Stability of a Continuous Baroclinic Flow in a Zonal Magnetic Field

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ABSTRACT

The properties of quasi-geostrophic, baroclinic flow in a zonal magnetic field, as formulated previously by the writer, are examined for a zonal flow profile of hyperbolic tangent form in the vertical coordinate. The stability problem is shown to be mathematically equivalent to a stratified shear flow problem considered by Kraus, with a magnetic Richardson number replacing the ordinary Richardson number of stratified shear flow. The marginal stability curve is found and indicates that, contrary to the two-layer case, there is a short-wave cutoff for unstable waves, and sufficient reduction of the channel width can render all waves stable.

The hyperbolic tangent profile represents essentially a thin baroclinic layer in a vertically unbounded atmosphere. A layer qualitatively of this type might be produced in the convective zone of the sun through the influence of rotation on the supergranulation or granulation scale motions, in the manner suggested by the writer and other authors previously.

1. Introduction

The problem of quasi-geostrophic, quasi-nondivergent, hydromagnetic perturbations of a coplanar baroclinic flow and zonal magnetic field has been examined recently by Gilman (1966, 1967a,b,c). Some general integral theorems were proved for both continuous and two-layer zonal flows and magnetic fields, and the stability properties, and structure and energetics of unstable disturbances were examined in the 2-layer case. However, no explicit continuous zonal flow profiles were studied. It is the purpose of the present brief article to examine the stability of one particular flow profile, the hyperbolic tangent profile, for which results may be found rather easily.

2. Perturbation equation and stability criteria

The basic state about which we wish to perturb is characterized by a zonal flow \( U = U(z) \), zonal magnetic field \( H = H(z) \) (both nondimensional, where \( z \) is the vertical coordinate), a horizontally averaged (dimensional) density \( \rho = \rho(z) \) that is at least quasi-Boussinesq (Charney and Stern, 1962) and a constant static stability, represented by \( \epsilon \) [see Charney and Stern (1962) or Gilman (1967a)]. The perturbation stream function for the quasi-nondivergent horizontal flow representing waves traveling in the zonal (\( x \) coordinate) direction between side walls at \( y = 0, \ell \) is given by

\[
\psi(x,y,z,t) = (\sin \gamma) \tilde{\psi}(z) e^{i(kx+\epsilon t)},
\]

where \( r = n \pi/\ell, \ n = 1,2,3, \ldots \), so that \( \psi \) vanishes at the side walls; \( k = 2\pi/\lambda \), the zonal wave number, and \( \lambda \) the zonal wavelength; and where \( c = c_r + ic_i \) is the phase velocity, complex for unstable waves (\( c_i > 0 \) gives the exponentially growing wave).

Because the basic state is independent of \( y \), the cross-channel coordinate, the potential vorticity equation for the perturbation stream function \( \tilde{\psi}(z) \) may be written in the form [easily found, for example, from Eq. (7.10) of Gilman (1967a)]

\[
\frac{1}{\rho} \frac{\partial}{\partial z} \left( \frac{\rho \partial \tilde{\psi}}{\partial z} \right) \left( k^2 + r^2 \frac{\partial q}{\partial y} \right) \frac{q}{U - c} \left( \frac{(k^2 + r^2)P}{U - c} \right) \tilde{\psi} = 0,
\]

where \( s = \epsilon \sin \gamma \phi \), the latitude at which the \( \beta \) plane expansion is made, and \( P = H^2M^2/4\pi \rho V^4 \), where \( M \) is the magnetic field scale in electromagnetism, and \( V \) is the scale velocity. \( \partial q/\partial y \) is the cross-channel gradient of potential vorticity of the basic state, given by

\[
\frac{\partial q}{\partial y} = \beta - \frac{1}{\rho} \frac{\partial}{\partial z} \left( \frac{\partial U}{\rho \partial z} \right),
\]

where \( \beta \), as usual, is the cross-channel gradient of the Coriolis parameter, in nondimensional form.

Let us assume that for this problem the fluid is Boussinesq rather than quasi-Boussinesq. For the present case, this is essentially assuming that the scale height for variation of mean density is much larger than for the mean flow profile \( U(z) \) and for the perturbation stream function \( \tilde{\psi}(z) \). Let us also take the mean zonal magnetic field \( H^2(z) \propto \rho \), so that the parameter \( P \) is constant. This is physically a reasonable restriction, as it makes the scale strength of mean magnetic and kinetic energies appear in constant ratio throughout the atmosphere. Finally, we neglect the \( \beta \) effect for the present problem.

With these conditions, (2) reduces to the form

\[
\frac{d^2\tilde{\psi}}{dz^2} - \left( k^2 + r^2 \frac{\partial U}{\partial z} \right) + \frac{Pe^2}{U - c} \tilde{\psi} = 0,
\]

where \( k^2 = (k^2 + r^2)/s \).
Eq. (4) has been studied in another physical context quite extensively. That is, (4) has the same mathematical form as that for perturbations on shear flow in a stratified incompressible fluid, for example, by Drazin (1958), Howard (1961), Drazin and Howard (1962), and Miles (1961, 1963). Here the product $Pa^2$ would be replaced by $J$, the Richardson number for stratified shear flow. $P$, then, may be looked on as a magnetic Richardson number. It should be kept in mind, though, that the two systems are physically rather different. For example, our system is hydrostatic, but the stratified shear flow (hereafter denoted by SSF) need not be. The static stability in our problem keeps the horizontal flow quasi-nondivergent whereas in the SSF case it acts to temper the Helmholtz instability of the vertical shear. In the present case, the magnetic field plays the role analogous to that of the static stability for SSF in decreasing the strength of the baroclinic instability.

Despite these physical differences, the mathematical properties carry over to the present problem, and can be used to considerable advantage. In particular, one profile for which (4) may be solved rather easily is the hyperbolic tangent flow,

$$U = U_0 \tanh(z/d),$$

studied by Drazin (1958) for SSF. This flow has most of its shear concentrated near $z=0$, with the flow reaching a maximum of $+U_0$ at $z=+\infty$ and a minimum of $-U_0$ at $z=-\infty$.

In the SSF problem, the dependent variable is the vertical motion $v$, and the boundary conditions chosen are that $w \to 0$ as $z \to +\infty$. In the present case, with the dependent variable $\tilde{\psi}$, the stream function for horizontal flow, we could take the condition

$$\lim_{z \to +\infty} \frac{d\tilde{\psi}^*}{dz} = 0,$$

where the asterisk denotes the complex conjugate, as stated by Gilman (1967a) and by Pedlosky (1964), for the analogous nonmagnetic baroclinic problem. However, for the present problem, we take instead a somewhat stronger condition, namely

$$\lim_{z \to +\infty} \tilde{\psi} \to 0.$$

This condition renders the present problem the same as for SSF in boundary conditions as well as in differential equation.

The solution of (4) with the limiting condition (7) for the hyperbolic tangent profile (5) has been given in detail in §104 of Chandrasekhar (1961), so the solution need only be sketched here.

First, it is evident that if $\tilde{\psi}$ is a solution to (4) with eigenvalue $\alpha$, then $\tilde{\psi}^*$ is a solution with eigenvalue $\alpha^*$. In addition, since $U$ is an odd function of $z$, $\tilde{\psi}(-z)$ is a solution with eigenvalue $-\alpha$. From these symmetry properties, and assuming that marginally stable waves [which separate stable (neutral) waves from unstable ones] are characterized by a unique eigenvalue $\alpha$, the principle of exchange of stabilities should be valid, so that the curve of marginal stability should be characterized by the condition $\alpha=0$. Hence, we should be able to find the marginal stability curve, and, therefore, the instability criteria, by solving (4) with $\alpha$ set equal to zero, i.e.,

$$\frac{d^2\tilde{\psi}}{dz^2} \left( \frac{d^2U/da^2}{U} + \frac{Pa^2}{U^2} \right) \tilde{\psi} = 0. \tag{8}$$

Following Chandrasekhar (1961, §104) we can show that a solution of (8) is given by

$$\tilde{\psi} = \text{constant} \cdot U^\alpha(1-U^{\alpha}), \tag{9}$$

where $\mu$ and $\nu$ satisfy the relations

$$\mu = \frac{1}{2} + \frac{1}{2}(1-4K)^{1/2}, \tag{10}$$

$$\nu = \frac{1}{2}(\alpha^2-K)^{1/2} \quad \text{(real part > 0)}, \tag{11}$$

$$2\nu + \mu - 1 = 0, \tag{12}$$

in which

$$K = \frac{\hbar^2 dP}{U^2}; \quad \alpha^2 = \hbar^2 dP. \tag{13}$$

Here the conditions (10) and (11) on $\mu$ and $\nu$ render the solution regular at the singular points $z=0, \pm \infty$. Furthermore, one can easily show that the solution (9) for $\tilde{\psi}$ satisfies the limiting conditions (7). The condition (12) determines the curve of marginal stability in $K$, $\alpha^2$ space. Thus, substituting from (10) and (11) into (12), and squaring, regrouping the result, and squaring again, gives

$$K = \alpha^2(1-\alpha^2). \tag{14}$$

So far we have found only the marginal stability curve, but no solutions for waves that are actually unstable. The question arises as to whether the neutral curve we have found indeed separates two regions, one unstable and the other stable with respect to normal mode disturbances. This is often not the case in other studies of baroclinic instability, as seen for example, in Burger (1962). In the present problem, however, it appears that the marginal stability curve (14) really does separate unstable waves below, i.e., for which

$$K < \alpha^2(1-\alpha^2), \tag{15}$$

from neutral oscillations above the curve. That is, mathematically speaking, the present stability problem is one which falls within a wider class of problems discussed by Miles (1963) for which the curves of marginal stability can be proven to separate stable and unstable regions.

For purposes of interpretation, and comparison with stability properties of the two-level case already ana-
lyzed by Gilman (1967b), we need to write (15) in terms of $k^2$, $r^2$, $s$, $d$, $P$, and $U_0^2$. That is, unstable waves will occur for

$$\frac{P}{U_0^2} < 1 - \frac{(k^2 + r^2)d^2}{s}. \quad (16)$$

The solution for $\tilde{\psi}(z)$ along the curve of marginal stability, for which (16) would contain an equality sign, is given by

$$\tilde{\psi}(z) = A \left( \text{sech}(z/d) \right)^{\frac{P}{U_0^2}} \left( \tanh(z/d) \right)^{\frac{2s}{P^2 - r^2}}, \quad (17)$$

where $A$ is a constant amplitude factor.

3. Discussion

From (16), it is evident that baroclinically unstable waves exist for continuous zonal flows in a zonal magnetic field as well as for the two-layer flows analyzed by Gilman (1967b,c). It follows also that the energetics of the unstable disturbances should be the same for both. That is, in each case the unstable waves convert available potential energy to kinetic energy of the disturbances, some of which is further converted into disturbance magnetic energy. These disturbances will also produce the weak vertical fields in a similar manner to the two-level disturbances, as shown particularly in Gilman (1967b).

It is certainly relevant to compare the stability properties in the two-layer and continuous cases. In doing so, however, there will inevitably be a certain amount of arbitrariness in the choice of values for scaling parameters, particularly the vertical length scale. In the two-layer problem, the writer (Gilman, 1967b) took $U_0$, rather than $2U_0$, as in the present case, as the total vertical variation in the zonal flow. The two layers were each taken to have a nondimensional depth of $\frac{1}{2}$.

To compare the stability criteria found there with (16), let us let $U_0 \rightarrow 2U_0$ in the two-layer stability criteria, and denote the nondimensional depth of each of the two layers by $b$. (The profiles and scaling parameters for the continuous and two-layer cases are presented in Fig. 1). The stability criteria for the two-layer case then become, in the present notation,

$$\frac{P}{U_0^2} < 1; \quad \frac{P}{U_0^2} > 1 - \frac{2s}{b^2(k^2 + r^2)} \quad (18)$$

for instability.

Several comparisons may be made between the stability criterion (16) for the continuous case, and (18) for the two-layer case. First of all, the minimum vertical shear in the zonal flow (and, consequently, the minimum meridional temperature gradient or available potential energy) required for instability for a given magnetic field strength is the same for both cases, i.e., $P/U_0^2 < 1$. However, the wavelength for the minimum required shear is vastly different; zero for the two-layer case, and infinity for the continuous case.

In addition, the hyperbolic tangent case has a short wavelength cutoff at

$$k^2 = \left( \frac{s}{d^2} \right) - r^2. \quad (19)$$

For $k$ larger ($\lambda$ smaller) than given by (19), all waves are stable no matter what the vertical shear or zonal magnetic field. As pointed out by Gilman (1967b), and evident from (18), the two-level model has no such cutoff.

It is also evident from (19) that if the channel width is narrow enough, i.e., if $r^2 > s/d^2$, there will be no unstable waves for any values of the vertical shear or zonal magnetic field. This is again contrary to the two-layer case, for which, as pointed out by Gilman (1967b) and easily deducible from (18), no matter how narrow

![Fig. 1. Zonal flow profiles for (a) two-layer and (b) continuous (hyperbolic tangent) cases.](image-url)
the channel, there will be unstable wavelengths for some finite range of magnetic field strengths and vertical shears in the zonal flow.

These differences between the two-layer and continuous cases imply that the destabilizing effects of the magnetic field for short waves in the two-layer case are probably due to the truncation errors introduced from taking such a small number of layers in the vertical.

If we choose explicit values for the parameters \( b, d, s \), and \( r \), we may compare graphically the regions of instability for the two cases in \( P/U_0^2, \lambda^2 \) space. Let us take \( s=1 \) and \( r=0 \) (infinite channel width) for simplicity and let \( b^2=1, d^2=1/4 \) (corresponding roughly to the relative scales for the two profiles in Fig. 1). Then in terms of \( \lambda^2 \), the stability criteria for the continuous (16) and two-layer (18) cases become, respectively,

\[
P/U_0^2 < 1 - 4\pi^2/\lambda^2, \tag{20}
\]

and

\[
P/U_0^2 < 1; \quad P/U_0^2 > 1 - \lambda^2/4\pi^2. \tag{21}
\]

The regions of instability defined by these criteria are represented graphically in Fig. 2. Obviously, the precise position of the boundary in Fig. 2 between the region for which only the two-layer model is unstable and that for which both are unstable, depends upon our particular choice for the parameters \( d \) and \( b \). The lack of a long-wave cutoff in Fig. 2 is due to our neglect of the \( \beta \) effect. It is clear that by including it, the continuous case we have studied will have both a short- and a long-wave cutoff.

Physically, the hyperbolic tangent profile we have just studied corresponds to a relatively narrow baroclinic layer in an unbounded atmosphere. That is, through the thermal wind relation, the meridional temperature gradient associated with the thin layer of strong vertical shear will be confined to a layer equally shallow in vertical extent. A baroclinic layer of this type is of relevance to the large scale solar general-circulation problem, as discussed in Gilman (1966, 1967a, c) and in earlier publications cited therein. That is, if the smaller scale motions in the sun, i.e., the granulation and supergranulation, are creating available potential energy for larger scale baroclinic overturnings, the meridional temperature gradient defining this available potential energy would presumably be confined to a layer of limited vertical extent within the convection zone (though admittedly not necessarily so limited as to be Boussinesq). As such, the meridional temperature gradient might be confined entirely to levels sufficiently below the photosphere so as to be incapable of detection at the surface. If this were the situation on the sun, it might explain why a meridional temperature gradient has not been conclusively demonstrated to exist at photospheric levels.

REFERENCES


