The Effect of Coalescence and Condensation on Rain Formation in a Cloud of Finite Vertical Extent

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ABSTRACT

The simultaneous effects of collisions between water droplets and their growth by condensation are investigated numerically for a cloud of finite vertical extent. The consequent changes in droplet distribution have been followed for time up to \( \sim 15 \) min. A comparison is made between geometric collision efficiency and the collision efficiency of Shafir and Neuburger. It is shown that the omission of growth by condensation may lead to considerable errors.

1. Introduction

The development of cloud droplet spectra with time has been treated by various authors. Tedford (1955) has considered growth by coalescence of an initial distribution consisting of two discrete sizes of droplets in a uniform cloud of infinite height. Twomey (1964, 1966) has set up an integral equation to describe the time development of a continuous droplet spectrum, again in a uniform and infinite cloud, and has then proceeded to solve this equation by numerical methods. Berry (1967) has considered both analytic and numerical solutions of the same equations. Warshaw (1967a,b) has added the effect of a finite vertical extent.

The purpose of this paper is to suggest a computation scheme to account simultaneously for gravity (in a cloud of finite vertical extent), collisions between droplets according to an arbitrarily given efficiency, growth by condensation at a rate which may have a general dependence on the droplet radius and on height within the cloud, and an updraft which may vary with height. It is hoped that the inclusion of all these effects in a relatively rapid computation will result in a more realistic evaluation of the process of rain formation in a warm cloud. The general partial differential equation describing the model is derived in Section 2 and this is followed by the computation scheme in Section 3, with results for four simple numerical examples in Section 4.

2. The differential equation

The rate of growth by condensation of a droplet of radius \( r \) at height \( h \) and time \( t \) has the general form

\[
\frac{dr}{dt} = f(r, h, t),
\]

where \( f \) depends on the coefficient of surface tension, the electric charge and quantity of salt in the droplet, and on the vapor concentration, temperature and pressure at height \( h \) and time \( t \). Since the droplet falls against an updraft, its vertical position is determined by

\[
\frac{dh}{dt} = W(h, t) - U(r),
\]

where \( W(h, t) \) is the updraft at height \( h \) and time \( t \), and \( U(r) \) is the terminal velocity of fall in still air of a droplet of radius \( r \).

Let \( N(r, h, t) \delta r \delta h \) be the number at time \( t \) of droplets with radii in the range \( \delta r \) around \( r \) and in a layer of thickness \( \delta h \) around \( h \). After a time \( \Delta t \) there will be \( N[r + \Delta r, h + \Delta h, t + \Delta t](\delta h + \Delta(\delta h))(\delta r + \Delta(\delta r)) \) droplets in the range \( \delta r + \Delta(\delta r) \) around \( r + \Delta r \) and in a layer of thickness \( \delta h + \Delta(\delta h) \) around \( h + \Delta h \). If new droplets are produced at a rate \( q(r, h, t) \delta r \delta h \) and droplets are being added by coalescence at a rate \( F(r, h, t) \delta r \delta h \) (\( F \) will usually be negative for small radii at which droplets are being depleted and positive for larger radii at which new droplets are formed by coalescence), then we have

\[
N[r + \Delta r, h + \Delta h, t + \Delta t] = N[r, h, t] + N[r, h, t + \Delta t] - N[r, h, t] - F(r, h, t) \delta r \delta h \Delta t. \quad (3)
\]

From Eq. (2) we then have

\[
\Delta h = [W(h, t) - U(r)] \Delta t; \quad \Delta(\delta h) = \delta(\Delta h) = \frac{\partial W}{\partial h},
\]

and from Eq. (1),

\[
\Delta r = f(r, h, t) \Delta t; \quad \Delta(\delta r) = \delta(\Delta r) = -\delta r \Delta t.
\]

Substituting these expressions into (3) leads to the
partial differential equation
\[
\frac{\partial N}{\partial t} = -\frac{\partial}{\partial h}[N(W-U)] + F \frac{\partial}{\partial r}(tN) + q. \tag{4}
\]

3. Scheme of computation

The model considered is a cloud of finite height in which the distribution \( N(r, h) \) of the number density of cloud droplets is initially given at discrete radii \( r_1, r_2, \ldots, r_M \), and heights \( h_1, h_2, \ldots, h_K \). The results of the calculation is the distribution \( N \) at successive times and at the above discrete radii and heights.

During a time step of size \( \Delta t \), the distribution of droplets \( N(r, h) \) is altered by a number of effects which are described below. Each one of these changes the distribution from \( N \) to a new distribution \( N_1 \). The various effects are as follows.

1) Falling of droplets:
\[
N_1 (r, h) = N[r, h + U(r) \Delta t - W(h,r) \Delta t]. \tag{5}
\]

In general, the height \( h + U(r) \Delta t - W(h,r) \Delta t \) will not be one of the heights \( h_1, h_2, \ldots, h_K \), at which the distribution is known. Hence, the distribution at this height is found by quadratic interpolation using three of the heights \( h_1, h_2, \ldots, h_3 \), nearest to this height. Special account is taken of the top and bottom of the cloud; these are computed at each time step in order to avoid interpolation between the cloud and the clear air outside it, a procedure which would lead to a spurious dispersal of the cloud.

2) Coalescence of droplets:
\[
N_1 (r, h) = N[r, h] - \sum_{i} \sum_{m=1}^{M} P(i, n) N(n, h) + \sum_{i} \sum_{m=1}^{M} B(n, m, i) P(n, m) N(r, h) N(m, h), \tag{6}
\]

where \( P(n, m) \) is the coalescence probability for droplets of radii \( r_n \) and \( r_m \)
\[
B(n, m, i) = \begin{cases} 
\frac{(r_n^2 + r_m^2 - r_{i-1}^2)/r_i^2 - r_{i-1}^2}{r_{i-1}^2 < r_n^2 + r_m^2 \leq r_i^2}, & \text{for } r_{i-1}^2 < r_n^2 + r_m^2 \leq r_i^2, \\
\frac{(r_{i+1}^2 - r_n^2 - r_m^2)/r_{i+1}^2 - r_{i+1}^2}{r_{i+1}^2 < r_n^2 + r_m^2 < r_{i+1}^2}, & \text{for } r_{i+1}^2 < r_n^2 + r_m^2 < r_{i+1}^2, \\
0, & \text{for } r_n^2 + r_m^2 \geq r_{i+1}^2 \text{ or } r_n^2 + r_m^2 \leq r_{i-1}^2.
\end{cases} \tag{7}
\]

The factor \( B(n, m, i) \) ensures that summation in the second term of (6) extends only over those pairs of droplets which combine to give droplets of radius \( r_i \). Since it almost never happens that \( r_n^2 + r_m^2 = r_i^2 \), each droplet formed by coalescence must be redistributed between the discrete radii adjacent to its radius. The redistribution effected by \( B(n, m, i) \) has been chosen because it is the one which preserves both the number of droplets and the volume of water concerned.

3) Growth of droplets due to condensation of water vapor:

The equation of growth has the general form, \( dr/dt = f(r, h, t) \).

\[
r(t + \Delta t) = r(t) + \Delta r = r(t) + f(r, h, t) \Delta t, \tag{8}
\]

and the change in the distribution is given by
\[
N_1 (r, h) = \sum_{n=1}^{i} G(n, i) N(n, h), \tag{9}
\]

where
\[
G(n, i) = \begin{cases} 
[(r_n + \Delta r_n)^2 - r_{i-1}^2]/[r_i^2 - r_{i-1}^2], & \text{for } r_{i-1}^2 < (r_n + \Delta r_n)^2 \leq r_i^2, \\
[r_{i+1}^2 - (r_n + \Delta r_n)^2]/[r_{i+1}^2 - r_{i+1}^2], & \text{for } r_{i+1}^2 < (r_n + \Delta r_n)^2 \leq r_{i+1}^2, \\
0, & \text{for } (r_n + \Delta r_n)^2 \geq r_{i+1}^2 \text{ or } (r_n + \Delta r_n)^2 \leq r_{i-1}^2.
\end{cases} \tag{10}
\]

The function \( G(n, i) \) serves the same purpose as the function \( B(n, m, i) \) in the coalescence process.

4) In the future it is planned to account for nucleation by adding new drops of small size at a rate which may depend on height.

The change in the distribution during a time step \( \Delta t \) is now taken to be the result of sequentially applying the above processes; i.e., we assume that during a single time step droplets first fall, then coalesce, and then grow by condensation. This is equivalent to calculating the time derivative of \( N \) by taking the various effects in the order in which they appear on the right hand side of (4).

The matrices \( P \), \( B \) and \( G \) are evaluated at the beginning of the computation and then stored for subsequent use. Thus, even the use of sophisticated collision probabilities and laws of growth will not essentially increase the time of computation. The stability of the computation scheme is discussed in the Appendix.

4. Numerical examples

Calculations were carried out for four simple cases. In all cases the initial situation was taken to be a cloud of height 500 m in which the distribution of droplets was uniform with height. The calculation was extended to include heights up to 50 m above and below the cloud and the height was subdivided into intervals of 10 m.

The updraft \( W(h,t) \) was taken to be 10 cm sec\(^{-1} \) at all heights and for all times. The initial distribution was of the form
\[
n(r) = 50(\sigma \sqrt{2\pi})^{-1} \exp \left[ - (r - \mu)^2 / (2\sigma^2) \right] \quad \text{[droplets cm}^{-3}\text{cm}^{-3}\text{]} \tag{11}
\]
with \( \sigma = 2.47 \times 10^{-4} \text{ cm and } \mu = 16.5 \times 10^{-4} \text{ cm} \), giving a total of 50 droplets cm\(^{-3} \) and a liquid water content of...
1 gm m\(^{-3}\). The function \(n(r)\) was discretized by defining

\[
\begin{align*}
  a_i &= (r_i+r_{i-1})/2, \\
  b_i &= (r_{i+1}+r_i)/2, \\
  N(r_i,h_i) &= \int_{a_i}^{b_i} n(r) dr
\end{align*}
\]

where \(N(r,h)\) was taken to be zero when the droplet concentration fell below 1 liter\(^{-1}\). The discrete radii were taken at intervals of 1 \(\mu\) from 6.5 to 27.5 \(\mu\) (the range in which the initial distribution was nonzero); for \(r > 27.5\ \mu\), the subdivision was defined by \(r_i = r_i + \alpha(r_i - r_{i-1})\) with \(\alpha = 1.15\). With 55 discrete radii this gives a maximum radius of 792 \(\mu\).

Terminal velocities \(U(r)\) were calculated according to Stokes' law for \(r \leq 40\ \mu\) and by linear interpolation in the results of Gunn and Kinzer (1949) for \(r > 40\ \mu\). The coalescence probability for droplets of radii \(r_n\) and \(r_m\) was calculated according to

\[
P(n,m) = \pi Y_e r_n^2 [U(r_n) - U(r_m)] t_i
\]

where \(r_n > r_m\) and \(Y_e\) is the linear collision efficiency. Two collision efficiencies were used: 1) geometric collision efficiency, \(Y_e = 1+r_m/r_n\); and 2) Berry's (1967) continuous approximation to the collision efficiencies of Shafrir and Neiburger (1963).

For each collision efficiency, the case in which droplets do not grow by condensation and the case in which they do were considered, giving a total of four cases. Where growth by condensation was included, droplets were assumed to grow according to the simple law, \(dr/dt = f(r,h,t) = 0.1 \ r^{-1}\) (\(r\) in microns), for all heights and times. [This assumes a constant and uniform supersaturation of the order of 0.1\% and typical values for the other parameters affecting the rate of growth (Fletcher 1962, pp. 163, 164).]

Parallel computations were carried out for an infinitely high homogeneous cloud for comparison, such a cloud being equivalent to a single layer with no net fallout and no updraft; formally, this follows from the vanishing of the first term on the right-hand side of (4), since there is no height dependence in an infinitely high homogenous cloud. The time step \(\Delta t\) in the calculations for the infinite cloud was controlled by comparing the result for a single time step against the result for two half-time steps and demanding that this agree to within a relative error of 0.005 for all radii less than the radius of the 100th largest droplet m\(^{-3}\). If this criterion was not met, the time step was halved and the check repeated. The time step was doubled if the foregoing relative error < 0.001. This check on the size of the time step was applied every 25 sec. For the cases with geometric collision efficiency, the time step started at 0.625 sec and increased to 10 sec after ~15 min. For the cases with the collision efficiency of Shafrir and Neiburger, the time step started at 2.5 sec and increased to 10 sec after ~15 min. For the cloud of finite height, the steps indicated by the automatic time step control for the cloud of infinite height were used since the use of automatic step control in a cloud of finite height would greatly increase the computing time and storage requirements.

The cases for the infinite cloud were rerun using a much finer subdivision of radius to check the error resulting from coarser subdivisions. This check indicated that for all four cases, the coarse subdivision used caused the radius of the 100th largest droplet m\(^{-3}\) to be ~10% too large after 900 sec. However, since the error was in the same direction and of nearly the same magnitude for each case, conclusions about the rate of growth of droplets in one case relative to the rates in other cases should be affected to a much lesser extent.

The single level cases were also rerun with much finer tolerances in the automatic time step control (0.002 and 0.0004 instead of 0.005 and 0.001). This caused the time step \(\Delta t\) to be about half as large as with the coarser tolerances, but changed the results after 900 sec (e.g., size of 100th largest droplet m\(^{-3}\) by only about 2–3\%).

The single level cases in which the effect of growth by condensation was included were also rerun, reversing the order in which condensation and coalescence were taken into account. This caused a negligible (less than 1\%) change in the results.

The finite cloud model in which the collision efficiencies of Shafrir and Neiburger were used and growth by condensation included was rerun with \(\Delta h = 20\ \text{m}\) (instead of 10 m). This caused a difference of ~5\% in the size of the 100th largest droplet m\(^{-3}\) after 1000 sec at the initial height of the cloud base.

Figs. 1 and 2 show the radius of the 100th largest droplet m\(^{-3}\) as a function of time for several cases.

![Fig. 1. Growth of the 100th largest droplet in an infinite cloud.](image-url)
Curves for the finite cloud are for the initial height of the cloud base. In Fig. 1 the results of Warshaw (1967b) are given for comparison. Fig. 3 shows the radius of the 100th largest droplet m⁻³ as a function of height for several times. The results in Fig. 3 are for the case in which Berry's (1967) approximation to the collision efficiency of Shafir and Neiburger was used and droplet growth by condensation included. The cloud base was initially at 500 m and the cloud top at 1000 m.

Table 1 shows the liquid water content of the cloud at several times for each case. For the finite cloud the amount of liquid water in [grams in a square meter vertical column] the cloud is given. For the infinite homogenous cloud the amount of liquid water in [gm
conclude that the quadratic interpolation for the falling also preserves the total amount of liquid water. Fig. 2 indicates that the question of including or ignoring growth by condensation is at least as important as the question of a realistic collision efficiency. The realism of a model which takes the initial distribution to be homogenous over a height of 500 m is of course questionable since the effect of falling is to produce inhomogeneity with respect to height. In the future it is planned to remedy this defect by taking into account the nucleation of small droplets near the cloud base instead of starting with an initial distribution of droplets. We hope to report on results of such calculations soon.

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APPENDIX

The Stability of the Computation Scheme

The effects of condensation and coalescence as treated in the computation scheme are equivalent to a nucleation term $Q$ which may be positive or negative. Since the updraft $W$ is constant for the cases we have considered, we may write the equation describing the system in the form

$$
\frac{\partial N}{\partial t} = Q(r, h, t) - (U - W) \frac{\partial N}{\partial h}.
$$

(A1)

We are solving this equation according to the scheme

$$
N(r, h_{j+1}, t + \Delta t) = \frac{1}{2} (\lambda - 1) (\lambda - 2) N(r, h_{j+1})
- \lambda (\lambda - 2) N(r, h_{j+1}, t) + \frac{1}{2} \lambda (\lambda - 1) N(r, h_{j-1}, t) + Q(r, h_{j+1}, t) \Delta t.
$$

(A2)

where $l = j$ is the greatest integer in $(U - W) \Delta t / \Delta h$ and $\lambda = [h_{j+1} + (U - W) \Delta t - h_{j+1}] / \Delta h$ (note that $0 \leq \lambda < 1$). The first three terms on the right-hand side of (A2) result from the quadratic interpolation for the falling referred

<table>
<thead>
<tr>
<th>$t$ (sec)</th>
<th>Geometric $Y_e$ No condensation</th>
<th>Geometric $Y_e$ Condensation</th>
<th>Shafir's $Y_e$ No condensation</th>
<th>Shafir's $Y_e$ Condensation</th>
</tr>
</thead>
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<td>50.0</td>
<td>50.0</td>
<td>50.0</td>
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<td>49.1</td>
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</tr>
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<td>4.5</td>
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<td>38.9</td>
</tr>
<tr>
<td>800</td>
<td>3.3</td>
<td>0.7</td>
<td>43.2</td>
<td>17.1</td>
</tr>
<tr>
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<td></td>
<td></td>
<td>35.1</td>
<td>3.0</td>
</tr>
</tbody>
</table>

Table 2. Number of droplets cm$^{-3}$ in the spectrum of the infinite cloud.

Table 1. Liquid water content of cloud at various times.

<table>
<thead>
<tr>
<th>$t$ (sec)</th>
<th>Geometric $Y_e$ No condensation</th>
<th>Geometric $Y_e$ Condensation</th>
<th>Shafir's $Y_e$ No condensation</th>
<th>Shafir's $Y_e$ Condensation</th>
</tr>
</thead>
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<td>0</td>
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<td>500</td>
<td>500</td>
<td>500</td>
</tr>
<tr>
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<td>500</td>
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</tr>
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<td>500</td>
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</tr>
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</tr>
<tr>
<td>1000</td>
<td>500</td>
<td>485</td>
<td>500</td>
<td>639</td>
</tr>
</tbody>
</table>

a. Finite cloud (gm m$^{-3}$ vertical column)

<table>
<thead>
<tr>
<th>$t$ (sec)</th>
<th>Geometric $Y_e$ No condensation</th>
<th>Geometric $Y_e$ Condensation</th>
<th>Shafir's $Y_e$ No condensation</th>
<th>Shafir's $Y_e$ Condensation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
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<td>200</td>
<td>1.00</td>
<td>1.20</td>
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<td>1.00</td>
<td>1.65</td>
</tr>
<tr>
<td>800</td>
<td>1.00</td>
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<td>1.00</td>
<td>1.80</td>
</tr>
<tr>
<td>1000</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.85</td>
</tr>
</tbody>
</table>

b. Infinite cloud (gm m$^{-3}$)
to in Section 3. We expand \( N \) and \( Q \) in Fourier series

\[
N(r, h, t) = \sum_k n_k(r, t) e^{ikh},
\]

\[
Q(r, h, t) = \sum_k q_k(r, t) e^{ikh}, \tag{A3}
\]

and substitute the expansions into (A2). Looking at the \( k \)th component, we obtain

\[
n_k(r, t+\Delta t) = \left[ \frac{1}{2} (\lambda - 1) (\lambda - 2) - \frac{1}{2} \lambda (\lambda - 2) e^{ik\Delta h} + \frac{1}{2} \lambda (\lambda - 1) e^{2ik\Delta h} \right] \times e^{ik(hl-h)\Delta t} n_k(r, t) + q_k(r, t) \Delta t
\]

\[
= D(\lambda, k) n_k(r, t) + q_k(r, t) \Delta t. \tag{A4}
\]

The criterion for the stability of this scheme is that the absolute value of the amplification factor \( D(\lambda, k) \) should not exceed 1 for \( 0 \leq \lambda < 1 \) and for all \( k \) (Richtmyer and Morton, 1967, p. 72). A simple calculation shows that

\[
|D|^2 = 1 + \lambda (\lambda - 1)^2 (\lambda - 2) \left( \frac{3}{2} + \frac{1}{2} \cos 2k\Delta h - 2 \cos k\Delta h \right),
\]

\[
|D|^2 = 1 + a(\lambda) b(k\Delta h), \tag{A5}
\]

Since \( a(\lambda) \leq 0 \) for \( 0 \leq \lambda < 1 \) and \( b(k\Delta h) \geq 0 \) for all \( k \), we may conclude that \( |D|^2 \leq 1 \) and our scheme is stable.

The consistency of the foregoing scheme, i.e., its behavior as \( \Delta t \to 0 \), and the truncation errors introduced by the use of finite \( \Delta h \) and \( \Delta t \), can be established by carrying out the appropriate expansions in (A2). Of course, these are essentially the very expansions which lead from (3) to (4) in Section 2.

REFERENCES


