THE FORMATION OF THE SEMIPERMANENT CENTERS OF ACTION IN RELATION TO THE HORIZONTAL SOLENOIDAL FIELD

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ABSTRACT

The perturbation of the middle cell of the general circulation in a baroclinic atmosphere is investigated under the assumption of horizontal, divergence-free, isothermal motion. With the introduction of the horizontal solenoidal field the velocity of the trough does not differ very much from that obtained by Rossby, but the existence of a critical wave length, beyond which the perturbation will become unstable, explains to some extent the flow pattern of upper-air motions during weak-circulation weather and also provides a means of explaining how the large-scale mixing process of air masses between different latitudes takes place.

It is shown that the assumption of non-divergent flow does not greatly affect the investigation of the disturbance of the horizontal motion. The inclusion of the divergence term in the perturbation equation adds to the frequency equation a term which may affect the magnitude of the critical wave length and the velocity of the disturbance by roughly ten per cent.

The approximate solution of a two-layer system leads to a trough velocity which, in weather of high zonal circulation, is greater than the zonal wind speed in the lower layer and less than the zonal wind speed in the upper layer, a phenomenon often observed in synoptic analysis.

The phase relationship between isobars and isotherms derived here is the same as that obtained by Rossby. The coincidence of the isobars and isotherms in the stationary disturbance eliminates any horizontal solenoidal field and leads to a stationary wave length equivalent to that in an autobarotropic atmosphere, namely \( L_s = \frac{2\pi}{\sqrt{U/\beta}} \). Here \( U \) is the speed of the undisturbed westerly flow and \( \beta \) is the derivative of the Coriolis parameter with respect to horizontal distance northward.

In the consideration of energy transformation an interrelationship between meridional and zonal circulation is established which is in fair accordance with the synoptic and statistical calculations by various authors.

1. Introduction

It was pointed out by J. Bjerknes [2] that, in a large-scale atmospheric disturbance, convergence and divergence must arise even if the wind field is gradient, owing to the increase of Coriolis force with latitude and to the variation of centrifugal force with the curvature of the isobars. From the amount of air transport across a section perpendicular to two consecutive sinusoidal isobars, it was found that the centrifugal force arising from the curved isobars causes the trough to travel eastward but that the variation of Coriolis force with latitude leads to a westward displacement of the perturbation. For perturbations actually observed in the free atmosphere these two effects are of approximately the same magnitude. It follows that long waves must travel westward and short waves eastward. There must be an intermediate wave length for which these two effects balance each other and the corresponding perturbation remains stationary.

\[ L_s = 2\pi \sqrt{\frac{U}{\beta}} \quad \beta = \frac{\partial f}{\partial \gamma} \]  

This follows from the integration of the perturbation equations for a homogeneous, incompressible, non-frictional atmosphere in purely horizontal motion. The result of this investigation, representing a first approximation only, affords a fairly good explanation of the formation and displacement of semipermanent centers of action.

In a later report [8], Rossby has superimposed a sinusoidal meridional disturbance of the form

\[ v = v_s \cos \left[ (2\pi/L)(x - ct) \right] \]

on the constant zonal motion

\[ U = \text{constant} \]

and thus derived the fundamental kinematic and hydrostatic characteristics of such long-wave perturbations by a very simple and ingenious method. The two types of waves he described differ markedly with respect
to the interrelationship between the fields of temperature and motion and apparently can, under favorable conditions, be recognized from aerological data obtained at a single station. From the good agreement between his results and the observational facts as shown by the upper-air charts, it is evident that his results offer a valuable means for single-station forecasting where adequate meteorological networks are not available.

It is important to note, however, that the kinematic features derived by Rossby [8] check fairly well with observations but fail to show how the change of vorticity in an individual air column,

$$\frac{d\zeta}{dt} = (c - U) \frac{4\pi^2}{L^3} \cos \frac{2\pi}{L} (x - ct),$$

(4)
could be maintained. The lack of such an explanation lies in the fact that the vorticity equation

$$\frac{d\zeta}{dt} + \frac{df}{dt} + (\zeta + f) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

$$= \frac{R}{\rho} \left[ \frac{\partial T}{\partial y} \frac{\partial \rho}{\partial x} - \frac{\partial T}{\partial x} \frac{\partial \rho}{\partial y} \right],$$

(5)
derived from the general equation of horizontal motion, reduces to the conservation of absolute vorticity in an individual air column,

$$(\zeta + f) = \text{constant},$$

(6)
when the atmosphere is autobarotropic. Actually, a horizontal solenoidal field always exists as a consequence of the uneven distribution of land and sea. Thus a possible explanation is provided for the change of vorticity in such long-wave perturbations. It will be shown in the following investigation, by the integration of the perturbation equations for a baroclinic atmosphere, that a suitable means is provided for connecting on a sounder dynamic basis the result of Rossby's two papers with the formation, as well as the deformation, of the semipermanent centers of action.

2. Perturbation equations in a baroclinic atmosphere

Before the perturbation equations are derived, it is necessary to consider the undisturbed current system. In the study of the general circulation of the atmosphere, especially in the zone of middle latitudes, the mean motion of the atmosphere is characterized by a zonal circular vortex rotating in the same direction as the earth's rotation. In the steady state the dynamic balance of Coriolis force, pressure gradient, and gravity is expressed by the geostrophic equation,

$$\frac{fU}{RT} = -\frac{\partial}{\partial y} \ln \rho,$$

(7)
and the hydrostatic equation,

$$\frac{g}{RT} = -\frac{\partial}{\partial z} \ln \rho.$$  

(8)
Here the coordinate system is a right-handed one with $x$ pointing eastward, $y$ northward, and $z$ upward; $U$ represents the speed of the west wind, $f$ the Coriolis parameter, $T$ the temperature, $\rho$ the pressure, $R$ the gas constant, and $g$ gravity. To have the pressure $p$ represent the same analytic function in both (7) and (8) requires that

$$-\frac{\partial}{\partial x} \left( \frac{fU}{RT} \right) = -\frac{\partial}{\partial y} \left( \frac{g}{RT} \right).$$

(9)
Hence

$$f \frac{\partial U}{\partial z} + \frac{fU}{T} \gamma_s = \frac{g}{T} \gamma_v,$$

(10)
where

$$\gamma_s = -\frac{\partial T}{\partial z}$$

and

$$\gamma_v = -\frac{\partial T}{\partial y}$$

(11)
are the vertical and horizontal temperature gradients respectively.

In the derivation of the perturbation equations, we start from the general equations

$$\frac{dW^*}{dt} = G - 2\Omega \times W^* - \frac{1}{\rho^*} \nabla \rho^*$$

(12)
and

$$\frac{\partial \rho^*}{\partial t} + \text{div} \rho^* W^* = 0,$$

(13)
or, in rectangular coordinates,

$$\frac{du^*}{dt} - f v^* = -\frac{1}{\rho^*} \frac{\partial \rho^*}{\partial x},$$

(14)
$$\frac{dv^*}{dt} + f u^* = -\frac{1}{\rho^*} \frac{\partial \rho^*}{\partial y}$$

and

$$\frac{\partial \rho^*}{\partial t} + \frac{\partial \rho^* u^*}{\partial x} + \frac{\partial \rho^* v^*}{\partial y} = 0,$$

(15)
in which the asterisk is used to denote that the variables are in the disturbed state. Instead of resolving the vector equation along rectangular coordinates, we can resolve $dW^*/dt$ into components along the direction of motion and perpendicular to it. If we denote the unit vector along the trajectory by $s$ and that along the normal by $n$, we have

$$\frac{dW^*}{dt} = \frac{dV^*}{dt} s + \frac{V^{*z}}{r} n$$

(16)
where $V^* = V^{ua} + V^{za}$ and $r$ is the radius of curvature. If this form of $dW^*/dt$ is substituted in (12) and the scalar product with $W^*$ is formed, it turns out
that for purely horizontal motion

\[ \mathbf{W}^* \cdot \nabla \rho^* = - \rho^* V^* \frac{dV^*}{dt}. \] (17)

It will be assumed that the undisturbed current is geostrophic, so that, in the equation above, \( \frac{dV^*}{dt} \) must vanish. The first two equations (14) will then also hold if we subtract from their left-hand members the vanishing quantities \( (dV^* / dt) s \cdot i \) and \( (dV^* / dt) s \cdot j \) respectively, where \( i \) and \( j \) stand for the unit vectors in the \( x \)- and \( y \)-direction. After eliminating \( \rho^* \) by means of the equation of state we have the following equations:

\[
\begin{align*}
\frac{\partial u^*}{\partial t} & = - \frac{\partial V^*}{\partial t} \frac{u^*}{V^*} + \frac{u^*}{\partial x} \frac{\partial u^*}{\partial x} + \frac{v^*}{\partial y} \frac{\partial u^*}{\partial y} - f u^*
\frac{\partial v^*}{\partial t} & = - R T^* \frac{\partial}{\partial x} \ln \rho^*
\end{align*}
\] (18)

Adding these terms follows the derivation of Philipps [6]. The advantage will become apparent when the perturbation equations are written down.

The introduction of a small perturbation can be carried out in the following manner. To the variables of the undisturbed state, \( p, T, W \), we add a small disturbance of pressure \( p' \), of temperature \( \tau \), and of the velocity \( \mathbf{v}' \), so that the disturbed state at time \( t \) is characterized by

\[
\begin{align*}
p^*(x, y, z, t) & = p(y, z) + p'(x, y, z, t) \\
T^*(x, y, z, t) & = T(y, z) + \tau(x, y, z, t) \\
u^*(x, y, z, t) & = U(x) + u'(x, y, z, t) \\
v^*(x, y, z, t) & = v'(x, y, z, t).
\end{align*}
\] (19)

In the following calculation, it is assumed that the perturbations \( p', \tau, u', v' \) are so small that in the perturbation equations only linear perturbation terms need be retained and all products between \( p', \tau, u', v' \), or their derivatives, may be neglected. Through this process of linearization and by means of the assumptions expressed in (19) that

\[
\begin{align*}
\frac{\partial T}{\partial x} & = \frac{\partial p}{\partial x} = \frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = 0,
\end{align*}
\] (20)

we find the following:

\[
\begin{align*}
T^* \frac{\partial}{\partial x} \ln \rho^* & = \frac{\partial}{\partial x} \ln (p + p') \\
& = (T + \tau) \frac{\partial}{\partial x} \ln p + T \frac{\partial}{\partial x} \ln \left( \frac{p'}{p} \right)
\end{align*}
\]

\[
\begin{align*}
T^* \frac{\partial}{\partial y} \ln \rho^* & = T \frac{\partial}{\partial y} \ln p + \tau \frac{\partial}{\partial y} \ln \left( \frac{p'}{p} \right)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial u^*}{\partial t} & - \frac{\partial V^*}{\partial t} \frac{u^*}{V^*} + \frac{u^*}{\partial x} \frac{\partial u^*}{\partial x} + \frac{v^*}{\partial y} \frac{\partial u^*}{\partial y} \\
& = \frac{\partial V^*}{\partial t} \frac{v^*}{V^*} + \frac{v^*}{\partial x} \frac{\partial v^*}{\partial x} + \frac{v^*}{\partial y} \frac{\partial v^*}{\partial y}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial u^*}{\partial x} + \frac{\partial u^*}{\partial y} & = \frac{\partial u'}{\partial x} + \frac{\partial u'}{\partial y} = \frac{\partial v'}{\partial x} + \frac{\partial v'}{\partial y}
\end{align*}
\]

Substituting (21) in (18) and remembering that for the undisturbed variables

\[ f U = - R T \partial \ln \rho / \partial y, \]

we have the perturbation equations of motion,

\[ \nu' = \frac{RT}{f} \frac{\partial}{\partial x} \left( \frac{p'}{p} \right) \] (22)

and

\[ \frac{\partial u'}{\partial t} + U \frac{\partial u'}{\partial x} + fu' = - R \tau \frac{\partial}{\partial y} \ln \rho - RT \frac{\partial}{\partial y} \ln \left( \frac{p'}{p} \right), \] (23)

and the equation of continuity,

\[ \frac{\partial}{\partial t} \left( \frac{p'}{p} \right) + U \frac{\partial}{\partial x} \left( \frac{p'}{p} \right) = - \frac{\partial}{\partial x} \left( \frac{\tau}{T} \right) - U \frac{\partial}{\partial x} \left( \frac{\tau}{T} \right) + \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) + v' \frac{\gamma_v}{T} \frac{fU}{RT} = 0. \] (24)

Since the motions are assumed to be purely horizontal and the perturbations in which we are interested are mostly long waves, it is permissible to assume that the thermodynamic process followed by an individual air quantum is isothermal. The condition is

\[ \frac{dT^*/dt}{d} = 0 \]

or

\[ \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} - v' \gamma_v = 0. \] (25)

We now have four equations which are sufficient
to solve for the four unknowns $u'$, $v'$, $p'$, and $r$. Since we are interested chiefly in wave perturbations, all variables are treated as functions of the form

$$Ae^{i(x-ct)}, \quad \sigma = 2\pi/L,$$

where $L$ is the wave length and $c$ the speed of propagation. We thus have another variable $c$ in addition to the four unknowns. It seems that an additional equation is necessary in order to solve for the five unknowns $u'$, $v'$, $p'$, $r$, and $c$. But, if we confine ourselves to the investigation of the frequency equation and do not determine the amplitude terms exactly, the four equations (22), (23), (24), and (25) suffice to give us information on the propagation of the wave perturbation. The amplitude factor of $u'$, $v'$, $r$, and $c$ is left in terms of the amplitude factor of $p'/p$, as will be shown below.

3. Integration of perturbation equation

Under the assumption that the perturbation field of motion is divergence-free, the equation of continuity is simplified to

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0,$$  \hspace{1cm} (26)

which can be satisfied by

$$u' = -\frac{\partial \psi}{\partial y}, \quad v' = \frac{\partial \psi}{\partial x},$$  \hspace{1cm} (27)

where $\psi$ is the stream function. Set

$$p'/p = Ae^{i(x-ct)+in\eta},$$  \hspace{1cm} (28)

$$\psi = Be^{i(x-ct)+in\eta},$$  \hspace{1cm} (29)

where $\eta = 2\pi/\delta$. The quantity $\delta$ is the width of the perturbation, the effect of which was studied by Haurwitz [4], and $A$ and $B$ are two amplitude factors. From (22)

$$\psi = \frac{RT}{f} p',$$  \hspace{1cm} (30)

and from (25)

$$\tau = \frac{\nu}{U - c} \psi.$$  \hspace{1cm} (31)

From (5) the equation of vorticity in the perturbed form can be written

$$\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \zeta' + \beta v' = \frac{R}{\rho} \left( \frac{\partial \tau}{\partial x} - \frac{\partial \psi}{\partial y} \right).$$  \hspace{1cm} (32)

where

$$\zeta' = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi.$$  \hspace{1cm} (33)

By means of (30) and (31) this reduces to

$$\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \nabla^2 \psi + \left( \beta + \frac{\gamma u}{T} \right) \frac{\partial \psi}{\partial x} = \frac{fU \nu}{T(U - c)} \frac{\partial \psi}{\partial x}.$$  \hspace{1cm} (34)

Substitution of (29) in (34) gives

$$(U - c)(\sigma^2 + \eta^2) \left( \beta + \frac{\gamma u}{T} \right) + \frac{fU \nu}{T(U - c)} = 0,$$  \hspace{1cm} (35)

whence

$$c = U,$$

$$\beta T + \frac{\gamma u}{\sigma^2 T} \left( 1 \pm \sqrt{1 - \frac{4U(\sigma^2 + \eta^2)fTrT}{(T\beta + fTr)^2}} \right).$$  \hspace{1cm} (36)

Equation (36) shows that, when the wave length of the perturbation is smaller than the critical wave length

$$L_c = 2\pi \sqrt{\frac{4UfTrT}{(T\beta + fTr)^2}} \sqrt{1 + \frac{L_c^2}{\delta^2}},$$  \hspace{1cm} (37)

the perturbation becomes unstable. The critical wave length decreases with increasing width until finally it reaches its minimum value $L_{cm}$ when $\delta$ becomes infinite, in which case the perturbation becomes independent of $\gamma$.

Professor B. Haurwitz, who independently reached the same result for the minimum critical wave length, has kindly made available to me the results of his numerical calculations. These results are represented in Figure 1. As an example of the use of this diagram, suppose that the temperature gradient $\gamma_0$ is 4 C/(1000 km), that the west wind is 4 m/sec, and that the minimum critical wave length is to be found at altitude 45°. In this case, one follows, in the lower part of the diagram, along the horizontal marked 45° to the point A where it intersects the line 4 m/sec. From there one follows the vertical through A to the point in the upper part of the diagram where the ordinate corresponds to the assumed meridional temperature gradient 4 C/(1000 km). It is seen that the value of the minimum critical wave length in this case is 1800 km. It is seen from the diagram or from equation (37) that under the same conditions, but at a higher latitude, the minimum critical wave length would be larger. Furthermore, at one and the same latitude the critical wave length increases with the meridional temperature gradient and with the wind velocity.
The existence of the critical wave length of the atmospheric disturbance in a baroclinic atmosphere can be explained by the following consideration. If we neglect for a moment the effect of the horizontal solenoidal field and consider only the effect of the variation of the Coriolis force with latitude, then from (5)

\[ \frac{d\psi'}{dt} = -8\nu', \]  

or the absolute vorticity of an air particle remains constant. If a current from the west be forced to move equatorward at some point A in its path (see Fig. 2), owing to the decrease of f with latitude the individual air particle continuously acquires cyclonic vorticity with respect to the earth, so that the cyclonic curvature of the current increases steadily, and eventually, after sufficient displacement to a point B, the current begins to bend back towards its equilibrium position.

As it moves away from the equator the effect of the variation of f will increase the anticyclonic vorticity and tend to bend the current equatorward. The net result will be a sinusoidal, or wavelike, oscillation of the west-wind belt around a certain mean latitude. Therefore the effect of the Coriolis force is that of a

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stabilizing agent which restores any meridional disturbance from the west-east zonal motion of middle latitudes.

If we now add the effect of the solenoidal term, it follows from (32), on rearrangement, that

$$\frac{d\gamma'}{dt} = -\beta v' + \left( \frac{c}{U - c} \right) \frac{f_\gamma u}{T} \gamma', \quad (39)$$

in which the first term of the right-hand side represents the stabilizing effect of the variation of the Coriolis force with latitude, while the second term represents the effect of the horizontal solenoidal field. Since $f_\gamma u/T$ is positive, the sign of the solenoidal effect on the individual change of vorticity depends only on the direction and magnitude of the wave velocity. For a retrograde wave or wave disturbance with velocity greater than the zonal wind velocity $U$ the sign of the second term becomes negative, so that this term behaves in the same way as the Coriolis term and consequently is a stabilizing agent. On the other hand, if the wave is progressive and the velocity of propagation is smaller than $U$, the second term is a positive quantity, which acts in a way opposing the effect of the first term and therefore is a destabilizing agent.

If we make use of the kinematic relation between the isobars and isotherms that will be derived later, we have

$$\frac{A_T}{A_p} = \frac{U}{U - c} \quad (40)$$

where $A_T$ and $A_p$ are the amplitudes of the isotherm and isobar respectively. Rearrangement of equation (39) in the form

$$\frac{d\gamma'}{dt} = -\left( \beta + \frac{f_\gamma u}{T} \right) v' + \frac{f_\gamma u}{T} \frac{U}{U - c} \gamma'$$

shows that the solenoidal field will behave as a destabilizing agent in the disturbance of zonal circulation when the amplitude of the isotherm is greater than that of the isobar. This can be shown schematically in Figure 3.

Let us now consider the effect of the solenoidal field alone. From (32), the vorticity equation can be written in the form

$$\frac{d\gamma'}{dt} = -\frac{R}{\rho} (\nabla p^* \times \nabla T^*)$$

$$= -\frac{R}{\rho} \left( -\frac{dp^*}{dn} \right) \frac{dT^*}{dn} \sin \theta \quad (41)$$

where $\theta$ is the angle between the descendent of the pressure $-dp^*/dn$ and the ascendent of the temperature $dT^*/dn$. At point A (see Fig. 3), since the air current moves towards the south, and since $\theta$ is smaller than $\pi$, the right-hand side of (41) is a negative quantity, so that the effect of the solenoidal field is to decrease the cyclonic or increase the anticyclonic vorticity of the air current. Conversely, if the air moves northward as at point C, the cyclonic vorticity increases. Therefore the effect of the solenoidal field in the slowly progressive wave acts in a direction opposite to that of the Coriolis effect. But the intensity of the solenoidal field is measured by the number of solenoids per unit area. It is therefore evident that this effect is greater for shorter wave lengths than for longer wave lengths. This shows that for disturbances with wave length shorter than the critical value the stabilizing effect of the Coriolis force is over-balanced by the destabilizing effect of the solenoidal field, with the result that the disturbances become unstable.

It is not difficult to show schematically the stabilizing effect of the solenoidal field on the retrograde wave, where the amplitude of the isobar is greater than that of the isotherm, or on the fast progressive wave, where isobars and isotherms are out of phase. But this is not important here, because in the first case, where the wave length is usually very long, the perturbation should by all means be stable, and in the second case, where the zonal circulation is strong, unstable perturbations are never observed in the upper air. Besides, in the initial stage of any disturbance in the atmosphere, the amplitude of isotherms must be greater than the amplitude of isobars, since the pressure at any place is controlled by the whole air column above. Pressure cannot be affected as much as temperature by the disturbance of a particular layer. This shows that atmospheric disturbances in their initial stage are characterized by slowly progressive speed, which indicates that the effect of the solenoidal field is destabilizing.
In (36), if we assume \( \delta \to \infty \) and \( \gamma_y \to 0 \), the positive sign of the square root gives Rossby's trough formula
\[
c = U - \beta / \sigma^2, \tag{42}
\]
and the negative sign gives
\[
c = U, \tag{43}
\]
which is obviously trivial.

The frequency equation (35) shows that the disturbances become stationary when
\[
L_0 = 2\pi \sqrt{\frac{U}{\beta} \sqrt{1 + \frac{L_0^2}{\delta^2}}}. \tag{44}
\]

For infinite width, this expression reduces to Rossby's formula
\[
L_0 = 2\pi \sqrt{U / \beta}. \tag{45}
\]

Up to this point the calculation has been based upon the assumption that the perturbation field is divergence-free, which is of course entirely arbitrary. In order to remove this restriction, we have to use the equation of continuity (24) instead of (26), so that
\[
\frac{\partial p'}{\partial x} + \frac{\partial v'}{\partial y} = \frac{1}{T} \frac{d\tau}{dt} - \frac{d}{dt} \left( \frac{p'}{p} \right) - v' \left( \frac{\gamma_y}{T} - \frac{fU}{RT} \right). \tag{46}
\]
For infinite width, \( \delta \to \infty \), we set
\[
p'/p = A e^{i(x-ct)} \tag{47}
\]
where \( A \) is the amplitude factor. From (22), (23), and (25), it follows that
\[
v' = i \sigma \frac{RT}{f} A e^{i(x-ct)} \tag{48}
\]
\[
\frac{\partial v'}{\partial y} = -i \sigma \left( \frac{RT}{f^2} \beta + \frac{R \gamma_y}{f} \right) A e^{i(x-ct)}, \tag{49}
\]
\[
\sigma = \frac{RT}{f} \frac{\gamma_y}{U-c} A e^{i(x-ct)} \tag{50}
\]
\[
u' = \left[ \sigma^2 \frac{RT}{f} (U-c) + \frac{R \gamma_y}{f} \frac{U}{U-c} \right] A e^{i(x-ct)}. \tag{51}
\]

The frequency equation, obtained by substituting (47), (48), (49), and (50) in (46), is
\[
\frac{RT}{f^2} \sigma^2 (U-c) - \frac{RT}{f^2} \beta - c + \frac{R \gamma_y}{f} \frac{U}{U-c} = 0 \tag{51}
\]
or
\[
\left( \frac{RT}{f^2} \sigma^2 + 1 \right) (U-c) - \left( \frac{RT}{f^2} \beta + \frac{R \gamma_y}{f} + U \right) + \frac{R \gamma_y}{f} \frac{U}{U-c} = 0. \tag{52}
\]

Neglecting unity compared with \( (RT/f^2) \sigma^2 \) in the first term and solving for \( c \), we have
\[
c = U - \frac{T \beta + f \gamma_y + f^2 U}{2 \sigma^2 T} \left[ 1 \pm \sqrt{1 - \frac{L_0^2}{L^2}} \right] \tag{53}
\]
where
\[
L_0 = 2\pi \sqrt{\frac{4 f \gamma_y T U}{T \beta + f \gamma_y + f^2 U / R}}. \tag{54}
\]

The order of magnitude of \( T \beta + f \gamma_y \) is \( 10^{-11} \) C cm\(^{-1}\) sec\(^{-1}\), while that of \( f^2 U / R \) is \( 10^{-17} \) C cm\(^{-1}\)sec\(^{-1}\); hence from (53) and (54) the inclusion of the divergence term will reduce the magnitude of the velocity of the trough and the critical wave length by roughly 10 per cent of the value calculated by (36) and (37). For simplicity of calculation in the following investigation we shall neglect the term -c arising from the divergence term.

If we consider the variation of the height of the tropopause, the equation of continuity in the disturbed state should take the form
\[
\frac{\partial \rho^*}{\partial t} + \frac{\partial \rho^* u^*}{\partial x} + \frac{\partial \rho^* v^*}{\partial y} = - \frac{\partial}{\partial t} \left( \rho^* \frac{\partial z^*}{\partial t} \right), \tag{55}
\]
or, after integrating from \( z = 0 \) to the height of the tropopause \( z = D^* \),
\[
\int_0^{D^*} \left( \frac{\partial \rho^*}{\partial t} + \frac{\partial \rho^* u^*}{\partial x} + \frac{\partial \rho^* v^*}{\partial y} \right) dz = - \frac{\partial}{\partial t} \left( \rho^* \frac{\partial z^*}{\partial t} \right)_0 = - \rho^* \frac{\partial h}{\partial t} \tag{56}
\]
where \( h \) is the perturbation of the tropopause height,
\[
D^* = D + h. \tag{57}
\]

Introducing the perturbation in (56) and making use of (48) and (50) we get
\[
\int \frac{\rho}{RT} \left( \frac{RT}{f^2} \sigma^2 (U-c) - \frac{RT \beta}{f^2} \right) - c \frac{R \gamma_y}{f} \frac{U}{U-c} \right) A \frac{dz}{dt} = - \rho_D \frac{\partial h}{\partial t}. \tag{58}
\]

We neglect the third term in the integrand as mentioned before. From the kinematic boundary condition at the tropopause,
\[
\frac{\partial D^*}{\partial t} + \mathbf{w} \cdot \nabla D^* = 0 \tag{58}
\]

whence
\[
h = \frac{\psi D}{U_D - \xi D} = \frac{RT_D}{f} \frac{\xi A_D}{U_D - \xi D} \tag{59}
\]
where \( \xi = - \partial D / \partial y \) is the slope of the tropopause.
The further evaluation of the frequency equation requires a knowledge of the variation of \( A \) with height. If we adopt the assumption of Rossby that the shape of the isotherms in the perturbed field is the same throughout the entire atmosphere, then, from (50) and the condition that

\[
\frac{A_D T_D}{U_D - c_D} = \frac{AT}{U - c},
\]

(60)
equation (57) is reduced to

\[
\int_0^D \frac{p}{RT} \left[ \frac{RT}{f^2} \beta (U - c)^2 - \frac{RT}{f^2} (U - c) - \frac{c \gamma_R R}{T_f} \right] dz = c \rho_D \frac{R}{f},
\]

(61)
Up to this point the west wind \( U \) has not been specified as a function of height. If we take the mean west wind as \( U \) and regard it as a constant with respect to height, then, according to (36), \( c \) is in fact dependent upon \( z \) but varies very slowly, so that at least for a first approximation we can consider it as constant. Equation (61) simplifies to

\[
\sigma^2 (U - c)^2 - \beta (U - c) - c \gamma_f / \tilde{T} = c \xi f \alpha
\]
where \( \tilde{T} \) is the mean temperature of the air column and

\[
\alpha = \frac{\rho_D}{\tilde{\rho}_D} = \frac{\rho_D}{\int_0^D \rho D}.
\]

(63)
Solving for \( c \), we get

\[
c = U - \frac{\tilde{T} \beta + f(\gamma_f - \xi \tilde{T} \alpha)}{2 \sigma^2 \tilde{T} \left[ 1 \pm \sqrt{1 - \frac{L_c^2}{L^2}} \right]},
\]

(64)
in which the critical wave length is given by

\[
L_c = 2 \pi \sqrt{\frac{4 U f \tilde{T} (\gamma_f - \xi \tilde{T} \alpha)}{(\tilde{T} \beta + f(\gamma_f - \xi \tilde{T} \alpha))^2}}.
\]

(65)
In the mean condition \( \xi = 1.7 \times 10^{-4} \) and \( \xi \tilde{T} \alpha \) is of the order of \( 10^{-9} \) C cm\(^{-1} \), while \( \gamma_f \) is of the order of \( 10^{-8} \) C cm\(^{-1} \). Since \( \gamma_f > \tilde{T} \alpha \), \( L_c \) is real and \( L_c^2 \) is positive. Hence the wave may become unstable when \( L < L_c \), but the critical wave length is shorter, and the speed of propagation greater, than in the case where the tropopause is horizontal.

According to the frequency equation (36) or (51) the speed of propagation can never be greater than the zonal circulation \( U \). However, it is known from synoptic experience that, in strong circulation weather, the cyclones in general travel at a speed greater than the west wind prevailing in the lower levels. In order to explain this observation, one has to integrate the perturbation equation in a two-layer system. It is assumed that the lower layer is separated from the upper by a discontinuity surface corresponding to a tropospheric front. If the indexes 1 and 2 are assigned to the lower and upper layer respectively and if the previous simplification is followed, the equation of continuity found for the lower layer is

\[
A_1 \left[ \frac{RT_1}{f^2} \sigma^2 (U_1 - c) - \frac{RT_1}{f^2} \beta + c \gamma_R R \frac{1}{U_1 - c} \right] = - \frac{\alpha_1}{D_1} \frac{\partial h_1}{\partial t},
\]

(66)
and that for the upper layer is

\[
A_2 \left[ \frac{RT_2}{f^2} \sigma^2 (U_2 - c) - \frac{RT_2}{f^2} \beta + c \gamma_R R \frac{1}{U_2 - c} \right] = - \frac{1}{D_2 - D_1} \left[ \frac{\partial h_2}{\partial t} - \frac{\alpha_2}{D_2} \frac{\partial h_1}{\partial t} \right],
\]

(67)
in which \( D_2 \) is the height of the tropopause and \( D_1 \) is the thickness of the tropospheric front, and

\[
\alpha_1 = \frac{\rho_D}{\tilde{\rho}_1}, \quad \tilde{\rho}_1 = \frac{1}{D_1} \int_0^{D_1} \rho_D \chi, \quad \rho_D = \frac{\alpha_1}{D_2 - D_1} \int_0^{D_1} \rho_D \chi, \quad \gamma_f = \gamma_f.
\]

(68)
Eliminating \( \partial h_1 / \partial t \) and making use of (59) we get

\[
\frac{D_1}{D_2 - D_1} \left[ \frac{RT_2}{f^2} \sigma^2 (U_1 - c) - \frac{RT_2}{f^2} \beta + c \gamma_R R \frac{1}{U_1 - c} \right] A_1
\]

\[
+ \left[ \frac{RT_2}{f^2} (U_2 - c) - \frac{RT_2}{f^2} \beta + c \gamma_R R \frac{1}{U_2 - c} \right] A_2 = 0.
\]

(70)
If we still retain the assumption that the shape of the isotherms in these two layers is the same, we have

\[
\frac{A_1}{A_2} = \frac{T_2}{T_1} U_1 - c.
\]

(71)
Substituting this in (70) and putting

\[
\kappa = \frac{D_1}{D_2 - D_1},
\]

\[
\frac{1}{T_s} = T_1 + \frac{1}{T_2},
\]

\[
U_s = \kappa U_1 - U_2
\]

(72)
and neglecting small terms, we find that

\[
c \leq U_s - \frac{T_2 \beta + f \gamma_f}{2 \sigma^2 T_s} \left[ 1 \pm \sqrt{1 - \frac{L_c^2}{L^2}} \right]
\]

(73)
where

\[
L_c = 2 \pi \sqrt{\frac{4 U f T_2 f \gamma_f}{(T_s \beta + f \gamma_f)^2}}.
\]

(74)
or
\[
c = U_1 + \frac{U_2 - U_1}{\kappa + 1} + \frac{T_0 \beta + f \gamma_0}{2 \sigma^2 T_0} \left[ 1 \pm \sqrt{1 - \frac{L_0^2}{L^2}} \right]
\]
\[
= U_2 - \frac{\epsilon(U_2 - U_1)}{\kappa + 1}
\]
\[- \frac{T_0 \beta + f \gamma_0}{2 \sigma^2 T_0} \left[ 1 \pm \sqrt{1 - \frac{L_0^2}{L^2}} \right].
\]
(75)

In the case of strong circulation under consideration the west wind increases rapidly with height, so that very probably
\[
\frac{U_2 - U_1}{\kappa + 1} > \frac{T_0 \beta + f \gamma_0}{2 \sigma^2 T_0}.
\]
(76)

Thus we may show from (75) and (76) that
\[
U_1 < c < U_2,
\]
in good agreement with the synoptic facts during strong-circulation weather. It must be noticed that the results obtained here are based upon the assumption that the shape of the isotherms remains unchanged, which provides an additional equation (71) for the elimination of the constants \(A_1\) and \(A_2\) in equation (70). It is therefore of interest to see whether or not this assumption checks with the boundary condition at the tropospheric front. This should give us another test of the validity of the preceding treatment. The kinematic boundary condition at the tropospheric front \(D_1\) is given by
\[
(U_1 - U_2) \frac{\partial h_1}{\partial y} + (v_1 - v_2') \frac{\partial D_1}{\partial y} = 0.
\]
(78)

Setting
\[
h_1 = He^{i(x-ct)}
\]
(79)

where \(H\) is the amplitude of the frontal disturbance, we find from (47) and (71) that
\[
\frac{\partial D_1}{\partial y} = \frac{H e^{i(x-ct)}}{v_1}.
\]
(80)

Since the speed of propagation \(c\) is always smaller than the west wind at higher levels, the slope of the tropospheric front is positive, i.e., the front ascends northward, which is the general characteristic of a polar front. If we further assume that \((U_2 - c) = 10\) m/sec, \(v_1 = 5\) m/sec, \(H = 1\) km, then
\[
\frac{\partial D_1}{\partial y} = 6 \times 10^{-3} \text{ for } L = 2000 \text{ km}
\]
and
\[
\frac{\partial D_1}{\partial y} = 3 \times 10^{-3} \text{ for } L = 4000 \text{ km},
\]
\footnote{For example, if we take \(U_1 - U_2 = 10\) m/sec, \(D_1 = 3\) km, \(D_0 = 12\) km, \(T_0 = 270\) C, \(\gamma_0 = 3\) C/(1000 km), \(\varphi = 45^\circ\), \(L = 3000\) km, then \(U_2 - U_1 = 7.5 \times 10^5\) cm/sec, \(\frac{T_0 \beta + f \gamma_0}{2 \sigma^2 T_0} = 2 \times 10^6\) cm/sec.}

which gives approximately the order of magnitude of the slope of quasi-stationary fronts.

4. Synoptic and kinematic characteristics of the disturbance

From the frequency equations derived under the various assumptions of the preceding paragraphs it is evident that the stationary wave length is equal to Rossby's value
\[
L_s = 2\pi \sqrt{U/\beta},
\]
which was derived for an autobarotropic atmosphere. This coincidence reveals that the horizontal solenoidal field must vanish in a stationary disturbance. The solenoidal field expressed on the right-hand side of the vorticity equation can be rearranged and reduced to
\[
\gamma \frac{\partial}{\partial y} \frac{c}{U - c} e^{i(x-ct)},
\]
(81)

which in fact vanishes when \(c = 0\). Kinematically this means that the isobars and isotherms should coincide in the stationary state. In order to investigate this fact, one has to find the equation for the isobars, isotherms, and streamlines in the disturbed field in which the pressure, temperature, and stream function are given by
\[
p^* = p_0 + \frac{\partial}{\partial y} y + \rho' = p_0 + \frac{\partial}{\partial y} y + \rho A e^{i(x-ct)},
\]
(82)

\[
T^* = T_0 + \frac{\partial T}{\partial y} y + \tau
\]

\[
= T_0 - \frac{\gamma_y y}{U - c} A e^{i(x-ct)},
\]
(83)

\[
\psi^* = -U y + \frac{RT}{f} \frac{\gamma_y}{U - c} A e^{i(x-ct)},
\]
(84)

where \(T_0\) and \(p_0\) are the temperature and the pressure of the undisturbed state at latitude \(y = 0\). The particular isobar, isotherm, and streamline on which \(p^* = p_0 = \text{constant}, T^* = T_0 = \text{constant}, and \psi^* = 0\), respectively, are then given by
\[
\gamma_p = \frac{RT \frac{A}{f}}{U} e^{i(x-ct)},
\]
(85)

\[
\gamma_T = \frac{RT \frac{A}{f}}{U - c} e^{i(x-ct)},
\]
(86)

\[
\gamma_\psi = \frac{RT \frac{A}{f}}{U} e^{i(x-ct)}.
\]
(87)

From these equations it is found that the ratio between the amplitudes of isobars and isotherms is given by
\[
\frac{A_p}{A_T} = \frac{U - c}{U},
\]
(88)
which shows that the isobars and isotherms should coincide when \( c = 0 \). Since this result is just the same as Rossby’s and since kinematic characteristics as well as synoptic evidence are extensively discussed in his report, further considerations are not presented here.

The important result of the present investigation is the introduction of a critical wave length \( L_0 \), which explains the deepening of the upper-air trough during low-index weather. As was pointed out by Rossby [9], a stable perturbation splits into two centers when the zonal index changes. At an earlier stage of this situation, provided that the zonal index changes from high to low, the disturbance moves westward, while the heterogeneity of the earth, in particular the asymmetric distribution of land and water, tends to plant the newly forced perturbation in the location of the semipermanent centers. It has been frequently observed from upper-air charts [5] that a new trough is formed in the Bay of Alaska, and that the corresponding wave length shortens abruptly and should be much smaller than the prevailing critical wave length. According to the present theory the disturbance becomes unstable and the amplitude factors of the pressure or temperature wave increase exponentially with time. The deepening of the upper trough and the increase in amplitude of isobars and isotherms during low-index weather supports our theoretical conclusion.

Theoretically the increase of amplitude of the unstable disturbance goes on indefinitely, but it must be remembered that the critical wave length is a function of zonal index \( U \). As will be shown later, during the unstable disturbance large-scale mixing of air between different latitudes takes place, leading to a continuous decrease of zonal index \( U \) and, consequently, of the critical wave length. Thus the critical value of the wave length which a stable disturbance should exceed becomes smaller and smaller during the deepening of the upper trough until a final stage is reached when the wave length equals the critical value corresponding to the prevailing zonal circulation. Then the deepening ceases and the disturbance becomes stable again.

In the discussion of the two-layer system we have made use of the assumption of the constancy of isotherm amplitude, which is given by

\[
\frac{A_1 T_1}{U_1 - c} = \frac{A_2 T_2}{U_2 - c} \quad (89)
\]

or, according to equation (85), by

\[
\frac{A_1 U_1}{U_1 - c} = \frac{A_2 U_2}{U_2 - c} \quad (90)
\]

In case of high-index weather, when \( c > U_1 \) and \( c < U_2 \), the isobars of the upper layer are 180° out of phase with those of the lower layer. Since, from (88), the isotherms of the lower layer are out of phase with the isobars, there must be a warm trough in the lower layer and a warm ridge above. This conforms with the configuration of flow patterns of general high-index weather as indicated by synoptic experience.

5. Energy transformation in middle latitudes

We assume that the isotherm travels without change in shape, that its deformation is the same at all heights, and hence that the waves consist essentially of a uniform riding north and south of a very deep layer. It is evident that there is no possibility of mixing of air in a stable disturbance of such simple wave motion. The trivial case \( c = U \) discussed by Rossby leads in fact to an amplitude factor increasing linearly with time and thus explains the possibility of setting forth the large-scale mixing process between different latitudes. It is, however, inconsistent dynamically with the frequency equation, according to which \( c \) can only be equal to \( U \) if the wave length approaches zero. Even if we disregard the dynamic equation and study the problem merely kinematically by assuming that the wave speed \( c \) is a constant and by allowing a variation of west wind \( U \) with height, the case \( c = U \) will nevertheless be confined to a certain upper level, while the large-scale mixing process in the atmosphere usually involves very thick layers of air.

The present investigation has shown the destabilizing effect of the horizontal solenoidal field for certain types of waves. The existence of a critical wave length provides a way to explain how the large-scale mixing process can be brought about. The deepening of the upper-air trough in low-index weather, as explained by the sudden shortening of the wave length when a new trough is formed by orographical means, gives us only a qualitative kinematic picture. In order to understand the mechanism of atmospheric disturbances dynamically, we have to consider the energy transformation in such a disturbance during large-scale mixing.

By means of equations (7) and (22) the vorticity equation can be transformed to

\[
\frac{\partial}{\partial x} \frac{dv'}{dt} = - \left( \beta + \frac{f \gamma_v}{T} \right) v' + \frac{fU}{T} \frac{\partial}{\partial x} \frac{v'}{dx} \quad (91)
\]

Multiplying this equation by \((dv'/dt)dx\) and integrating along a latitude circle, one finds

\[
\oint \frac{dv'}{dt} \frac{dv'}{dx} \frac{dx}{dt} = - \left( \beta + \frac{f \gamma_v}{T} \right) \oint v' \frac{dv'}{dt} \frac{dx}{dt} + \frac{fU}{T} \oint \frac{\partial}{\partial x} \frac{dv'}{dt} \frac{dx}{dt} \quad (92)
\]

in which the integral of the left-hand side is obviously zero, and on the right
\[-\left(\beta + \frac{f \gamma_y}{T}\right) \oint \nu' \frac{dv'}{dt} \, dx\]
\[-= \left(\beta + \frac{f \gamma_y}{T}\right) \left[ \oint \nu' \frac{dv'}{dt} \, dx + \oint \nu' \frac{dv'}{\partial x} \, dx \right],\]
in which the second integral again vanishes. Finally, by means of integration by parts, the last integral of (92) reduces to
\[\frac{fU}{T} \oint \frac{\partial \tau}{\partial x} \frac{dv'}{dt} \, dx\]
\[= \frac{fU}{T} \left[ \tau \frac{dv'}{dt} \right] - \oint \tau \frac{\partial}{\partial x} \left( \frac{dv'}{dt} \right) \, dx\]
\[= -\frac{fU}{T} \oint \tau \frac{\partial}{\partial x} \left( \frac{dv'}{dt} \right) \, dx\]

since the contribution of \(\tau \frac{dv'}{dt}\) integrated around a closed path is zero. We have then
\[\left(\beta + \frac{f \gamma_y}{T}\right) \frac{\partial}{\partial t} \oint \frac{\nu'^2}{2} \, dx = -\frac{fU}{T} \oint \tau \frac{\partial}{\partial x} \left( \frac{dv'}{dt} \right) \, dx.\]

If we substitute the vorticity equation (91) again in the integral of the right-hand side and remember that
\[\oint \tau \frac{\partial}{\partial x} \, dx = 0,\]
the energy integral is further simplified to
\[\frac{\partial}{\partial t} \oint \frac{\nu'^2}{2} \, dx = \frac{fU}{T} \oint \tau \nu' dx.\]
(93)

After taking a space mean along the whole latitude circle, and multiplying by density \(\rho\), we obtain
\[\frac{\rho}{\partial t} \oint \frac{\nu'^2}{2} \, dx = \frac{fU}{T} \oint \rho \tau \nu' \, dx.\]
(94)

Following Prandtl, we may denote the mixing length of the large-scale mixing by \(l'\),
\[\tau = l' \frac{\partial T}{\partial y}.\]
(95)

Since in general \(\nu'\) and \(\tau\) are positively correlated, i.e., warm air in the southern and cold air in the northern current in the northern hemisphere, one finds
\[\rho \frac{\partial}{\partial t} \oint \frac{\nu'^2}{2} \, dx = \frac{fU \gamma_y}{T} A\]
(96)

where
\[A = \rho \nu' l'.\]
(97)
is the macroscopic Austausch coefficient in the sense of Defant [3]. This shows that in the case of an unstable disturbance, in which large-scale mixing occurs, the kinetic energy of the north-south currents increases. As pointed out by Haurwitz [4], the effect of a horizontal solenoidal field can only change the vorticity, or, as pointed out by Brunt [1], it can only produce a wind shear and thus only causes instability in the system. If we regard the atmosphere as an incompressible fluid, the meridional distribution of temperature is equivalent to a system with higher density in the north and lower density in the south. If we consider a thin layer of air with \(\rho\) a function of \(y\), the potential energy per unit width at the initial stage is
\[(P.E.)_1 = \frac{H}{2} \int_0^Y \rho \, dy\]
(98)

where \(H\) is the thickness. At the final stage, when the density is equalized by mixing, the potential energy will be
\[(P.E.)_1 = \frac{1}{2} H \int_Y \rho \, dy,\]
(99)

but
\[\int_0^Y \rho \, dy,\]
or
\[(P.E.)_1 = (P.E.)_1.\]
(100)

Therefore the potential energy of the system is not changed by the mixing process, which shows that the energy of the north-south current cannot derive from the solenoidal field.

Now consider the equation of motion in the \(x\)-direction,
\[\frac{\partial u^*}{\partial t} + u^* \frac{\partial u^*}{\partial x} + v^* \frac{\partial u^*}{\partial y} - f v^* = -\frac{1}{\rho^*} \frac{\partial \rho^*}{\partial x},\]
(101)
or the form,
\[\frac{\partial u^*}{\partial t} - v^* (\dot{\tau}^* + f) + \frac{\partial}{\partial x} \left( \frac{u^{*2} + v^{*2}}{2} \right) = -\frac{1}{\rho^*} \frac{\partial \rho^*}{\partial x}\]
(102)
in which \(u^* = U(x, t) + u', v^* = v'\), and \(\dot{\tau}^* = \dot{\tau}'\). Multiplying (102) by \(\rho^*\) and integrating around the whole latitude circle, since
\[\oint \rho^* \frac{\partial}{\partial x} \left( \frac{u^{*2} + v^{*2}}{2} \right) \, dx = 0\]
(103)
and
\[\oint \frac{\partial \rho^*}{\partial x} \, dx = 0,\]
(104)
we obtain
\[\oint \rho^* \frac{\partial u^*}{\partial t} \, dx = \oint \rho^* \nu' \dot{\tau}' \, dx + f \oint \rho^* v' \, dx.\]
(105)
The last integral represents the total mass exchange in the north-south direction. If we neglect the effect of vertical eddy viscosity, the large-scale horizontal mixing process produces a zero transport of mass in that direction, so that this integral is zero. Then, if we take the space mean of (105),
\[\rho \frac{\partial U}{\partial t} = \frac{1}{\rho^*} \frac{\partial \rho^*}{\partial x}.\]
(106)
In the mean condition the relative vorticity decreases with increasing latitude at the rate \( \beta = \partial \omega / \partial y \). If the mixing length \( l' \) is introduced, we find, since \( l' \) and \( \nu' \) are negatively correlated, that the equation above reduces to
\[
\rho \frac{\partial U}{\partial t} = - \beta \rho \nu' \frac{l'}{U} = - \beta A, \tag{107}
\]
or the rate of change of kinetic energy in the west wind is
\[
\rho \frac{\partial (\frac{1}{2} U^2)}{\partial t} = - \beta UA. \tag{108}
\]
This shows that during the large-scale mixing process the kinetic energy of the west wind decreases. In other words, the kinetic energy of the meridional circulation is derived merely at the expense of the energy of zonal circulation. The total change of the kinetic energy of the whole system is
\[
\rho \frac{\partial}{\partial t} \left( \frac{U^2 + \omega'^2}{2} \right) = - \left( \beta - \frac{f \gamma_q}{T} \right) UA, \tag{109}
\]
and this should represent the energy dissipation by the lateral mixing process. For middle latitudes \( f = 10^{-4} \text{ sec}^{-1} \). If we assume \( U = 10 \text{ m/sec}, \gamma_q = 5 \text{ C.}/(1000 \text{ km}), T = 300 \text{ C}, A = 5 \times 10^2 \text{ g cm}^{-1} \text{ sec}^{-1} \) [3], and if the mixing process extends to a height of 10 km, the energy of dissipation is \( 7 \times 10^4 \text{ watt/cm}^2 \). This checks fairly well with the average energy transformation of middle-latitude cyclones calculated by Schröder [10].

According to the present investigation the speed of the westerly current should decrease when the north-south interchange is large, and increase when it is small. A number of correlations between meridional indices at different latitudes and subsequent values of the maximum-minimum and 35°-55° zonal indices were computed at Massachusetts Institute of Technology [11]. Although only a few of the coefficients appear to be significant individually, every one of the 22 coefficients is negative, which gives strong support to the present theory.

A dynamic system with continuously decreasing kinetic energy cannot last indefinitely. As was mentioned before, the large-scale mixing process can only occur in the unstable disturbances of the atmosphere characterized by \( L < L_4 \); but from (37) the critical wave length itself is a function of the west wind speed \( U \). Therefore, during the large-scale mixing process with decreasing zonal index, the critical wave length \( L_4 \) that a stable wave length should exceed becomes smaller and smaller until finally the condition \( L > L_4 \) is reached, when the wave disturbance is stable again and consequently all mixing processes cease.

The simple model of the present theory explains to a certain extent the interrelationship between meridional and zonal circulation of middle latitudes accompanying the large-scale mixing process. However, it still lacks an adequate physical explanation as to how the zonal index can increase after the mixing process has ceased. According to Rossby’s scheme of general circulation [7], the release of latent heat of condensation at the polar front is the principal heat source which directly drives the polar cell and indirectly drives the middle cell or zonal westerlies of the general circulation. It is obvious that the readjustment of zonal circulation can only be explained after a model of a polar cell of the general circulation has been constructed, which is beyond the scope of the present investigation.

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