ \tau/\rho = u^* = -\bar{u}\bar{w}' = p \left( \frac{\partial \bar{u}}{\partial Z} \right) \frac{\partial \bar{u}}{\partial Z}. \]

(b) The von Karman model: Dimensional and similarity arguments yield

\[ l_Z = k \left| \frac{\partial \bar{u}}{\partial Z} \right| \left( \frac{\partial^3 \bar{u}}{\partial Z^3} \right)^{-1} \]

where \(k\) is von Karman’s constant and it is assumed that derivatives higher than the second may be neglected.

(c) The Goldstein model: The vertical flux \(F_\phi\) of some transferable quantity \(S\) is given by

\[ F_\phi = -\rho \bar{w} l_Z \frac{\partial S}{\partial Z} \]

In the case of momentum, this becomes

\[ -\tau = -\rho \bar{w} l_Z \frac{\partial \bar{u}}{\partial Z} \]

It is assumed that a mean mixing length \(l_Z\) may be defined in such a way that

\[ \bar{w} l_Z = (\bar{w}^2)^{1/2} l_Z. \]

Then

\[ \tau/\rho = u^* = (\bar{w}^2)^{1/2} l_Z \frac{\partial \bar{u}}{\partial Z}. \]

In this derivation, nothing has been said about \(w'\) or its relation to \(w\). However, it is implicit that \(u^* = -\bar{w}'\) as in eq (1).

As noted by Sutton (1953), eq (1), (2) and (3) are merely a series of definitions. The various mixing lengths may therefore differ by constants of proportionality.

In adiabatic flow, the additional assumption is made that \(l = kZ\). Eq (1) and (2) then lead to the familiar logarithmic wind profile.
In the more general case of non-adiabatic flow, \( l \neq kZ \). Alternate expressions for \( l \) have been suggested involving the Richardson number. However, attention will be directed here mainly to the plausibility of eq (1), (2) and (3) rather than to the functional form of \( l \).

Eq (1) is not likely to hold in diabatic flow because of the assumption that \( l_x = l_z \). For example, during inversions, vertical turbulence will be suppressed but there is no similar dampening effect in the horizontal. Eq (3), on the other hand, is derived from the concept of vertical flux of momentum and there appears to be no implicit restriction to adiabatic flow. This view is shared, for example, by Yamamoto (1959), who states that (3) is formally correct in any lapse rate.

Eq (2) is based on dimensional reasoning and the simple postulate that the mixing length depends on the variation of wind speed through the boundary layer. This is a perfectly general argument that can be extended to non-adiabatic flow.

Eq (2) and (3) will therefore be the basic assumptions of this paper. However, to ensure equivalence of mixing lengths, an additional constant of proportionality will be included.

\[
l_y = c_l_k
\]

where \( l_y \) and \( l_z \) are the Goldstein and von Karman mixing lengths and \( c \) is a constant. Thus,

\[
u^2 = \frac{1}{\langle w^2 \rangle k c} \left| \frac{\partial \bar{\alpha}}{\partial Z} \right| \left. \frac{\partial \bar{\alpha}}{\partial \bar{\phi}} \right| Z = \frac{1}{\langle w^2 \rangle k c}.
\]

Monin and Obukhov (1954) have reasoned dimensionally that

\[
\frac{\partial \bar{\alpha}}{\partial Z} = \frac{u^2}{kZ} \phi
\]

where \( \phi = 1 \) in adiabatic conditions but otherwise is an unknown function of \( Z/L \) where

\[
L = - u^2 T \rho c_p / k g H.
\]

\( \Pi \) is the heat flux, and the other symbols are standard.

It might be thought that (5) and (2) are inconsistent in as much as (5) implies that \( \partial \bar{\alpha}/\partial Z \neq 0 \) while the term has been omitted from (2). However, all that has been postulated is that \( \partial \alpha/\partial Z \) and higher derivatives have negligible effect on the magnitude of \( L \). It is not suggested that these derivatives vanish. A similar argument is generally accepted in the familiar adiabatic case; the neglect of \( \partial \bar{\alpha}/\partial Z \) in (2) leads to

\[
\frac{k}{u^2} \frac{\bar{\alpha}}{\partial Z} = \frac{1}{Z}
\]

from which it is seen that the higher derivatives do not vanish. From (5)

\[
\frac{\partial^2 \bar{\alpha}}{\partial Z^2} = \frac{u^2}{kZ} \frac{\partial \phi}{\partial Z} - \frac{u^2}{kZ^2}
\]

Substitution of (5) and (6) in (4) leads to

\[
u^2 \frac{\partial \phi}{\partial Z} = c \langle w^2 \rangle k \phi.
\]

This may be rearranged in the form

\[
\langle w^2 \rangle k = \frac{1}{c} \frac{d(Z/\phi)}{dZ}.
\]

By integration from \( Z_0 \) to \( Z \),

\[
u^2 \int_{Z_0}^{Z} d(Z/\phi) = c \int_{Z_0}^{Z} \langle w^2 \rangle \bar{\phi} dZ.
\]

From dimensional arguments, it is generally accepted that, for small \( Z \), the wind profile approaches the logarithmic adiabatic form (Ellison 1959, Shepard 1958). We may then reason that \( \phi = 1 \) at \( Z = Z_0 \) in any lapse rate. Thus,

\[
u^2 \int_{Z_0}^{Z} \langle w^2 \rangle \bar{\phi} dZ = c \int_{Z_0}^{Z} \langle w^2 \rangle dZ
\]

and

\[
\frac{\phi}{Z} = \frac{1}{Z_0 + \frac{c}{\nu^2 \int_{Z_0}^{Z} \langle w^2 \rangle dZ}}.
\]

Therefore, from (5),

\[
k \frac{\bar{\alpha}}{\nu^2} = \int_{Z_0}^{Z} \frac{\phi}{Z} dZ = \int_{Z_0}^{Z} \frac{dZ}{Z_0 + \frac{c}{\nu^2 \int_{Z_0}^{Z} \langle w^2 \rangle dZ}}.
\]

3. Discussion

It is seen from (8) or (10) that under adiabatic conditions \( \phi = 1 \),

\[
\frac{c}{\nu^2} \langle w^2 \rangle = 1.
\]

\( \langle w^2 \rangle \) is then constant with height. This fact has been noted by Sutton (1953).

In other lapse rates, from (8), \( \langle w^2 \rangle = 1 \) appears to vary with height. The nature of the variation can be seen qualitatively by using the approximation \( \phi = 1 + aZ/L \) for \( Z \ll |L| \) and \( a \) a positive empirical constant (Monin and Obukhov 1954). Although there is some question about the value of \( a \) (Taylor 1960), the following qualitative argument is not affected.

Substitution in (8) yields

\[
\langle w^2 \rangle = \frac{\nu^2}{c(1 + aZ/L)^2}.
\]
In superadiabatic conditions, \((1 + aZ/L) < 1\) and decreases with height; then \((\bar{w}^2)^{1/4}\) must increase with height. During inversions, \((1 + aZ/L) > 1\) and increases with height; \((\bar{w}^2)^{1/4}\) must therefore decrease with height. These predictions may be tested experimentally.

The theory might also be tested indirectly from eq (10); i.e., if measurements of \((\bar{w}^2)^{1/4}\) were available at different heights, a wind-profile relation could be determined by numerical integration of the right side of (10).

4. Comparison with O’Neill, Nebraska 1953 data

Fig. 1 from a paper by Cramer (1957) shows the diurnal variations of the standard deviations of elevation and azimuth angles at two levels, 1.5 and 12 m, obtained from records of the Gill bivane at O’Neill, Nebraska in 1953.

The standard deviation of the elevation angle \(\sigma_E\) is closely related to \((\bar{w}^2)^{1/4}\). For \(\theta < 20^\circ\), \((\bar{w}^2)^{1/4}/\bar{u} = \tan \sigma_E\) approximately. Hence, \(\sigma_E\) may be used as a qualitative measure of \((\bar{w}^2)^{1/4}\). Fig. 1 shows that \((\bar{w}^2)^{1/4}\) decreases with height at night (inversions) and increases with height during the day (superadiabatic) as predicted by theory. Such behaviour is reasonable on physical grounds; stable air suppresses turbulence, while unstable air, on the other hand, promotes the vertical development of eddies.

The equations make no predictions concerning \((\bar{w}^2)^{1/4}\) or \(\sigma_A\) (standard deviation of the azimuth angle). It is seen from fig. 1 that the diurnal cycle of \(\sigma_A\) is quite different from that of \(\sigma_E\).

The height variations of many micrometeorological elements are approximated by power-law profiles.

If it may be assumed that this can be done for \((\bar{w}^2)^{1/4}\), then

\[
(\bar{w}^2)^{1/4} / (\bar{w}^2)^{1/4} = (Z_1/Z_2)^m \cong \tan \sigma_E_1 / \tan \sigma_E_2
\]

where suffixes 1 and 2 denote heights \(Z_1\) and \(Z_2\). When \(\phi = 1, m = 0\). Also, a power law can be justified on dimensional grounds for the special case of free convection; Sheppard (1958) points out that in such circumstances \(m = \frac{1}{4}\).

The data in fig. 1 are sufficient to determine \(m\), and the results are plotted in fig. 2. At night, \(m\) fell to \(-0.14\) followed by a rise to \(+0.15\) in the afternoon.

With the power-law assumption, let

\[
\frac{c}{\bar{u}} (\bar{w}^2)^{1/4} = c_1 Z^m
\]

where \(c_1\) is a new constant.

Eq (10) becomes

\[
k(\bar{u} - u_*) = \frac{\int_{Z_1}^{Z} \frac{dZ}{Z_0 + \frac{c_1}{m+1} (Z^{m+1} - Z_0^{m+1})}}{u_*}.
\]

The integral cannot be solved directly. It is therefore necessary to assign specific values to the parameters and solve graphically or numerically. For the present qualitative purposes, it is sufficiently accurate to assume \(Z_0 = 1\) cm, \(c = 1\), and \(\bar{u}/u_* = 10\). Then

\[
c_1 = c \frac{\bar{u}}{u_*} \frac{\tan \sigma_E}{Z^n} = 10 \tan \sigma_E/Z^n.
\]

For the data in fig. 1, this yields values of \(c_1\) ranging from 1.5 at night to 0.4 in the afternoon. For hours when \(m = 0\) (the adiabatic case), \(c_1\) is near unity so
that eq (11) then reduces to the standard logarithmic profile. In other lapse rates, the apparent variation in \( c_1 \) may be caused by the fact that \( \bar{u}/u_\ast \) has been held constant whereas, in fact, the ratio is larger in lapse conditions than during inversions.

In any event, eq (11) has been solved graphically for \( m = -0.14 \) (night-time), \( m = 0 \) (adiabatic) and \( m = \pm 0.14 \) (afternoon), assuming \( c_1 = 1.5 \) at night, \( c_1 = 1 \) in the adiabatic case and \( c_1 = 0.4 \) in the afternoon, and \( c_1 = 1 \) throughout.

The derived wind profiles are displayed in fig. 3. Qualitatively, the curves show the characteristic curvatures of low-level wind profiles (convex to the log Z axis at night, concave to the log Z axis in the afternoon). This indirectly supports the view that the height variation of \( \langle w^2 \rangle \) can be approximated by a power law.

No quantitative conclusions should be drawn from fig. 3. However, the analysis suggests that it would be useful to observe vertical profiles of \( \langle w^2 \rangle \) in more detail.

5. Comparison with other experimental results

Panofsky and Deland (1959) have examined the height variation of \( \langle w^2 \rangle \) by using Brookhaven and 1956 Project Prairie Grass data. They conclude that "There is considerable evidence that the variance of vertical velocity in the lowest 100 m tends to decrease upward in stable air and increase upward slowly in unstable air." The present paper provides a theoretical justification for this experimental result.

Monin (1959) discusses the relation between \( \langle w^2 \rangle \) and \( u_\ast \) in adiabatic flow. Experiments indicate that

\[
\langle w^2 \rangle = 0.86 u_\ast \quad \text{or} \quad c = 1/0.86 = 1.16.
\]

On the other hand, Panofsky and McCormick (1960) report that in adiabatic conditions

\[
\langle w^2 \rangle = 1.25 u_\ast \quad \text{or} \quad c = 0.8.
\]

The variation in \( c \) may be caused by differences in terrain or differences in instrumental response to high-frequency eddy fluctuations.

6. Comparison with other theoretical models

Businger (1959) has derived an expression for \( \langle w^2 \rangle \) in terms of \( u_\ast \) and the Richardson number. His analysis is based on an equation of energy decay and leads to

\[
\frac{\langle w^2 \rangle}{u_\ast^2} = \frac{A}{2} \frac{Ri}{1 + \frac{1}{4} [(ARi)^2 + 4(1 - ARi)^2]} \quad (12)
\]

where \( A \) is a positive constant. When \( Ri = 0 \) (adiabatic conditions), eq (12) reduces to

\[
\frac{\langle w^2 \rangle}{u_\ast^2} = 1 \quad \text{or} \quad c = 1.
\]

In other lapse rates, noting that \( |Ri| \) increases with height, Businger's fig. 5 leads to the same qualitative predictions as in the present paper; i.e., \( \langle w^2 \rangle \) increases with height in superadiabatic conditions and decreases during inversions.

Panofsky and McCormick (1960), without introducing mixing-length concepts, have argued dimensionally that

\[
\langle w^2 \rangle = B u_\ast (\phi - \delta Z/L)^0, \quad (13)
\]

using the sign notation of the present paper, where \( B \) and \( \delta \) are arbitrary positive dimensionless quantities.

Now, \( B = 1/c \). Hence, eq (13) will lead to the same predictions as eq (8), provided

\[
\frac{d(\phi/L)}{dZ} = (\phi - \delta Z/L)^0. \quad (14)
\]
Eq (14) may be solved for $\phi$. However, this does not provide a verification of eq (8), (13) and (14) because the analytic form of $\phi$ is not known.

On a qualitative basis, Panofsky and McCormick's fig. 4 predicts that $\left(\frac{w}{z}\right)^{1/4}$ increases with height during superadiabatic lapse rates. No information is available for inversion conditions.

7. Conclusion

By assuming steady-state conditions and constant shearing stress, a theoretical expression has been found for $\left(\frac{w}{z}\right)^{1/4}$ (eq 8).

Qualitatively, $\left(\frac{w}{z}\right)^{1/4}$ increases with height under superadiabatic conditions, is constant with height when the lapse rate is adiabatic, and decreases with height under inversions. These predictions are confirmed by data from O'Neil, Nebraska.

If the variation with height of $\left(\frac{w}{z}\right)^{1/4}$ is assumed to follow a power law, then it is possible to solve a wind-profile equation. This has been done for the O'Neil bivane data, and it is shown that the resulting wind profiles exhibit the classical variation of curvature with lapse rate. This indirectly supports the view that the height variation of $\left(\frac{w}{z}\right)^{1/4}$ can be approximated by a power law.

REFERENCES


