An Analysis of Cloud Drop Growth by Collection:
Part IV. A New Parameterization

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ABSTRACT

A new parameterization of cloud particle growth combines the effects of accretion and self-collection, and frees the large-hydrometeor water mass and spectral moments to grow independently. The large-hydrometeor water mass grows through accretion in proportion to the cloud water mass. The spectral moments grow through self-collection in proportion to the large-hydrometeor water content. Accretion is also dependent upon the magnitude of the spectral moments and thus an important feedback loop occurs.

The spreading or narrowing of the drop spectrum is dependent upon the collection kernel and the relative magnitudes of accretion and self-collection in the region of interest. These effects are included in the parameterization. In order for the spectrum to spread there must exist both the stochastic mode and a sufficiently rapid increase in the collection kernel with drop size.

1. Introduction

Part I of this series illustrated the strong differences produced in the growth of a droplet spectrum by the accretion and self-collection modes of stochastic collection. The accretion mode, using the continuous approximation of having the larger (collecting) drops a diameter or more larger than the smaller (collected) droplets, produced a growing but narrowing spectrum, while the self-collection mode, drawing upon the interaction among drops more nearly their own size, produced a rapidly spreading spectrum. It was anticipated that the spreading or narrowing was also a result of certain properties of the collection kernel. This paper will analyze the effect of the kernel on each mode of collection.

Part II pointed out regularities associated with the development of several initial spectra. The increase of the predominant radius $r_p$ up to about 50 $\mu$m was slower for those spectra with smaller kernels at their points of origin and with stronger initial spreading. Once beyond 50 $\mu$m, the rate of change of $r_p$ was essentially the same for all spectra independent of the amount of water mass on the large-hydrometeor portion of the spectrum. Also, all broadened spectra then began to assume similar Golovin-type shapes. This paper will show some mathematical reasons for this behavior.

Part III laid the mathematical basis for the parameterization of the growth of the large-hydrometeor spectrum. It was shown how accretion and self-collection could be treated separately and their effects superimposed through parameterized rate equations with defined rate coefficients. Values of these rate coefficients were calculated and plotted (reproduced here in Fig. 1). These provide the basis for the discussion in this paper of some fundamental features of drop growth by collection.

2. Fundamentals of spectrum spreading

Part III showed the derivation of the four basic equations for the parameterized drop growth, i.e., the growth rates of $x_2$ and $x_f$ for both continuous and stochastic modes. These equations are:

\[ \frac{1}{x_{2_2}} \frac{dx_{2_2}}{dt} = \frac{L_1}{L_2} \int_{S_2} V(x_2|x_1) \left( \frac{2x_2}{x_{2_2}} - 1 \right) f(x_2) dx_2, \quad (1) \]

\[ \frac{1}{x_f} \frac{dx_f}{dt} = \frac{L_1}{L_2} \int_{S_2} V(x_2|x_1) f(x_2) dx_2, \quad (2) \]

\[ \frac{1}{x_{2_2}} \frac{dx_{2_2}}{dt} = \frac{1}{Z_2} \int_{S_2} dx \int_{S_2} dx' x_2 f(x) V(x|x') f(x'), \quad (3) \]

\[ \frac{1}{x_f} \frac{dx_f}{dt} = \frac{1}{2N_2} \int_{S_2} dx \int_{S_2} dx' f(x) V(x|x') f(x'), \quad (4) \]

where the subscript $c$ refers to the continuous or ac-

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cretion mode and to the stochastic or self-collection mode.

Using the three analytic kernels, namely,

\[
V_1(x|\alpha') = b_1\alpha x', \quad V_2(x|\alpha') = b_2(x+x'), \quad V_3(x|\alpha') = b_3,
\]

in Eqs. (1)–(4), we derive the growth rates for \(x_a\) and \(x_f\) under each of the six situations:

\[
\begin{align*}
\frac{1}{x_{a_1}} \frac{dx_{a_1}}{dt} &= x_{a_1}b_1 L_1, \\
\frac{1}{x_{a_2}} \frac{dx_{a_2}}{dt} &= \left(1 + \frac{2x_1}{x_{a_2}}\right) b_2 L_1, \\
\frac{1}{x_{a_3}} \frac{dx_{a_3}}{dt} &= \left(\frac{2}{x_{a_3}} - 1\right) b_3 L_1, \\
\frac{1}{x_{f_2}} \frac{dx_{f_2}}{dt} &= x_{f_2}b_1 L_1, \\
\frac{1}{x_{f_3}} \frac{dx_{f_3}}{dt} &= b_2 L_1, \\
\frac{1}{x_{f_2}} \frac{dx_{f_2}}{dt} &= \frac{1}{x_{f_2}} b_3 L_1, \\
\frac{1}{x_{a_1}} \frac{dx_{a_1}}{dt} &= x_{a_1}b_1 L_2, \\
\frac{1}{x_{a_2}} \frac{dx_{a_2}}{dt} &= \frac{2}{x_{a_2}} b_2 L_2, \\
\frac{1}{x_{a_3}} \frac{dx_{a_3}}{dt} &= \frac{1}{x_{a_3}} b_3 L_2, \\
\frac{1}{x_{f_2}} \frac{dx_{f_2}}{dt} &= 2b_2 L_2, \\
\frac{1}{x_{f_2}} \frac{dx_{f_2}}{dt} &= \frac{1}{x_{f_2}} b_3 L_2.
\end{align*}
\]

Excluding the second-order correction proportional to \(x_1\) in (6b), the continuous mode changes \(x_a\) and \(x_{f_2}\) equally for kernels 1 and 2, which are the more important kernels for cloud physics applications. For kernel 3, \(x_{f_2}\) increases as a function of \(b_3 L_1/x_{f_2}\), while \(x_a\) tracks \(x_{f_2}\), decreasing when it is larger than \(2x_{f_2}\), and growing when it is smaller \([(6c)]\) but not fast enough to prevent \(x_{f_2}\) from closing in. As \(x_{f_2}\) approaches \(x_a\), the growth rate of \(x_{f_2}\) increases until it is the same as that of \(x_{f_2}\), forming a delta function, and thereafter the tracking is exact and the delta function cannot spread.

For the stochastic mode the growth rates of \(x_{f_2}\) are just one-half those for \(x_a\) when \(x_{f_2}\) and \(x_a\) are equal. For comparison between kernels 1, 2, 3, we arbitrarily set the growth rates of \(x_a\) equal at the boundaries between 1 and 2, and 2 and 3. Therefore,

\[
\begin{align*}
[x_a b_1 &= b_2 = b_3/x_a], \\
[x_a b_1 &= 2b_2 = b_3/x_a],
\end{align*}
\]

where \(x_a\) and \(x_a\) are the lower and upper bounds of region 2, and apply to both \(x_{f_2}\) and \(x_{f_2}\).

We are now in a position to investigate the spectrum spreading due to the stochastic and continuous modes for the three analytic kernels. We distinguish five general ranges of collection depending upon whether \(x_a\) and \(x_{f_2}\) fall into regions 1, 2 or 3, of the analytic kernels, and designate these ranges by the subscripts 11, 21, 22, 32, 33, respectively. Using

\[
\frac{d}{dt} (\text{var } x) = \left(\frac{1}{x_a} \frac{dx_a}{dt} + \frac{1}{x_f} \frac{dx_f}{dt}\right).
\]

we evaluate the spreading rate for the five ranges for both continuous and stochastic modes:

\[
\begin{align*}
\frac{d}{dt} (\text{var } x)_{11} &= 0, \\
\frac{d}{dt} (\text{var } x)_{21} &= \frac{x_{f_2}}{x_{f_2}} \left(\frac{2}{x_{f_2}} - 1\right) b_2 L_1, \\
\frac{d}{dt} (\text{var } x)_{22} &= \frac{x_{f_2}}{x_{f_2}} \frac{2}{x_{f_2}} b_2 L_1, \\
\frac{d}{dt} (\text{var } x)_{32} &= \frac{x_{f_2}}{x_{f_2}} \frac{1}{x_{f_2}} b_3 L_1, \\
\frac{d}{dt} (\text{var } x)_{33} &= \frac{x_{f_2}}{x_{f_2}} \left(\frac{2}{x_{f_2}} - 1\right) b_3 L_1.
\end{align*}
\]
\begin{equation}
\frac{d}{dt} \text{var} x_{s,3a} = \frac{x_{s2}}{x_{f2}} \left( \frac{1}{x_{f2}} - \frac{1}{2x_{f2}} \right) b_3 L_2.
\end{equation}

(14e)

Both the growth and spreading of the continuous mode depend upon \( L_1 \) while those for the stochastic mode depend upon \( L_2 \). For the stochastic mode at constant \( L_2 \), these ranges represent successive, gradual slowings of the rate of spreading.

The spreading produced by the continuous mode for kernel 1 is zero while that for kernel 2 produces an approach to \( \text{var} x = 1 \), although slowly by virtue of the small coefficient \( x_1 \). That for kernel 3 is negative going to zero when the spectrum approaches the delta function \( (x_{s2} = x_{f2} \text{ and } \text{var} x = 0) \) [(13e)]. In the first overlapping range [(13b)] the spreading is the same as for kernel 2. In the second overlapping range [(13d)], the spreading is, by virtue of our discussion of (6c), negative for all but the delta function, for which the spreading is zero.

For the stochastic mode the spreading rate of (14a) is smallest, but non-zero, when \( x_{s2} = x_{f2} \text{ and increases as } x_{s2} \text{ increases. When } x_{s2} \text{ enters the second region [(14b)] the spreading rate gradually diminishes as } x_{s2} \text{ increases until } x_{f2} b_1 = 2b_2, \text{ as in (11). Then (14c) applies and the spreading rate depends only on the ratio } x_{s2}/x_{f2} \text{ (assuming } L_2 \text{ constant). Eventually } 2b_2 = b_3/x_{f2} \text{ and growth enters region 4 [(14d)] and the spreading rate gradually decreases. It goes to zero when } x_{s2} = b_3/b_2, \text{ and negative for } x_{s2} > b_3/b_2 \text{ (because } x_{f2} \text{ continues to increase rapidly while } x_{s2} \text{ slows down). When } 2b_2 = b_3/x_{f2} \text{ (which, incidently, never happened in the calculations performed) we are already in the area of negative spreading [(14e)]. It continues negative as } 2x_{f2} \text{ closes in on } x_{s2} \text{ where an equilibrium state is reached.}

This latter equilibrium state of (14e) corresponds to \( \text{var} x = 1 \) which is the shape found in some initial distributions of Part II. It is a self-preserving form on the \( g(\text{ln} r) \) plot for the constant kernel. Similarly, (14a) and (14c) are the formulas for the spreading resulting from kernels 1 and 2, respectively, as discussed in Part III. The spreading rate of (14c) also has been shown (Berry, 1967) to produce a self-preserving form on the \( g(\text{ln} r) \) plot.

These relationships may be observed in yet another way. We define \( R_{s,i} \) and \( R_{c,i} \) as the ratios of the spreading to the growth rates for each mode and for each analytic kernel, where \( i = 1, 2, 3 \). Thus,

\begin{equation}
R_{s,i} = \frac{d}{dt} \frac{\text{var} x_{s,i}}{x_{s2}},
\end{equation}

(15)

and similarly for \( R_{c,i} \). Then from the above we have the following:

\begin{align}
R_{s,1} &= 0 \tag{16a} \\
R_{c,2} &= \frac{x_1}{x_{f2}} (1 - \text{var} x) \tag{16b} \\
R_{c,3} &= -2 \text{ var } x (1 + \text{var } x)/(1 - \text{var } x) \tag{16c} \\
R_{s,1} &= \frac{1}{2} (1 + 2 \text{ var } x) \tag{17a} \\
R_{s,2} &= \frac{1}{2} (1 + \text{var } x) \tag{17b} \\
R_{s,3} &= \frac{1}{2} (1 + \text{var } x)/(1 - \text{var } x). \tag{17c}
\end{align}

The greatest relative spreading rate is given by \( R_{s,1} \), followed by \( R_{s,2} \). \( R_{s,3} \) seeks the equilibrium state \( \text{var} x = 1 \). The continuous mode shows no spreading for \( R_{c,1} \) while \( R_{c,3} \) shows a very slow approach to \( \text{var} x = 1 \). The numerator of \( R_{c,3} \) is the spreading rate, which is always negative until \( \text{var} x = 0 \). The denominator of \( R_{c,3} \) is the growth rate of \( x_{s2} \) which is negative for \( \text{var} x > 1 \) and positive for \( \text{var} x < 1 \). Thus, if the distribution is very broad (\( \text{var} x > 1 \)) \( R_{c,3} \) shows that \( x_{s2} \) decreases in value simultaneously with the decrease in \( \text{var} x \). Once \( \text{var} x < 1 \), the \( x_{s2} \) increases while \( \text{var} x \) continues to decrease to a delta.

There is no important spreading produced by the continuous mode. The spectrum variance remains unchanged by this process for kernel 1, goes slowly to 1 for kernel 2 and to the delta function for kernel 3. This is the reason for the narrowing of \( S2 \) seen in Part I.

The stochastic mode is necessary but not sufficient for spectrum spreading. Given the stochastic mode, spreading is highly dependent upon the rate of increase with \( x \) and \( x' \) of the collection kernel as shown in (5). When this increase is large as it is for kernel 1 the spreading rate is large [see Eq. (14a)]. When it is moderate, as it is for kernel 2, the spreading rate is also moderate [(14c)]. When it is zero, as it is for kernel 3, the spreading is negative if \( \text{var} x > 1 \), positive for \( \text{var} x < 1 \), and tends to zero as \( \text{var} x \) approaches 1 [(14e)].

Thus, any complete explanation of the spreading of \( S2 \), as shown in Part I must include the effect of the rate of increase of the kernel with droplet size. Were it not for the fact that the kernel increases very rapidly in the region of the \( S2 \) tail in Fig. 6 (Part I), then indeed such spreading would not occur. Were the kernel in this region roughly \( b_2(x+x') \), as it is for the intermediate-sized drops, the \( S2 \) would not have so lengthened its tail but would rather have grown in the fashion of the intermediate drops, keeping its shape similar to the Golovin asymptotic form.

3. Simultaneous effects of accretion and self-collection

In order to set the parameterized equations in a more usable form we have defined in Part III spe-
specialized rate coefficients in terms of the following equations:

\[ \frac{1}{x_{q2}} \frac{dx_{q2}}{dt} = \dot{b}_{q}(x_{q2}, x_{1}) L_{1} \]  \hspace{1cm} (18)

\[ \frac{1}{x_{q2}} \frac{dx_{q2}}{dt} = \dot{b}_{q}^{*}(x_{q2}, x_{1}) L_{1} \]  \hspace{1cm} (19)

\[ \frac{1}{x_{q2}} \frac{dx_{q2}}{dt} = \dot{b}_{q}^{*}(x_{q2}) L_{2} \]  \hspace{1cm} (20)

\[ \frac{1}{x_{q3}} \frac{dx_{q3}}{dt} = \dot{b}_{q}^{*}(x_{q3}, x_{2}) L_{2}. \]  \hspace{1cm} (21)

By comparison to (1), (2), (3) and (4) it is evident that these specialized rate coefficients take on the following values:

\[ \dot{b}_{q}(x_{q2}, x_{1}) = \frac{1}{L_{2}} \int_{x_{q2}}^{2x_{2}} V(x_{2} | x_{1}) \left( \frac{2x_{2}}{x_{q2}} - 1 \right) f(x_{2}) dx_{2} \]  \hspace{1cm} (22)

\[ \dot{b}_{q}^{*}(x_{q2}, x_{1}) = \frac{1}{L_{2}} \int_{x_{q2}}^{x_{1}} V(x_{2} | x_{1}) f(x_{2}) dx_{2} \]  \hspace{1cm} (23)

\[ \dot{b}_{q}^{*}(x_{q2}) = \frac{1}{L_{2} Z_{2}} \int_{x_{q2}}^{2x_{2}} dx \int_{x_{q2}}^{x_{2}} dx' x' f(x) V(x | x') f(x') \]  \hspace{1cm} (24)

The kernel \( V \) is conveniently related in Part III to the accretion mode rate coefficient \( b_{q} \) as

\[ b_{q}(x_{q2}, x_{1}) = V(x_{2} | x_{1}) / (x_{2} + x_{1}). \]  \hspace{1cm} (26)

The values of \( b_{q} \) and \( \dot{b}_{q}^{*} \) as calculated in Part III are displayed in Fig. 1.

In the region where \( b_{q} \) is almost a constant, i.e., for \( 50 \mu m \leq r_{2} \leq 200 \mu m \) and \( r_{q1} \geq 8 \mu m \), Eqs. (22) and (23) reduce to the approximations

\[ \dot{b}_{q}^{*}(x_{q2}, x_{1}) \approx b_{q}(x_{q2}, x_{1}), \]  \hspace{1cm} (27)

\[ \dot{b}_{q}^{*}(x_{q2}, x_{1}) \approx b_{q}(x_{q2}, x_{1}). \]  \hspace{1cm} (28)

Although (24) can be readily reduced to

\[ \dot{b}_{q}^{*}(x_{q2}) = \frac{2}{L_{2} Z_{2}} \int_{x_{q2}}^{x_{2}} x f(x) dx \int_{x_{q2}}^{x_{2}} b_{q}(x, x') x' f(x') dx', \]  \hspace{1cm} (29)

it is difficult to derive analytically an approximation for \( \dot{b}_{q}^{*} \). However, such an approximation was derived empirically in Part III, as shown in Fig. 1; the result was

\[ \dot{b}_{q}^{*}(x_{q2}) \approx 2b_{q}(3.4x_{q2}, x_{q2}), \]  \hspace{1cm} (30a)

\[ \dot{b}_{q}^{*}(r_{q2}) \approx 2b_{q}(1.5r_{q2}, r_{q2}). \]  \hspace{1cm} (30b)
Table 1. Calculations of growth rates of \( r_e \) and \( r_g \) for Figs. 1–6 of Part II at the moment when \( r_e = 50 \mu m \). The rates are to be compared with curves on Fig. 7 of Part II.

| Figure no. | \( r_f \) (\( 10^9 \)) | \( r_e \) (\( 10^9 \)) | \( r_g \) (\( 10^9 \)) | \( r_g \) (\( 10^9 \)) | \( L_f \) (\( 10^6 \)) | \( b_e \) (\( 10^{-6} \)) | \( b_g \) (\( 10^{-6} \)) | \( L_e b_e \) (\( 10^9 \)) | \( L_g b_g \) (\( 10^9 \)) | \( \left( \frac{1}{r_f} \right) \) (\( 10^9 \)) | \( \left( \frac{1}{r_e} \right) \) (\( 10^9 \)) | \( \left( \frac{1}{r_g} \right) \) (\( 10^9 \)) | \( \left( \frac{1}{r_f} \right) \) (\( 10^9 \)) | \( \left( \frac{1}{r_e} \right) \) (\( 10^9 \)) | \( \left( \frac{1}{r_g} \right) \) (\( 10^9 \)) |
|------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1          | 10              | 10              | 12.5            | 110             | 0.91            | 0.094           | 0.47            | 0.97            | 0.43            | 0.09            | 0.52            | 0.95            |                   |                   |                   |                   |
| 2          | 14              | 11.1            | 15.0            | 100             | 0.87            | 0.125           | 0.50            | 0.96            | 0.42            | 0.12            | 0.54            | 0.96            |                   |                   |                   |                   |
| 3          | 14              | 12              | 15.0            | 90              | 0.83            | 0.17            | 0.50            | 0.95            | 0.42            | 0.16            | 0.57            | 0.98            |                   |                   |                   |                   |
| 4          | 14              | 14              | 17.5            | 75              | 0.71            | 0.29            | 0.51            | 0.92            | 0.36            | 0.27            | 0.63            | 0.97            |                   |                   |                   |                   |
| 5          | 18              | 14.3            | 19.5            | 73              | 0.68            | 0.32            | 0.58            | 0.92            | 0.36            | 0.30            | 0.66            | 1.02            |                   |                   |                   |                   |
| 6          | 18              | 18              | 22.5            | 60              | 0.43            | 0.57            | 0.63            | 0.90            | 0.27            | 0.51            | 0.78            | 1.05            |                   |                   |                   |                   |

A similar approximation is derivable for \( b_{g0}^* \):

\[
\frac{b_{g0}^*}{b_{g0}^*} = \frac{1}{N_s L_z} \int x f(x) dx \int b_e(x, x') f(x') dx',
\]

(31)

\[
b_{g0}^* = b_e(x_{g0}, x_{g2}),
\]

(32)

but the limits to this approximation have not yet been checked.

Having achieved a basic parameterization for each collection mode separately, we may superimpose the respective equations to gain a combined parameterization:

\[
\frac{d x_{g2}}{x_{g2} dt} = \frac{b_{g0}^* L_1 + b_{g0}^* L_2}{e + s},
\]

(33)

\[
\frac{d x_{g1}}{x_{g1} dt} = \frac{b_{g0}^* L_1 + b_{g0}^* L_2}{e + s}.
\]

(34)

In order to explain the spectrum development shown in Part II, we define a total \( b_g(x_g) \) to account for the growth of the \( x_g \) of the combined \( S1 \) and \( S2 \) spectrum:

\[
\frac{d x_g}{x_g dt} = b_g(x_g) L.
\]

(35)

As shown in Part III,

\[
b_g = \frac{b_{g0}^* + b_{g0}^*}{(L_1/L) + b_{g0}^*(L_2/L)}.
\]

(36)

Using (27), (28) and (36) we write (35) and (33) as

\[
\frac{d x_{g2}}{x_{g2} dt} = 2 b_e L_1 + b_{g0}^* L_2,
\]

(37)

\[
\frac{d x_{g1}}{x_{g1} dt} = b_e L_1 + b_{g0}^* L_2.
\]

(38)

Once \( r_e \) is beyond 50 \( \mu m \), rates of growth are nearly the same because \( b_{g0}^* \approx 2 b_e \), in which case, according to (37), the growth rate of \( r_e \) will not be highly dependent on the relative values of \( L_1 \) and \( L_2 \).

It is of interest in this regard to calculate the growth rates for both \( r_g \) and \( r_e \) for Figs. 1–6 of Part II at the moment \( r_e = 50 \mu m \). The data are shown in Table I. The \( b_e \) and \( b_{g0}^* \) values are taken from Fig. 1. The \( r_g \) values are estimated from the figures, and the \( L_g \) are calculated from this estimate. It can be shown that the final answer is rather insensitive to error in estimating \( r_g \).

Table 1 shows that the growth rate of \( r_g \) is much greater for those spectra with steeper \( S2 \) slopes and larger \( L_g \). The values of \( b_{g0}^* \) are nearly the same in all cases, while the values of \( b_e \) increase with larger \( r_f \). It is easily seen that the dominating factor is \( L_g \). Its effect is partially masked in the growth rates of the \( r_g \)'s, where twice the value of \( L_g b_e \) is contributed. The \( r_g \) growth rates only gradually increase with \( b_e \) to give approximately equal sloped lines in Fig. 7, Part II. We conclude in either case, however, that the rapid achievement of high \( L_g \) is the most important single factor in obtaining a fast growth rate by stochastic collection.

As a final demonstration of the importance of this effect, we show the development of several spectra where autoconversion is unimportant but accretion and self-collection are. Fig. 2 shows several initial spectra with various \( S2 \) and the same \( S1 \) with the total \( L \) adjusted to 1 g/m\(^2\). Fig. 3 shows the development of each case at \( t = 0, 10 \) and 20 min.

We see that the two upper displays began with too large an \( r_{g2} \) and subsequently did not have \( b_e \) large enough to gather very much water from \( L_1 \). The two middle displays show that a small amount of water added near \( r_{g2} = 50 \mu m \) accumulates \( L_1 \) rapidly and thereby increases \( L_2 \) so that self-collection can proceed at a rapid rate. The result is far more water on large drops in a shorter time than the first two instances could provide. The lower displays which began with \( r_{g2} = 20 \mu m \) are at the end only some 8 min behind the middle two experiments.

We conclude that a proper recognition of the feedback between \( L_2 \) and \( r_{g2} \) is fundamental to parameterized reasoning of precipitation development. The behavior in Fig. 3 can be very closely predicted by the parameterizations in (37) and (38).
4. In retrospect

The first general parameterization of collection seems to have been that of Kessler (1969). He postulated a cloud model which consists of two size categories of drops: cloud droplets with insignificant fallspeeds, and hydrometeors, and used the terms autoconversion and accretion to describe the two primary mechanisms of drop growth as he perceived it.

It is interesting to note that we did not set out to put our collection parameterizations in terms of cloud water, hydrometeor water, autoconversion and accretion. We tried every other way we could think of, but the results of our calculations kept telling us to break spectrum at \( r_e = 50 \mu m \), and once we did, the pieces began to fall into place.

For convenience we will summarize the essential ingredients of Kessler's and our parameterization forms of hydrometeor collection. In our notation the Kessler method is

\[
\frac{1}{L_2} \frac{dL_2}{dt} = K_2 L_1 L_2^{-1/3}, \tag{39a}
\]
where \( K_2 \) is a constant, \( D_0 \) the median volume diameter, and Eq. \( (39b) \) results from the required fit to the Marshall-Palmer spectrum.

The method of parameterization that results from the present analysis is

\[
\frac{1}{L_2} \frac{dL_2}{dt} = b_{q2} L_1, \quad (40a)
\]

where the first term on the right-hand side is due to accretion and the second to self-collection. The rate
coefficients are as follows:

\begin{equation}
\begin{align*}
\beta_c^* &= \frac{1}{L_2} \int_{s_2} b_c(x_2, x_1) x_2 f(x_2) dx_2 
&\approx b_c(x_2, x_1) (x_2) \frac{2 x_2}{x_2} (x_2) \frac{2 x_2}{x_2} \frac{1}{x_2} \frac{d x_2}{x_2} \\
&\approx b_c(x_2, x_1) \quad (41a) \\
\beta_s^* &= \frac{1}{L_2} \int_{s_2} b_c(x_2, x_1) \left( \frac{2 x_2}{x_2} - 1 \right) x_2 f(x_2) dx_2 \\
&\approx b_c(x_2, x_1) \quad (41b) \\
\beta_r^* &= \frac{x_2}{L_2} \int_{s_2} x_2 f(x_2) dx_2 
&\approx b_c(x_2, x_1) \quad (41c) \\
\beta_{r2} &= \frac{2}{x_2 L_2} \int_{s_2} x_2 x_2 f(x_2) dx_2 
&\approx 2 b_c(3, 4 x_2, x_2) \quad (41d). \\
b_c(r_2, r_1) &= \left( \frac{r_2^3}{r_2^3 + r_1^3} \right) \Delta v(r_2, r_1) V^2(r_2, r_1). \quad (42)
\end{align*}
\end{equation}

By way of comparison we notice that \( b_c^* \) in (40a) replaces \( K_2 L_2^{-1/4} \) in (39a) but this difference is unimportant since both quantities are calculated from the same accretion equation. Comparing the accretion portion of (40b) with (40a) we see that

\begin{equation}
\begin{align*}
\frac{1}{x_2} \frac{d x_2}{dt} &= \frac{1}{L_2} \frac{d L_2}{dt}, \\
\frac{1}{r_2} \frac{d r_2}{dt} &= \frac{1}{L_2} \frac{d L_2}{dt}. \quad (43b)
\end{align*}
\end{equation}

Comparing (43b) to (39b) we see that \( D_2 \) is constrained to grow more slowly than \( r_2 \) by accretion. This, however, could well better account for the effect of drop breakup which is necessary to produce the Marshall-Palmer spectrum and which must yet be added to the present parameterization. The key difference between the two methods is in the self-collection. Absent in the older method, this process serves in the newer method to free the dependence of the mean mass \( x_f \), \( x_2 \) or \( D_2^2 \) on the total mass \( L_2 \), and opens the parameterization to the inclusion of important microphysical feedback effects. Two spectral moments are used which together contain information about the variance of the spectrum.

The radar reflectivity is readily seen to be the same equation in both cases. The distinction in the new parameterization lies in the independent treatment of added water mass due to accretion and increased predominant mass due to self-collection.

5. Conclusion

The spreading or narrowing of a droplet spectrum undergoing collection is dependent upon the kernel in the region of interest and the mode of collection that is in effect. The accretion process produces no spreading for rapidly increasing kernels, a slow approach to \( \var r = 1 \) for a moderately increasing kernel, and progressive narrowing for a constant kernel.

The stochastic mode is a necessary but not a sufficient condition for rapid spreading of the spectrum. It is also necessary that the collection kernel increase sufficiently rapidly with drop size. It seems more correct to say that this kernel increase is the cause of spectrum spreading and stochastic collection is the mode through which the kernel operates. In the stochastic mode a rapidly increasing kernel produces rapid spreading, an intermediate kernel produces moderate spreading, and a constant kernel drives the shape toward \( \var r = 1 \).

A new parameterization of cloud particle growth has been described which superimposes the effects of accretion and self-collection and thereby allows water mass and spectral moments to change independent of one another. Drop breakup is not included at this stage. The parameterization can readily be extended to include the ice phase and will include feedback processes essential to the modelling and explanation of cloud particle growth.

Growth rates due to accretion are proportional to cloud water content; those of self-collection, to large hydrometeor water content. Therefore, the rapid accumulation of large hydrometeor water content will do more toward the generation of large hydrometeors than the maintenance of high cloud water content. Since the shape of the large hydrometeor distribution is dependent upon the relative magnitudes of accretion and self-collection (as well as breakup which is neglected here), this shape is therefore a clue to the physical processes operating within the cloud.

APPENDIX

List of Symbols

b \quad \text{total rate coefficient, accretion plus self-collection}
\hline
b_c \quad \text{rate coefficient for accretion only}
\hline
b_s \quad \text{rate coefficient for self-collection only}
\hline
f \quad \text{number density function}
\hline
g \quad \text{mass density function}
\hline
L \quad \text{liquid water content}
\hline
N \quad \text{total number density}
\hline
r \quad \text{droplet radius}
\hline
r_f \quad \text{radius corresponding to } x_f = L/N
\hline
r_g \quad \text{radius corresponding to } x_g = Z/L
\hline
S1 \quad \text{the small hydrometeor or “cloud water” portion of the spectrum}
\hline
S2 \quad \text{the large hydrometeor portion of the spectrum}
\hline
V \quad \text{collection kernel}
\hline
\var r \quad \text{relative variance with respect to mass}
\hline
\left( \frac{x_g}{x_f} - 1 \right)
$x$  droplet mass

$x_f$  mean mass of the number density function

$x_c$  mean mass of the mass density function

$Z$  spectral radar reflectivity, the second mass moment of the number density function.

REFERENCES


