Frictionally Controlled, Thermally Driven Circulations in a Circular Vortex with Application to Tropical Cyclones

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ABSTRACT

Based upon an argument by Charney and Eliassen, a tropical cyclone is envisaged as a combined system of a quasi-gradient circular vortex and a slow meridional circulation. The driving mechanism of this circulation is the release of latent heat, which is in turn controlled by the mass convergence in the surface boundary layer. A consistent set of dynamic equations is derived from scale and energy considerations, followed by the presentation of a two-level approximation in which potential temperature is specified only at the mid-tropospheric level. A perturbation analysis based on the linearized system shows that the exponential growth rates are of the correct order of magnitude.

A numerical integration of this set of equations is performed, starting from hypothetical initial distributions of tangential velocities. Unlike the too rapid and too intense development of meridional circulations observed in the numerical integrations of some previous hurricane models, the result obtained here shows a slowly developing circulation; the maximum tangential velocity is increased from 3 m sec⁻¹ (initial value) to 40 m sec⁻¹ in 60 hours and this velocity is twice as large as the maximum inflow. However, the circulation does not seem to approach a steady state. Other shortcomings of the model as revealed by the numerical integration are discussed.

1. Introduction

In recent articles, Charney and Eliassen (1964), Kuo (1965) and Ooyama (1964) have proposed dynamical models of a tropical cyclone in its developing stage. These models are distinctively different from those proposed in the past in one major point: differences in scale (and, consequently, in dynamical characteristics) between individual cumulus cells and tropical cyclones are clearly recognized. This led these authors to modify the basic hydro-thermodynamic equations in two respects. The first is that the energy releasing micro-scale motion (individual cumulus) is separated dynamically from the macro-scale motion, taking into account, however, their thermodynamic interaction in a parametric fashion. The other is that the wind and temperature changes in the macroscale motions take place in such a way that the horizontal circulatory motion preserves a quasi-gradient balance at all times and places.

According to their reasoning, the physical processes involved in the development of a tropical cyclone may be depicted in the following way. Consider a pre-existing weak horizontal, non-divergent, and quasi-geostrophic (or quasi-gradient) circulatory motion. Since the effect of friction is most pronounced near the ground surface, the tangential motion brings about a cross-isobaric motion toward low pressure (and consequently a mass convergence) in the planetary boundary layer. This mass convergence must be compensated by an upward transport of air through the top of the boundary layer. Since the density stratification in the lower portion of the tropical troposphere is conditionally unstable, this upward mass transport takes place most likely in the form of cumulonimbus clouds [the “hot towers” described by Riehl and Malkus (1961)]. A collective effect of the latent heat thus released inside the clouds is the production of a macro-scale meridional circulation, which consists of inflow in the lower troposphere and outflow in the upper troposphere. This circulation, presumably weaker than the primary circulatory motion, gives rise to a tangential velocity in the lower troposphere, because the absolute angular momentum of an air parcel is approximately conserved. The vertical mass transport through the top of the boundary layer may thereby be intensified, and this process can continue to develop.

The purpose of this paper is to investigate the evolution with time of this type of slow thermally and frictionally driven circulation in a circular vortex. To do this, a scale analysis will first be made, and a consistent set of equations will be derived. The analysis covers not only motions associated with a weak tropical depression (at small Rossby number), as was done by Charney and Eliassen (1964), but also covers motions in a well-developed tropical cyclone (at large Rossby number), and also motions both in the surface boundary layer and in the free atmosphere. A two-level representation will
then be introduced. The resulting equations are similar to those applied by Charney and Eliassen (1964) or Ooyama (1964) in their essential physical mechanism, but the problem is formulated here in the simplest possible way. The combined system of dynamic equations in the free atmosphere and in the boundary layer possesses a proper energy integral. A linear perturbation analysis of the system shows that the exponential growth rate for disturbances of small amplitude is of a realistic order of magnitude, \( 10^{-6} \) to \( 10^{-5} \) sec\(^{-1} \). Finally the result of a numerical integration of the non-linear equations is presented.

2. Scale analysis of the motion in the free atmosphere

Assuming axial-symmetry of the motion, we may write the basic equations in cylindrical coordinates as:

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{1}{r} \frac{\partial}{\partial \phi} (rv) = -\frac{\partial \phi}{\partial \phi} - \frac{\partial \tau_{r\phi}}{\partial \phi},
\]

(2.1)

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{1}{r} \frac{\partial}{\partial \phi} (rv) = -\frac{\partial \phi}{\partial \phi} - \frac{\partial \tau_{r\phi}}{\partial r},
\]

(2.2)

\[
\frac{\partial \phi}{\partial \phi} = -\alpha,
\]

(2.3)

\[
\frac{\partial (ru)}{\partial r} + \frac{\partial \phi}{\partial \phi} = 0,
\]

(2.4)

\[
\frac{\partial \theta}{\partial t} + \frac{1}{\kappa} \frac{\partial \theta}{\partial r} + \frac{1}{\kappa} \frac{\partial \theta}{\partial \phi} = -\frac{\rho}{\kappa c_p T} Q.
\]

(2.5)

The first two equations represent the equations of motion in the tangential and radial directions, respectively. The third equation represents the hydrostatic approximation, which we assume to be valid because the horizontal dimension (\( \sim 100 \) km) of the macro-scale motion considered here is much larger than the vertical dimension (\( \sim 10 \) km). The fourth equation is the conservation equation of mass, and the fifth is the first law of thermodynamics. Symbols used in (2.1)-(2.5) are:

- \( u \) = radial component of velocity,
- \( v \) = tangential component of velocity,
- \( \omega \) = vertical \( \phi \)-velocity (= \( \partial \phi / \partial t \)),
- \( f \) = Coriolis parameter (a constant),
- \( \phi \) = geopotential,
- \( \theta \) = potential temperature,
- \( T \) = absolute temperature,
- \( R = c_p - c_o \), \( \alpha \) = specific volume of the air,
- \( \tau_{r\phi} \) and \( \tau_{r\phi} \) = tangential and radial components of vertical stresses.

We note that horizontal stresses and turbulent mixing are ignored in (2.1), (2.2) and (2.5). This has been done because no definite magnitudes of these quantities are known and we intend to investigate motions which do not depend critically upon these quantities. We shall further ignore \( \tau_{r\phi} \) and \( \tau_{r\phi} \) for motions in the free atmosphere, retaining them, however, when in the next section we discuss the motion in the surface boundary layer.

A scale analysis for the motion associated with a hurricane is complicated because there appear to be three distinctive horizontal length scales; one is the scale corresponding to the size of an individual cumulonimbus cloud observed in a hurricane core (\( 1 \sim 10 \) km); another is a characteristic length scale corresponding to the typical observed size of a core itself (or the distance from the center of a hurricane to the point of maximum tangential velocity). We may take 100 km as a typical value of this length scale. The last length scale is the one which may be determined in terms of static stability and \( f \) from a dimensional argument, that is, \( L = [R(\Delta \theta)]/f \), where \( \Delta \theta \) is the range of variation of potential temperature in a mean atmosphere. This length \( L \) is essentially the "radius of deformation" defined by Rossby (1938), which occurs in problems of quasi-geostrophic motion. In addition to the variety of length scales, a characteristic tangential velocity, \( V \), also varies from 1 m sec\(^{-1} \) to 50 m sec\(^{-1} \), depending upon the stage of development of a hurricane.

In order to simplify the analysis, we will consider two extreme cases. The case (A) will correspond to a tropical depression for which we may take \( V \sim 5 \) m sec\(^{-1} \) and \( L = [R(\Delta \theta)]/f \). The reason for taking this particular length scale will be given in Section 5 when a perturbation analysis of the linearized system of equations is presented. If we take a latitude of 20 deg, \( f \) is \( 5 \times 10^{-5} \) sec\(^{-1} \). A value of \( \Delta \theta = 60 \) C is typical of the interval between ground and tropopause, and this gives a value of about 2600 km for \( L \). The relative vorticity \( \delta (\partial \phi / \partial r) / \partial r \) is then found to be much smaller than \( f \), and the value of the Rossby number \((V f^{-1} L^{-1})\) is about \( 10^{-1} \).

The case (B) will correspond to a well-developed tropical cyclone, for which we take \( V \sim 50 \) m sec\(^{-1} \). Since the rotation of the earth is expected to give only a secondary effect on this strong circulatory motion, the length \( L \) used in case (A) may not be a significant length scale here. Instead, we take an "observed" value, \( L = 100 \) km. The Rossby number in this case is then 10, very large compared to that in case (A) or to the value typical of traveling cyclones in middle latitudes. We mention, however, that this large Rossby number appears not to be of any particular significance for the motions in the free atmosphere considered here.

Since the effect of deep cumulus convection is going to be parameterized, the length scale corresponding to this type of motion does not come into our analysis. The vertical length scale may be taken equal to the height of the homogeneous atmosphere \( H (\sim 8 \) km) or, equivalently, to the standard surface pressure \( P (\sim 1000 \) mb).

Table 1 gives typical values of some variables in both cases (A) and (B) in terms of the basic parameters, \( V \), \( L \), \( H \) (or \( P \)), \( f \), \( \Delta \theta \) and a parameter which represents the effect of surface friction. Among them, the most important parameter is the typical value of \( \omega \) (denoted by
\( \Omega \) which is assumed to be given by the value of \( \omega \) at the top of the surface boundary layer, \( \omega_a \). [The subscript 4 is introduced in anticipation of the two-level representation given in Section 4.] The above assumption implies that \( \omega \) is at largest of the order of \( \omega_a \) in magnitude throughout the troposphere. The length \( h \) is the thickness of the surface boundary layer.

The vertical motion induced by mass convergence in a surface boundary layer was discussed by Charney and Eliassen (1949) and Syöno (1950, 1951). The mathematical expressions for \( \omega_a \) derived by them are slightly different from each other, depending upon how the inertia terms are approximated and how the constant stress layer is incorporated with the planetary boundary layer. We shall take here the following form (see Section 3):

\[
\omega_a = - \frac{\rho_s g \partial}{r \partial r} \left( \frac{r \tau_s}{f + \frac{\partial (r \omega_a)}{\partial r}} \right),
\]

(2.6)

where \( \rho_s \) = density of the air at the sea surface, \( \tau_s \) = surface stress, and \( \omega = v \) at the top of the boundary layer. The surface stress may be expressed as a function of surface wind speed \( v \), and a drag coefficient, \( c_D \):

\[
\tau_s = c_D \rho_s v_a |v_a|.
\]

(2.7)

To date, values of \( c_D \) estimated over a free water surface range from \( 2 \times 10^{-4} \) to \( 10^{-2} \) (Stommel, 1958, p. 82). In the following discussion, we take \( c_D = 3 \times 10^{-3} \).

The form of (2.6), when combined with (2.7), is rather inconvenient to use in a linear perturbation problem. It is also not consistent with geostrophic flow. For these reasons, we may also apply the following form [Charney and Eliassen (1949, 1964)]:

\[
\omega_a = - \frac{\rho_s gh}{2 r \partial r} \sin 2 \alpha_i,
\]

(2.8)

where \( \alpha \) is the angle between the surface wind and the wind above the boundary layer. In terms of the eddy kinetic viscosity coefficient \( v \), \( h \) is expressed as \( h = (2v/f)^{1/2} \).

We shall use (2.8) in case (A) and (2.6) in case (B) in evaluating the magnitude of \( \omega_a \). In the latter case, the typical value of the vertical component of relative velocity \( \zeta = (\partial \omega_a)/\partial r \) is of the order \( V/L \) and is much larger than \( f \). However, a typical radial distribution of \( \zeta \) estimated from the observed distribution of tangential velocity indicates that, while \( \zeta \) is much larger than \( f \) near the center of a tropical cyclone, it is comparable in magnitude with \( f \) near the point where the tangential velocity assumes the maximum value. It would therefore be reasonable to take \( f \) instead of \( V/L \), in the denominator of the right hand side of (2.6) in evaluating \( \Omega \). If we do this, \( \Omega \) is given by \( c_D P V^2/H L f \).

A typical value of the radial component of velocity, \( U_j \), is evaluated from \( \Omega \) by using the equation of mass continuity (2.4). The choice of time scale is less straightforward. An obvious choice might be \( L/V \). However, this time scale is the one which an air parcel takes to rotate once around the axis of rotation and would not be of any significance for the development of a hurricane. Instead, we may equate (in order of magnitude) the term \((\partial \zeta / \partial h) \) with \( fu \) in case (A) and with \( we/r \) in case (B), because the local variation of \( v \) with time is expected to be accounted for by the conservation of absolute angular momentum.

In Table 1, \( \Phi \) is a typical value of the variable part of geopotential. In case (A), \( \Phi \) is scaled so that quasi-geostrophic balance is implied, while in case (B), the pressure force is comparable with the radial acceleration. \( \Theta \) is a scale for the variable part of potential temperature determined from (2.3) and the ideal gas law. The quantity \( \theta_0 = \Theta \) denotes a typical value of potential temperature and \( Fr \) is the Froude number defined by \( V^2/g H \).

Using the scale factors thus determined, we now make the equations dimensionless. We divide the potential temperature \( \theta \) into a variable part and a part \( \tilde{\theta} \) representing the mean state of the atmosphere. Denoting the
dimensionless variables by a prime, (2.1), (2.2) and (2.5) may then be rewritten in case (A) as
\[
\frac{\partial \nu'}{\partial t'} + u' \frac{\partial \nu'}{\partial r'} + \omega \frac{\partial \nu'}{\partial \theta'} + \frac{\nu'^2}{r'} = 0,
\]
and in case (B):
\[
\frac{\partial \nu'}{\partial t'} + u' \frac{\partial \nu'}{\partial r'} + \frac{\nu'^2}{r'} - \frac{\partial \phi'}{\partial \theta'} = 0.
\]

3. Scale analysis of the motion in the boundary layer

Instead of the height of a homogeneous atmosphere \((H)\), the depth of the boundary layer \((b)\) is taken as a vertical length scale in a scale analysis of motions in the boundary layer. The theory of the boundary layer for case (A) (small \(\text{Ro}\)) is well known (Ekman, 1906). For case (B) the problem is not so clear, however. The following dimensionless forms of (2.1) and (2.2) apply in case (B):
\[
\left(\frac{h}{H}\right)^2 \left[\frac{\partial \nu'}{\partial t'} + u' \frac{\partial \nu'}{\partial r'} + \frac{\nu'^2}{r'} - \frac{\partial \phi'}{\partial \theta'}\right] = \frac{Q}{(\Delta \theta)V \alpha_p T},
\]
where \(\tau_\theta\) and \(\tau_r\), have been scaled by \(\rho g c_D V^2\) and \(\rho g c_D UV\), respectively. The results in case (A) are essentially the same as above. We observe in (3.1) and (3.2) that the time derivative terms may be ignored because \((h/H) \sim 10^{-1}\).

The terms inside the square brackets on the left side of (3.2) may not be ignored, because \((c_D V/h) \sim 10^{-1}\). If we nevertheless ignore them, (3.2) simplifies to
\[
\frac{\nu'^2}{r'} + \frac{\partial \phi'}{\partial r'} = 0.
\]

We may further assume, as is normally done in boundary layer analysis, that \(\phi'\) is independent of \(r'\) throughout the boundary layer. In view of (3.3), this approximation implies that \(\phi'\) is also independent of \(r'\) in the boundary layer. We may therefore ignore the term \(\omega' (\partial \omega' / \partial \phi')\) in (3.1), retaining a non-vanishing \(\tau_r\) at the ground surface, however. If we do this, and if we assume that the stresses vanish at the top of the boundary layer and that the vertical velocity vanishes at the ground surface, the equation (2.6) for \(\omega'\) follows from (3.1) and (2.4). It must be emphasized, however, that the argument for neglecting the bracketed terms in (3.2) is one of convenience, and is not based on their being negligible; a complete boundary layer theory has not been developed here for the case of large Rossby number.

The starting point of our analysis was the equation for \(\omega'\), (2.6) or (2.8). A question may then be asked: how relevant is the \(\omega'\) given by (2.6) or (2.8) for the development of a hurricane? To answer this question, we may
refer to the papers by Syōno (1950, 1951). He proposed that the rate of precipitation associated with a typhoon be expressed as a product of the surface mixing ratio of water vapor and the vertical velocity, the latter being given in terms of the tangential velocity:

$$w = A \frac{\partial (\nu_v)}{\partial r} \left[ f + \frac{\partial (\nu_v)}{\partial r} \right]^{-1}.$$  \hspace{1cm} (3.4)

The subscripts $s$ and 4 in this expression again denote quantities at the ground surface and at the top of the Ekman boundary layer, respectively. $A$ represents the combination

$$A = \sqrt{\nu} \sin \alpha / 2 \cos (\alpha + \pi / 4).$$

When $\frac{\partial (\nu_v)}{\partial r}$ is very small compared to $f$, (3.4) is almost the same as (2.8). By taking a proper value for $A$, Syōno showed that the radial distribution of the rate of precipitation thus calculated agreed fairly well with that observed. Fig. 1 shows the result of a similar type of comparison made by Syōno, Ogura, Gambo, and Kasahara (1951). The dotted lines represent rates of precipitation observed at three different observation stations during the near-by passage of Typhoon Jane. The dashed lines represent those calculated from observed tangential velocities on the basis of (3.4). Both the rate of precipitation and tangential velocities were observed as functions of time at each station. From a knowledge of the position of the typhoon center at successive times and the assumption of axial-symmetry, the variations of meteorological elements with time were converted to variations with distance relative to the typhoon center. In spite of the inevitable inaccuracy of this procedure, the agreement between observed and calculated rates of precipitation appears reasonable.

The scale analysis developed above indicates that in the free atmosphere the radial component of velocity is considerably smaller than the tangential velocity, whereas in the Ekman boundary layer the two are comparable in magnitude. Since the mixing ratio of water vapor decreases rapidly with height, we may infer that convergence of moisture takes place mostly in the boundary layer. The moisture thus accumulated in the boundary layer will then be transported upward in the form of the “hot towers.”

It is reasonable to assume that there is no change in water content in the boundary layer, with the excess moisture transported upward and ultimately precipitated inside the cumulus clouds. The quantity $\omega_4$ is then a measure of the amount of precipitation, as was in fact demonstrated by Fig. 1. $\omega_4$ may then be considered as the controlling factor of the whole circulation, because a hurricane is envisaged as being driven entirely by release of latent heat.

4. A two-level representation and the energy integral

For maximum simplicity in mathematical analysis, without loss of the essential physical processes, a two-level representation of the atmosphere will be useful. Let us denote quantities at $p = n p_4 / 4 \ (n = 0, 1, 2, 3, 4)$ by the subscript $n$, where $p_4$ is the pressure at the top of the boundary layer. In this two-level model, $u$ and $v$ are specified at two levels ($n = 1$ and 3) whereas $\theta$ is specified only at the mid-level ($n = 2$) of the atmosphere. We introduce as new dependent variables $M (= \text{square of the absolute angular momentum per unit mass})$ and $\chi$:

$$M = \left( r \nu v + \frac{f a^2}{2} \right),$$  \hspace{1cm} (4.1)

$$\chi = \phi - \frac{f a^2}{8}.$$  \hspace{1cm} (4.2)

The equations for $v$ and $u$ applied at level 1 and 3 may then be written as

$$\frac{\partial M_2}{\partial t} + \frac{\partial (\nu_v M_2)}{\partial r} + \frac{\omega_2 (M_1 + M_3)}{2 \Delta p} = 0, \hspace{1cm} (4.3)$$

$$\frac{\partial M_3}{\partial t} + \frac{\partial (\nu_v M_3)}{\partial r} + \frac{1}{2 \Delta p} \left[ 2 \omega_2 M_3 - \omega_2 (M_1 + M_3) \right] = 0, \hspace{1cm} (4.4)$$

$$\frac{M_1}{r^2} \frac{\partial \chi_1}{\partial r} - \frac{M_3}{r^2} \frac{\partial \chi_3}{\partial r} = 0, \hspace{1cm} (4.5)$$
The relations $M_2 = (M_1 + M_3)/2$ and $\omega_0 = 0$ have been assumed in deriving (4.3) and (4.4). In addition, $M_3 = M_4$ has been assumed for simplicity.

The hydrostatic equation is given by

$$\frac{x_2 - x_3}{\Delta \rho} = \frac{R \left( \frac{\rho_2}{\rho_3} \right) \theta_2}{\rho_3 \phi_4},$$

where $\kappa = R/c_p$. In terms of a stream function $\psi$, $u$ and $\omega$ are written as

$$ru_1 = -\frac{\psi_2}{\Delta \rho}, \quad ru_3 = -\frac{(\psi_4 - \psi_2)}{\Delta \rho},$$

$$r\omega_2 = -\frac{\partial \psi_2}{\partial r}, \quad r\omega_3 = -\frac{\partial \psi_4}{\partial r}.$$

Combining (2.6) with (4.8), we see that $\psi_4$ is given by

$$\psi_4 = -\rho_c g D \left( \sqrt{M_4 - \frac{f \phi_4^2}{2}} \right) \frac{\partial \sqrt{M_4}}{\partial r}.$$

It is not possible in our model to take into account the variation of static stability with time and space because $\theta$ is specified only at one level. We therefore write (2.5) in the form

$$\frac{\partial \theta_2}{\partial t} + \frac{\partial \psi_2}{\partial \rho} = \frac{\partial}{\partial \rho} \left( \frac{\kappa_2 T}{\rho_2} \right) Q,$$

with a constant $\partial \theta/\partial \rho$. It is noted that we have ignored not only the third term in the left side of (2.5)'s, but also the second term $\omega_2(\partial \psi_2/\partial t)$. The main reason for ignoring the second term is that otherwise the set of equations proposed here would not allow a proper energy integral [See the discussion following equation (4.13)].

With regard to $Q$, it was assumed by Charney and Eliassen (1964) that the amount of heat liberated in a vertical unit column is proportional to the total convergence of moisture. From the consideration described in the last part of the preceding section, however, we will simplify their assumption so that the amount of heat liberated is proportional to $\omega_2$ if $\omega_2 < 0$. This assumption was applied also by Oyama (1964) in his dynamical model of a hurricane. An individual cumulus cell in the convection zone of a tropical cyclone is observed to extend vertically through most of the troposphere [Malkus, Ronne and Chaifee (1961), and Riehl and Malkus (1961)]. With typical vertical velocities inside cumulonimbus clouds, an air parcel starting from the top of the boundary layer reaches the tropopause in 30 to 60 minutes. Comparing this time scale with that for the large-scale motion (Table 1), we may assume that the release of latent heat takes place instantly all the way through most of the troposphere. Since we specify temperature only at one level, it is also not necessary to specify the manner in which moisture is transported vertically inside an individual cumulus. We therefore write

$$\left( \frac{\partial}{\partial \rho} \right) Q = \frac{\partial \theta}{\partial \rho} - \eta \omega_2, \quad \text{if} \omega_2 < 0,$$

$$= 0, \quad \text{if} \omega_2 > 0.$$

The factor $\partial \theta/\partial \rho$ has been introduced into (4.11) simply to make the proportionality constant $\eta$ nondimensional. With typical values of $\partial \theta/\partial \rho (\approx 60 \text{C}/1000 \text{mb})$ and the mixing ratio of water vapor at the top of the boundary layer ($\approx 10 \text{ g kg}^{-1}$), the magnitude of $\eta$ can be roughly estimated as 3 to 4. The distinction between one group of theoretical models of hurricanes and another group may be seen most clearly in (4.11) or the corresponding mathematical formulation of $Q$. In one group [Haque (1952), Syööno (1955, 1962), Lilly (1960), Kuo (1961), Kasahara (1961), Estoque (1962) and Rosenthal (1964)], $Q$ at any height in the free atmosphere is assumed to be proportional to the vertical velocity at that height, whereas it is not so in another group [Charney and Eliassen (1964), Kuo (1964), Oyama (1964) and Kasahara$^*$.]

We may eliminate $x_4$ and $x_5$ from (4.3), (4.4), (4.5), (4.6) and (4.10) to obtain the diagnostic equation for $\psi_2$:

$$\frac{\partial}{\partial \rho} \frac{\partial}{\partial r} \left( \frac{\partial \psi_2}{\partial r} \right) + \frac{\partial}{\partial \rho} \left( \frac{\partial \psi_2}{\partial \rho} \right) \frac{\partial (M_1 + M_2)}{\partial \rho}$$

$$= \frac{\partial}{\partial \rho} \left( \frac{\partial \psi_2}{\partial \rho} \right) \frac{\partial \theta_2}{\partial \rho} + \frac{\partial}{\partial \rho} \left( \frac{\partial \psi_2}{\partial \rho} \right) \frac{\partial Q}{\partial \rho}.$$

With proper lateral boundary conditions, equations (4.3) [or (4.6)], (4.4) and the equations (4.7) through (4.12) then constitute our complete set of equations.

We shall now show that the above set of equations for a two-level model gives an energy integral. First, we get from (4.3) and (4.4) the rate of change of circulatory kinetic energy:

$$\frac{\partial}{\partial t} \left( \int_0^a \frac{1}{2}(v_1^2 + v_2^2) \, r dr \right) = -\int_0^a \omega_2(\phi_2) \, r dr$$

$$- \frac{1}{\Delta \rho} \int_0^a \omega_1(\frac{3}{2}v_1^2 + \phi_2) \, r dr.$$

Here it is assumed that both $u_1$ and $v_2$ vanish at a certain radius, $a$. Multiplication of (4.10) by $\theta_2$, followed by integration with respect to $r$ from 0 to $a$, yields the

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following equation:
\[
\frac{\partial}{\partial t} \int_0^a \frac{1}{2} \theta^2 r dr = -\frac{\partial}{\partial r} \frac{\theta}{R(p_2)} \int_0^a \omega_{a0} r dr + \int_0^a \left( \frac{\partial}{\partial r} \right) Q \theta r dr. \tag{4.14}
\]

If we multiply (3.1) [without the term \( \partial \theta / \partial t \)] by \( v \), (3.3) by \(-u\), add the resulting equations and then integrate with respect to \( p \) from \( p_k \) to \( p_e \) we obtain
\[
\frac{1}{\Delta p} \int_0^a \omega_k (\frac{1}{2} v^2 + v \phi_r) r dr - \frac{1}{h} \int_0^a v \omega_d r dr + \frac{1}{\Delta p} \int_0^a \frac{\partial v}{\partial r} \tau r r dr dp = 0. \tag{4.15}
\]

Combining (4.13) through (4.15), we finally obtain
\[
\frac{1}{2} \frac{\partial}{\partial t} \int_0^a \left[ \left( v_1^2 + v_2^2 \right) - \frac{1}{\Delta p} \frac{\partial v}{\partial p} \frac{\partial v}{\partial p} \right] r dr = \frac{1}{\Delta p} \frac{\partial}{\partial p} \frac{R(p_2)}{p_2} \frac{\partial}{\partial p} \frac{\partial}{\partial p} + \frac{1}{\Delta p} \int_0^a \frac{\partial v}{\partial r} \tau r r dr dp. \tag{4.16}
\]

The physical interpretation of this energy integral is as follows. The first and second term in the left side denote the rate of change with time of kinetic energy (measured only by the tangential velocity component) and available potential energy, respectively. The first term on the right side is the non-adiabatic heating rate, while the second and third terms represent the dissipation of energy by surface friction and by internal friction within the boundary layer. The last term is zero if we wish to be consistent with the approximation following (3.3) that \( \partial v / \partial p \) is zero in the boundary layer. The loss of energy is then due only to surface stresses. If we keep the terms \( u_0(\partial u_0 / \partial r) + \omega_0(\partial u_0 / \partial p) \) in (3.2) or the term \( u_0^2(\partial \theta_0 / \partial r) \) in (4.11), this type of energy integral would not follow.

It should be noted, however, that because of the formulation (4.11) for the heating mechanism, the total energy of our system is not really limited. (We have ignored the question of supplying the moist air to the boundary layer, as shown clearly by the omission of an equation of continuity for water substance.) Aside from the effects of non-adiabatic heating and boundary layer friction, the equations do correspond to one of the energetically consistent systems discussed in more general terms by Lorenz (1960).

5. A linear perturbation analysis of the model

In order to examine some dynamical characteristics of the model, a perturbation analysis based on the linearized system of equations is now presented. In doing this, the equations for conservation of absolute angular momentum are written as
\[
\frac{\partial v_1}{\partial t} + f u_1 = 0, \tag{5.1}
\]
\[
\frac{\partial v_3}{\partial t} + f u_3 = 0. \tag{5.2}
\]

Equations (4.5) and (4.6) are combined to give the thermal wind equation:
\[
f(v_1 - v_3) = \frac{\partial \theta_2}{\partial r}, \tag{5.3}
\]
where \( \gamma = R(\Delta p) p_2^{-1}(p_2 / \rho_2)^{\gamma} \). All the equations written above may readily be obtained by ignoring the terms containing \( R_0 \) in the corresponding equations in case (A) given in Section 2.

The continuity equations are expressed by
\[
\frac{\partial u_1}{\partial t} = \frac{\omega_2}{r \Delta p}, \tag{5.4}
\]
\[
\frac{\partial u_3}{\partial t} = \frac{1}{r \Delta p} (\omega_2 - \omega_1). \tag{5.5}
\]

The first law of thermodynamics [(4.10) with (4.11)] and equation (2.8) for \( \omega_4 \) are linear and do not need to be simplified.

Before obtaining eigen solutions of this set of linear equations, it is instructive to consider a simple case in which the relation \( (\theta / \omega_0 T) Q = \omega_0(\partial \theta / \partial p) \) is assumed to hold for both positive and negative values of \( \omega_4 \). It is possible in this simple case to derive both a necessary condition for instability and an upper bound of the exponential growth rate without finding eigen solutions.

We take \( \exp(\sigma t) \) multiplied by an \( r \)-varying amplitude factor as the form of solution. The exponential growth rate \( \sigma \) is in general complex. We may write this explicitly as \( \sigma = \sigma_r + i \sigma_i \), with \( \sigma_r \) and \( \sigma_i \) real. If \( \sigma_r > 0 \), the perturbation grows exponentially with time and the flow is unstable.

After an elementary calculation, we derive from the set of equations the following equation for the \( r \)-varying amplitude factor of \( u_4 \) (denoted by \( U_3 \))
\[
\left[ \frac{K}{\sigma} - (\eta - 1) - 1 \right] \frac{\partial}{\partial r} \left( \frac{\partial U_3}{\partial r} \right) + \lambda \left( \frac{K}{2 \sigma} + 1 \right) U_3 = 0, \tag{5.6}
\]
where
\[
K = \frac{f(h/H)}{\sin 2\alpha}, \tag{5.7}
\]
and
\[ \lambda^2 = 4 f^2 \omega_0^2 / \bar{T}_2 R (\Delta \theta). \] (5.8)

We multiply (5.6) by \( r U_s \) (\( U_s = \) complex conjugate of \( U_3 \)) and then integrate with respect to \( r \) from 0 to \( a \) (where we assume \( \omega_0 \) vanishes). Since \( \sigma \) and \( U_s \) are complex, the real and imaginary part of the resulting equation must vanish separately. The real part gives
\[ \left[ \frac{\sigma K}{|\sigma|^2} \right] \left( \eta - 1 \right) I_1 - \lambda^2 I_2 = \lambda^2 I_2 + I_1, \] (5.9)
where
\[ I_1 = \int_0^a \left| \frac{\partial r U_s}{r \sigma} \right|^2 r dr, \quad I_2 = \int_0^a |U_3|^2 r dr. \]

Since \( \lambda^2 > 0 \), the right side of (5.9) is positive. Consequently \( \sigma_r \) can be positive only if
\[ \eta > 1. \] (5.10)

This gives a necessary condition for instability. Conversely, a sufficient condition for stability is that \( \eta < 1 \).

We can also derive an upper bound for \( \sigma_r \) from (5.9). We first rewrite (5.9) as
\[ I_1 \left[ \frac{K \sigma_r}{|\sigma|^2} (\eta - 1) - 1 \right] = \lambda^2 I_2 \left[ \frac{K \sigma_r}{2 |\sigma|^2} + 1 \right]. \] (5.11)

When \( \sigma_r > 0 \), the right side of (5.11) is positive. Therefore we have
\[ \frac{K \sigma_r (\eta - 1)}{|\sigma|^2} > 1. \] (5.12)

Recalling that \( |\sigma|^2 \geq \sigma_r^2 \), it follows from (5.12) that
\[ \sigma_r < (\eta - 1) K. \] (5.13)

This upper bound indicates that the exponential growth rate is always smaller than \( f(h/H)(\eta - 1) \).

We shall now return to the case of a conditional heat source, where we consider that non-adiabatic heating is present only in the region extending from \( r = 0 \) to a certain radius, \( r_0 \), where \( \omega_0 < 0 \). The eigen solutions in this case are expressed in terms of Bessel functions of the first kind for the inner region \( 0 \leq r \leq r_0 \) and by Hankel functions for the outer region \( r_0 \leq a \leq \alpha \). By demanding that both the potential temperature and the radial component of velocity be continuous at \( r = r_0 \), we get the following equation for \( \sigma \):
\[ \frac{1}{nr_0} \frac{J_0(nr_0)}{J_1(nr_0)} = \frac{i}{nr_0} \frac{H_0^{(1)}(ivr_0)}{H_1^{(1)}(ivr_0)}, \] (5.14)
where
\[ (nr_0)^2 = (\alpha r_0)^2 \left( \frac{\sigma}{\eta - 1} \right)^2, \]
\[ 1 + \frac{K}{\sigma} \]
\[ (nr_0)^2 = (\alpha r_0)^2 \left( \frac{1}{\eta - 1} \right)^2, \]
\[ 1 + \frac{K}{\sigma}. \]

It is interesting to note that (5.14) is exactly the same as that derived by Haque (1952), if we replace \( gB \) (negative static stability in the inner region), \( gB' \) (positive static stability in the outer region), \( \alpha^2 \) (square of the growth rate) and \( \beta \) in his equation (30) by our dimensionless parameters, \( -(\eta - 1) \), 1, \( \sigma/K \) and (1/2), respectively. In spite of this resemblance in analytical form, however, the essential differences in physical processes involved in these two models should again be stressed: a tropical depression in Haque’s analysis is a free circulation driven by unbalanced buoyancy forces in a conditionally unstable atmosphere, whereas it is viewed in this study as a forced circulation driven by the heat released in organized deep cumulus convection.

In Fig. 2, \( \sigma \) is given from (5.14)–(5.15) as a function of \( r \), for several values of \( \eta \). The values of parameters used in this calculation are: \( f = 5 \times 10^{-6} \text{sec}^{-1}, h = 730 \text{m}, \) \( H = 8 \text{km}, \alpha = 15 \text{degrees} \) and \( \Delta \theta = 60 \text{C} \). These values give \( K = 2.28 \times 10^{-6} \text{sec}^{-1} \) and \( (1/\lambda) = 1180 \text{km} \). We observe in Fig. 2 that the maximum values of \( \sigma \) are in the range of \( 10^{-8} - 10^{-6} \text{sec}^{-1} \) for reasonable values of \( \eta \). It may also be noted that Charney-Eliassen’s analysis (Fig. 1 of their paper) shows a similar type of growth rate.

As described in Section 2, we have taken \( (R \Delta \theta)^{1/\lambda} \) as the horizontal length scale in our scale analysis for case (A) (a weak tropical depression). The reason for this is the following. Suppose we take instead an arbi-
trary length $L$ as a horizontal scale. From (5.3), the scale for the variable part of the potential temperature, $\Theta$, is given by $\Theta \sim fVL/R$. On the other hand, we may have $\Theta \sim (\Delta \theta) V/\ell L$, from (4.10), if the time scale is assumed to be given by $H/\ell f$ [see Table 1 and (5.13)]. These two temperature scales must of course be equal to each other and this requirement is met only if $L^2 \sim \lambda^2 \sim R \Delta \theta/f^2$.

6. Results of numerical integrations

The set of equations derived in Section 4 has been integrated numerically, starting from hypothetical initial conditions. In the finite difference approximations, $\psi_2$ and $\psi_4$ are represented at gridpoints $r = i/(\Delta r)$, while $M_1$, $M_3$ and $\theta_2$ are at $r = (i+\frac{1}{2})/(\Delta r)$, where $i = 0, 1, 2, \ldots, I$. The space increment $\Delta r$ is 5 km and $I = 200$, and the radial component of velocity is assumed to vanish at $r = I \Delta r = 1000$ km. For initial conditions $v_1$ and $v_2$ were equal to

$$v_1 = v_2 = (V_0/b) \exp[-(r/b)^2]$$

(6.1)

where $V_0 = 1.17$ m sec$^{-1}$ and $b = 141$ km. These give a maximum tangential velocity of 5 m sec$^{-1}$ at $r = 100$ km at $t = 0$. The identical distributions $v_1$ and $v_2$ in (6.1) imply that there is no horizontal temperature gradient initially. The initial distributions of $\psi_2$ and $\psi_4$ (and consequently those of $M_1$, $M_3$, $\omega_2$, and $\omega_4$) are then determined diagnostically from (4.12) and (4.9), respectively. The values of the other parameters applied here are $\Delta \theta/\partial \rho = -60^\circ/1000$ mb, $c_p = 3 \times 10^{-8}$ and $\eta = 3$.

A quasi-Lagrangian method (one-sided upstream space-difference with a forward time-difference) is used in the integration. The time increment used in the integration is chosen so that a linear computational stability criterion is satisfied. (It was 15 minutes at the beginning and was reduced to 3 minutes in the final steps of integration.) The computation has been discontinued at 60 hours of physical time because it became apparent by this time what this model could (or, more discouragingly, could not) describe. The total machine time used for the 60-hour prediction was about 50 minutes on an IBM 7090 computer.

A word of caution should be added in connection with (4.12). Suppose that $M_1$, $M_3$ and $\theta_2$ are given at a certain time $t_0$. Equation (4.12) then gives the $\psi_2$-field at that time. [The method described in Richtmyer's book (p. 103) is used in solving this equation.] We may then compute $M_3$ and $\theta_2$ at a later time $t_0 + \Delta t$ from (4.4) and (4.10). The new value of $M_3$ is finally computed diagnostically from (4.5) and (4.6) using the new values of $M_3$ and $\theta_2$.

Figs. 3 through 6 depict some of the results of this numerical integration. We observe in Fig. 3 that the maximum tangential velocity at level 3 increases with time, exceeding 40 m sec$^{-1}$ around 60 hours of the elapsed time. An anticyclonic circulation develops at the upper level ($v_1$) at values of $r$ greater than several hundred km. Fig. 4 shows that the growth of the meridional circulation is very slow initially, is gradually accelerated as $v_2$ increases, and is then catastrophically accelerated. This is because the surface stress, and consequently $\omega_4$, is proportional to the square of $v_2$. We mention also that, although the intensity of the inflow is considerably smaller than that of the horizontal circulatory motion and although it agrees approximately with that predicted from the scale analysis (Table 1), it seems to be stronger than might be expected from observations. In Hurricane Daisy, inflow was observed up to about 500 mb (Riehl and Malkus, 1961). In contrast, the analysis of the momentum and energy balance of hurricane Helene made by Miller (1962) indicates the presence of a deep layer of no inflow extending from 700 mb to 400 mb. Riehl (1963) postulated in his study of a steady state hurricane a meridional circulation which appeared to consist of inflow in the boundary layer (a layer directly affected by the surface friction), outflow in the upper troposphere and no radial motion at all between them.

It is interesting to compare the magnitude of the computed $\omega_4$ (or $\omega_2$) in Fig. 5 with some values estimated from observed data. Riehl and Malkus (1961) estimated the rate of precipitation for Hurricane Daisy on August 27 as 85 cm day$^{-1}$ near the hurricane center. If we take $13 \text{ g kg}^{-1}$ for the mixing ratio of water vapor at the top of the boundary layer, this rate of precipitation corresponds to 80 cm sec$^{-1}$ of vertical velocity at that height. Krishnamurti (1961) also estimated the vertical velocity for Hurricane Cleo from the observed distribution of tangential velocity. His result (Fig. 6 of his paper) shows that the vertical velocity assumes its maximum value of $-250$ mb hour$^{-1}$ (1.4 m sec$^{-1}$) near the 400-mb level. Compared to these results, the values of $\omega_2$ and $\omega_4$ obtained here are not unrealistic. However, the radial distributions of $\omega_2$ and $\omega_4$, and their variations with time are far from satisfactory; the area of active cloud convection is confined in a very narrow region and this area does not change its radial position with time, or even shows a tendency to move outwards instead of moving inwards to form a concentrated hurricane core.

Fig. 6 shows the variation with time of the radial distribution of potential temperature anomaly, defined as the difference from potential temperature at $r = 1000$ km. (It is the horizontal gradient of potential temperature, not the potential temperature itself, which counts.) For reference, we mention that the maximum temperature anomaly observed in hurricanes ranges from 5 to 10 deg, according to analyses by Colon and staff (1961) and by others.

7. Concluding remarks

Numerical integrations of unbalanced models of tropical cyclones have been performed in the past by Kasahara (1961) and Syöö (1962), and recently by Rosenthal (1964). By taking a large horizontal grid
Fig. 3. Radial distributions of tangential velocity at the upper level ($v_1$) and at the lower level ($v_2$) at various times.

Fig. 4. Radial distributions of radial velocity at the upper level ($u_1$) and at the lower level (plotted as minus $u_2$) at various times.

Fig. 5. Radial distributions of vertical $\rho$-velocity ($\omega_2$) at the mid-tropospheric level ($\omega_2$) and at the top of the boundary layer ($\omega_3$) at various times.

Fig. 6. Radial distribution of potential temperature anomaly at the mid-tropospheric level at various times.
mesh (thereby introducing a large damping effect), Rosenthal was able to carry his computation up to 10 hours of physical time. The result was that a vortex which was initially weak intensified and surpassed hurricane intensity by a considerable margin. This intensification occurred, however, much too rapidly and the meridional circulation which evolved was too intense; in fact, the maximum inflow ($u$) exceeded the maximum tangential wind ($v$) in most of his numerical experiments.

In contrast, the balanced model developed in this paper produces a slowly intensifying circulation, at least in the initial stage; the maximum tangential velocity exceeds 40 m sec$^{-1}$ only after 60 hours and the maximum inflow is considerably smaller than the maximum tangential velocity during the course of development. Aside from these points, however, the present result is by no means satisfactory if intended specifically to describe the detailed development of a tropical cyclone.

One of the most unsatisfactory features may be that the growth of the meridional circulation is eventually accelerated catastrophically and that there is no tendency to approach a steady state. The same feature was observed also by Ooyama (1964), who formulated the problem in terms of two incompressible homogeneous layers of different densities. We may attribute this feature in part to the simplifications introduced into the thermodynamic equation (4.10); the horizontal advective term is ignored and both the static stability and $\eta$ are fixed. These assumptions imply that $\theta_z$ has to increase as $\omega_4$ increases.

This shortcoming may relatively easily be remedied by representing the potential temperature at two levels and imposing the constraint that the potential temperature at each level does not exceed a prescribed value, (for example, the value corresponding to the equivalent potential temperature at the surface under saturation conditions). However, even if these modifications were made, the tangential velocity would likely continue to increase. This is because the time-independent temperature in the updraft region with $\partial\theta/\partial p<0$ still implies the presence of liberation of latent heat and this would continue to produce the inflow (and consequently the increase of tangential velocity) in the lower troposphere.

One possible process through which the tangential motion may approach a steady state is that an air ring ascends as it approaches the center of a tropical cyclone, moves outwards at the upper level and then descends, effecting thereby a recirculation of dried air. In view of the fact that the stratification of the tropical atmosphere is stable for unsaturated descending motion and the intensity of a downdraft is therefore very weak [as indicated in Fig. 5 or in the writer's paper on cumulus type convection (1963)], this process is possible but would take too much time to be effective.

At this point, it should be recalled that our formulation of the boundary layer was such that the vertical Reynolds stresses vanish at the top of the boundary layer. This implies that surface friction has no direct effect on the circular motion at level 3. In contrast to this, Sheppard, Charnock and Francis (1952) have shown that the vertical gradient of wind speed in the first few hundred meters over the northeastern Atlantic was of the same order at all levels and there was no identifiable transition from a layer of frictional influence to a layer of friction-free wind.

We have already mentioned other unsatisfactory aspects of our results; the inflow seems too strong and the position of the maximum $\omega_4$ or $v_3$ tends to shift outwards with time, instead of shifting inwards as we might expect. Apparently this tendency resulted from our formulation (2.6) for $\omega_4$; $\omega_4$ is quite sensitive to $v_3$ and $\partial(rv_3)/\partial r$. These considerations suggest that a more proper formulation of the boundary layer associated with a tropical cyclone is required for improvement of the present model.

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