Resonant Topographic Waves in Barotropic and Baroclinic Flows

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ABSTRACT

The problem of resonant, topographic, quasi-geostrophic waves is examined analytically by exploiting simplifications that arise when the flow is nearly resonant. The barotropic and (two-layer) baroclinic problems are studied. In each case the topographic linear stability problem is solved explicitly and analytic expressions are given for the growth rate. The bifurcation problem in finite amplitude also is described.

Some differences with earlier treatments are noted. In particular, in the barotropic problem subresonant instability may occur if the zonal wavelength is long enough. In both the barotropic and baroclinic problems the critical point at which multiple equilibria occur does not correspond to the stability thresholds of the linear problem. In the baroclinic problem Reynolds stresses are found to be of equal importance with eddy heat fluxes in altering the zonal flow although only the latter can transfer energy to the wave field for the zonal velocity profile considered.

Analysis of the marginally stable wave at the minimum critical shear of the ordinary baroclinic stability problem shows that topography is stabilizing.

1. Introduction

In a series of recent papers (Charney and Devore, 1979; Charney and Strauss, 1980; Hart, 1979) the notion of topographic instability has been developed and described. A zonal flow on the β plane, otherwise stable, is shown to become destabilized in the presence of sinusoidal bottom topography that undulates in the direction of flow. Associated with the instability in finite amplitude is the existence of multiple, linearly stable equilibria and the authors cited above have suggested the identity between certain of these states and so-called “blocking pattern” states of atmospheric flow. [Of course, multiple equilibria quite generally occur in forced, unstable systems (e.g., Pedlosky, 1981) and the identification of topographic instability as the underlying cause of atmospheric blocking requires more than the simple existence of high and low wave amplitude equilibria.]

A common feature of the models in the papers cited above is the use of ad hoc equations of motion. In Charney and Devore (1979) and Charney and Strauss (1980) a severely truncated spectral representation of the fields is used and it remains unclear precisely what assumptions are involved for the validity of the model although the authors themselves point out several deficiencies (e.g., the artificial absence of Reynolds stresses due to the truncation). The paper of Hart’s avoids truncation but obtains the necessary simplification of the equations of motion by restricting attention to topographic features which are of unrealistic infinite meridional extent.

In both of these treatments a major result deduced from the simplified equations is that topographic instability and the associated multiple equilibria are associated with the flow parametric conditions appropriate for linear resonance of stationary topographic waves. To some extent this might be anticipated since for parameter conditions removed from resonance the linear, forced solution is expected to obtain with none of the strong feedback to the zonal flow that characterizes topographic instability. However, at resonance, questions about the validity of severe truncation or of the ordering based on the single parameter associated with large meridional scale become even more troubling. Nevertheless, the physical processes discovered in these early papers seem fundamental and are likely to transcend the limitations of the analysis.

The purpose of the present paper is to present an alternative analysis of topographic instability which is asymptotic and deductive in character and which will hopefully add to our understanding of the phenomenon. The point of view taken here is first, that the identification of topographic instability with flow states near linear resonance is a feature which should be exploited in the analysis and second, the importance of linear resonance discovered earlier, points to the quasi-linear nature of the physics. That is, flow over very strong, jagged topography of great
amplitude would undoubtedly lose the sensitivity to resonance required by the theory. Indeed, the results of Hart (1979) emphasize the broadening of the parameter domain of multiple equilibria as the topographic amplitude diminishes.

These two facts lead to the great simplification in the following analysis that, to lowest order, the wavy flow has the form of a free, stationary wave and only its amplitude need be determined by fundamentally nonlinear processes. A similar approach has been independently taken by Plumb (1981) in his analysis of forced baroclinic waves with, however, no feedback affecting the topographic forcing.

Two separate problems are discussed in the remainder of this paper. They are the topographic instability and multiple equilibria of first barotropic and then baroclinic flows. The scaling relations between amplitude and topography are different in the two cases and the discussion of the barotropic problem is presented first in the belief that its relative simplicity will be helpful in allowing the reader to follow the more complex analysis of the baroclinic case.

There is one feature of the baroclinic problem that deserved particular comment from the viewpoint of Physical Oceanography. As Charney and Strauss (1980) discovered, the baroclinic topographic instability occurs for values of the vertical shear that are subcritical with regard to the classical threshold of baroclinic instability. Furthermore, the topography actually raises the threshold of the ordinary baroclinic instability mode. Hence, baroclinic topographic instability may well be a source of fluctuation activity for flows that have weak vertical shears, as in the mid-ocean where the threshold for ordinary baroclinic instability may not be met.

2. The barotropic model

The quasi-geostrophic vorticity equation for a homogeneous, barotropic fluid on the beta plane can be written in nondimensional form as (Pedlosky, 1979)

\[
\left( \frac{\partial}{\partial t} + \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} \right) (\nabla^2 \psi + \beta y + \eta_B) = -r \nabla^2 \psi + Q(y) \tag{2.1}
\]

where \( \psi \) is the geostrophic streamfunction whose \( x \) and \( y \) derivatives give \( v \) and \(-u\), respectively, while \( \beta \) is the planetary vorticity gradient \( \beta_0 \) scaled by a typical value of the relative vorticity gradient \( \bar{U}/L^2 \), where \( \bar{U} \) is a characteristic horizontal velocity and \( L \) is a characteristic horizontal length. On the right-hand side of (2.1), the dissipation of vorticity is assumed to be due to the action of the Ekman layer (or layers if the upper surface of the layer is rigid). \( \bar{U} \) and \( L \) have been used to nondimensionalize horizontal velocities and lengths. The bottom topography enters as the function

\[
\eta_B = \frac{h_s}{DU} f L, \tag{2.2}
\]

where \( h_s \) is the elevation of the topography, \( D \) the total depth of the layer and \( f \) is the Coriolis parameter. The function \( Q(y) \) represents some external vorticity source which, in the absence of topography, would drive the purely steady zonal flow \( U(y) \), i.e.

\[
-r \frac{dU}{dy} = Q(y). \tag{2.3}
\]

It is convenient to partition \( \psi \) between \( U \) and the disturbance field due to \( \eta_B \) as follows:

\[
\psi = -\int U(y') dy' + \epsilon \phi, \tag{2.4}
\]

in terms of which (2.1) becomes

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla^2 \phi + \frac{\partial \phi}{\partial x} \left( \beta - \frac{dU}{dy^2} \right) + \epsilon J(\phi, \nabla^2 \phi) + J(\phi, \eta_B) = -r \nabla^2 \phi - \frac{U}{\epsilon} \frac{\partial \eta_B}{\partial x}, \tag{2.5}
\]

the standard notation

\[
J(a, b) = \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} - \frac{\partial a}{\partial y} \frac{\partial b}{\partial x} \tag{2.6}
\]

has been employed in (2.5).

The flow is contained within a channel of width \( L \) on whose boundaries, \( y = 0, 1 \)

\[
\frac{\partial \phi}{\partial x} = 0, \tag{2.7}
\]

while any \( x \)-independent portion of \( \phi \) must satisfy

\[
\frac{\partial \phi}{\partial y} = 0. \tag{2.8}
\]

In particular, I will restrict attention in this paper to the case where

\[
\eta_B = h_0 \frac{e^{ikx}}{2} \sin ly + \ast, \tag{2.9}
\]

where an asterisk denotes the complex conjugate of the preceding term and where \( l \) is an integral multiple of \( \pi \). Furthermore, \( h_0 \) will be considered to be small and as will be shown, the natural ordering at resonance between \( h_0 \) and \( \epsilon \) is \( \epsilon = O(h_0^{1/3}) \). Eq. (2.5) then represents a fluid system forced initially by the interaction of the zonal flow \( U \) and the topography and resonance can occur for those values of \( k, l \) and \( \beta \) for which the left-hand side has stationary, normal-mode solutions. To make matters as simple as possible, the case is considered where \( U \) is inde-
pendent of \( y \). Then resonance will occur when the Rossby stationary wave criterion is met, i.e., when

\[
U = \frac{\beta}{a^2} ; \quad a^2 = k^2 + l^2. \tag{2.10}
\]

I will insist that \( U \) be near the resonance point so that

\[
U = \frac{\beta}{a^2} + \Delta, \quad \Delta \ll 1. \tag{2.11}
\]

Indeed, it will also become apparent that \( \epsilon = O(\Delta^{1/2}) \). It is now only necessary to select the proper scaling for the development time for the instability and the appropriate scaling relationship between \( r \) and \( \epsilon \).

The time scale used to derive (2.1) is the advective time \( L/U \). The results of Charney and Devore (1979) suggest that development occurs on the longer time scale \( L/(UA^{1/2} \epsilon) \). The time is therefore rescaled by introducing the new variable

\[
T = \mu t; \quad \mu = O(\epsilon^3),
\]

and \( r = O(\epsilon^3) \) is also chosen for reasons which will become clear later.

When all these choices are made (2.5) becomes

\[
\left[ \mu - \frac{\partial}{\partial T} + \left( \frac{\beta}{a^2} + \Delta \right) \frac{\partial}{\partial x} \right] \nabla^2 \phi + \beta \frac{\partial \phi}{\partial x} + \epsilon J(\phi, \nabla^2 \phi) + J(\phi, \eta_\theta) + r \nabla^2 \phi = -\left( \frac{\beta}{a^2} + \Delta \right) \epsilon^{-1} \frac{\partial \eta_\theta}{\partial x} \tag{2.13}
\]

whose solution will be sought in the asymptotic series

\[
\phi = \phi_0 + \epsilon \phi_1 + \epsilon^2 \phi_2 + \cdots. \tag{2.14}
\]

It is useful first, however, to consider the \( x \) average of (2.13). Denoting the \( x \) average of any quantity by an overbar, we obtain

\[
\left( \mu - \frac{\partial}{\partial T} + r \right) \bar{\nabla}^2 \tilde{\phi} = - \frac{\partial}{\partial y} \left[ \frac{\partial \phi}{\partial x} \left( \eta_\theta + \epsilon \nabla^2 \phi \right) \right]. \tag{2.15}
\]

The \( O(1) \) problem which results from inserting (2.14) into (2.13) is simply

\[
\frac{\beta}{a^2} \frac{\partial}{\partial x} (\nabla^2 \phi_0 + a^2 \phi_0) = 0. \tag{2.16}
\]

whose solution is

\[
\phi_0 = A W + *, \tag{2.17}
\]

where

\[
W = \frac{e^{ikx}}{2} \sin ly. \tag{2.18}
\]

The solution \( \phi_0 \) is simply a nearly stationary Rossby wave whose amplitude \( A(T) \) is a slowly varying function of time and whose magnitude will depend on the size of the topographic forcing and the nonlinear, near resonance dynamics. The evolution equation for \( A \) is derived by considering the higher order problems for \( \phi_1 \) and \( \phi_2 \).

Since \( \mu = O(\Delta) = O(\epsilon) = O(h_0/e) = O(\epsilon^3) \), the order \( \epsilon \) problem is simply

\[
\frac{\beta}{a^2} \frac{\partial}{\partial x} (\nabla^2 \phi_1 + a^2 \phi_2) = -J(\phi_0, \nabla^2 \phi_0) = 0 \tag{2.19}
\]

since \( \nabla^2 \phi_0 \) is a constant multiple of \( \phi_0 \). The solutions for \( \phi_2 \) may be either a multiple of \( \phi_0 \), which can be eliminated by merely renormalizing \( A(T) \) to include it, or a zonal flow correction which is nontrivial. Thus, I take

\[
\phi_1 = \Phi_1(y, T). \tag{2.20}
\]

At this stage \( \Phi_1 \) is an undetermined function. The \( O(\epsilon^3) \) problem for \( \phi_2 \) is

\[
\beta \frac{\partial}{\partial x} (\nabla^2 \phi_2 + a^2 \phi_2)
\]

\[
= -\frac{\mu}{\epsilon^2} \frac{\partial}{\partial T} \nabla^2 \phi_0 - \Delta \frac{\partial}{\partial x} \nabla^2 \phi_0 - \frac{r}{\epsilon^2} \nabla^2 \phi_0
\]

\[
- \frac{\partial \phi_0}{\partial x} \frac{\partial \Phi_1}{\partial y} + \frac{\partial \Phi_1}{\partial y} \frac{\partial}{\partial x} \nabla^2 \phi_0 - \frac{\beta}{a^2 \epsilon^2} \frac{\partial \eta_\theta}{\partial x}
\]

\[
= a^2 \left( \frac{\mu}{\epsilon^2} \frac{\partial \phi_0}{\partial T} + \frac{r}{\epsilon^2} \phi_0 + \Delta \frac{\partial \phi_0}{\partial x} \right)
\]

\[
- \frac{\partial \phi_0}{\partial x} \left( \frac{\partial \Phi_1}{\partial y} + a^2 \frac{\partial \Phi_1}{\partial y} \right) - \frac{\beta}{a^2 \epsilon^2} \frac{\partial \eta_\theta}{\partial x}. \tag{2.21}
\]

Portions of the right-hand side of (2.21), unless otherwise constrained, have projections on the homogeneous solutions of the left-hand side of (2.21). To keep \( \phi_2 \) bounded we insist that the right-hand side have no projection on the homogeneous solutions (which are proportional to \( W \)) which leads to an equation for \( A \) immediately, i.e.,

\[
\frac{\mu}{\epsilon^2} \frac{dA}{dT} + \frac{r}{\epsilon^2} A + i k \frac{\Delta}{\epsilon^2} A
\]

\[
+ i \frac{kl}{a^2} A \int_0^1 \sin ly \left( \frac{\partial \Phi_1}{\partial y} + a^2 \Phi_1 \right) dy
\]

\[
= \frac{\beta}{a^2 \epsilon^2} ikh_0. \tag{2.22}
\]

To close the problem a connection between \( \Phi_1 \) and \( A \) must be found. For this we may use (2.15), wherein \( \phi = e^{i \Phi_1} \), so that the right-hand side must be calculated to \( O(\mu \epsilon) = O(\epsilon^3) \). In particular, the term \( \epsilon (\partial \phi/\partial x) \nabla^2 \phi \) requires the \( x \) average of products of the \( O(1) \) and \( O(\epsilon^3) \) fields \( \phi_0 \) and \( \phi_2 \). However, it is not necessary to calculate \( \phi_2 \) in detail since it may easily be shown that
\[ \frac{\partial \phi}{\partial x} \nabla^2 \phi = e^3 \left( \frac{\partial \phi_1}{\partial x} \nabla^2 \phi_2 + \frac{\partial \phi_2}{\partial x} \nabla^2 \phi_1 \right) \]
\[ = e^3 \left( \frac{\partial \phi_1}{\partial x} \nabla^2 \phi_2 - a^2 \frac{\partial \phi_2}{\partial x} \frac{\partial \phi_1}{\partial x} \right) \]
\[ = -e^3 \phi_0 \frac{\partial}{\partial x} \left( \nabla^2 \phi_2 + a^2 \phi_2 \right) \] equation (2.23).

This can be easily evaluated with the aid of equation (2.21) directly to yield
\[ e \frac{\partial \phi}{\partial x} \nabla^2 \phi \]
\[ = -e^3 \frac{a^4}{2\beta} \left( \frac{\partial \phi_0}{\partial T} + 2r \phi_0^2 \right) + \phi_0 \frac{\partial \eta_x}{\partial x}. \] equation (2.24).

Thus, equation (2.15) becomes
\[ \left( \frac{\mu}{e^2} \frac{\partial}{\partial T} + r \right) \frac{\partial^2 \Phi_1}{\partial y^2} = \frac{a^4}{4\beta} \left( \frac{d}{d|A|^2} + 2r|A|^2 \right) \sin 2ly, \] equation (2.25)

where $|A|$ is the absolute value of the complex amplitude $A$.

Eqs. (2.22) and (2.25) close the system for the determination of $A$ and the correction to the nearly resonant zonal flow. Note that topography is not explicitly a forcing term for the zonal flow. In agreement with the ideas of wave mean flow interaction theories only the secular change of $|A|$ and frictional dissipation in the wave field alter the zonal flow.

However, by consideration of (2.22) and its complex conjugate it is a simple matter to show that
\[ \frac{\mu}{e^2} \frac{d}{dT} |A|^2 + 2r |A|^2 = \frac{\beta}{e^3} \left( h_0 A - h_0^* A \right), \] equation (2.26)

so that equation (2.25) could be rewritten
\[ \left( \frac{\mu}{e^2} \frac{\partial}{\partial T} + \frac{r}{e^2} \right) \frac{\partial^2 \Phi_1}{\partial y^2} = i \frac{k l}{4e^3} \left( h_0 A - h_0^* A \right) \sin 2ly. \] equation (2.27)

The first integral in $y$ of equation (2.27) yields the equation for the zonal momentum correction $U_1 = -\partial \Phi_1 / \partial y$, i.e., noting (2.8),
\[ \left( \frac{\mu}{e^2} \frac{\partial}{\partial T} + \frac{r}{e^2} \right) U_1 \]
\[ = -i \frac{k}{e^3} \left( h_0 A - h_0^* A \right) (1 - \cos 2ly), \] equation (2.28)

the right-hand side of which is directly proportional to the wave drag on the bottom topography. Thus, the wave drag is responsible for the alteration of the mean flow insofar as it yields a nonzero value of $\mu d |A|^2 / dT + 2r |A|^2$. According to (2.28), this will be due entirely to that part of $A$ which is 90° out of phase in $x$ with the mountain range. That is, if we take $h_0$ to be strictly real (without any loss of generality) equation (2.27) becomes
\[ \left( \frac{\mu}{e^2} \frac{\partial}{\partial T} + \frac{r}{e^2} \right) \frac{\partial^2 \Phi_1}{\partial y^2} = \frac{k l}{2e^3} h_0 A \sin 2ly, \] equation (2.29)

where $A_1$ is the imaginary part of $A$, $(A = A_r + iA_i)$.

3. Inviscid topographic instability (barotropic)

In the absence of friction, a steady solution for $A$ may be easily found since by equation (2.25) $\Phi_1$ is then identically zero and
\[ \epsilon A_0 = \epsilon A_{0r}, \quad \frac{\beta}{a^2} \Delta = \frac{h_0}{U} \frac{(U - a^2 - 1)}{\beta}. \] equation (3.1)

Of course, this solution is not finite at resonance where $\Delta \rightarrow 0$ and the nonlinear theory is required to determine the amplitude then. For the moment, I exclude the point $\Delta = 0$.

The stability of equation (3.1) to small perturbations may be easily studied by writing
\[ A = A_0 + A', \quad \Phi_1 = \Phi_1', \] equation (3.2)

where $A'$ and $\Phi_1'$ are small. Then the linearized form of equation (2.29) (or 2.25) yields (in the absence of friction)
\[ \Phi_1' = A_0 A_1' \left[ 2l \left(y - \frac{1}{2}\right) \sin 2ly \right] \] equation (3.3)

which when inserted into the linearized forms of equation (2.22) yields (after separation into real and imaginary parts)
\[ \frac{dA_1'}{dT} - \frac{k \Delta}{\mu} A_1' = 0, \] equation (3.4a)
\[ \frac{dA_1'}{dT} + k \frac{\Delta}{\mu} A_1' - \frac{k e^2 \Delta}{8 \mu} A_r (3k^2 - l^2) = 0 \] equation (3.4b)

or
\[ \mu^2 \frac{d^2 A_1'}{dt^2} = k^2 A_1' \left[ \Delta \frac{e^2 \Delta}{8} (3k^2 - l^2) - \Delta^2 \right]. \] equation (3.5)

Thus,
\[ A_1' = A_1'(0) e^{\sigma T / \mu}, \] equation (3.6)

where
\[ \sigma^2 = k^2 \left[ \Delta \frac{(e^2 \Delta)}{8} (3k^2 - l^2) - \Delta^2 \right]. \] equation (3.7)

As $l \rightarrow 0$, the result in equation (3.7) agrees with the result of Charney and Flierl (1981) who redis the meridi-
onally infinite example of Hart somewhat more explicitly, if terms which are small near resonance are ignored in their result. The factor \((3k^2 - l^2)^2\) which occurs in (3.7) follows from the use of (3.3) in (2.2) and yields a qualitative result which differs from both the severely truncated models and the meridionally infinite case \((l \to 0)\). Namely, topographic instability will occur only for super-resonant flows \((\Delta > 0)\) as long as \(3k^2 > l^2\). This condition, in fact, has been the case considered in all the earlier studies. However, for sufficiently long waves for which \(k < l/\sqrt{3}\) the instability will occur only on the sub-resonant side \((\Delta < 0)\). In each case the range of instability is
\[
(U - \beta/a^2)(3k^2 - l^2) > 0
\]
and
\[
|U - \beta/a^2| < \frac{\epsilon^2A_\infty^2}{8} |3k^2 - l^2|.
\]

4. Multiple equilibria (barotropic)

Let
\[
\Phi_1 = \frac{kh_0}{8l\epsilon\mu} [\sin2ly - 2l(\gamma - 1/2)]\psi(t)
\]
and choose \(\epsilon\) and \(\mu\) so that
\[
\epsilon = \left[ \frac{8h_0}{(3k^2 - l^2)} A_\infty^2 \right]^{1/3},
\]
\[
\mu = \frac{\beta}{a^4} \frac{kh_0}{\epsilon} \left[ \frac{\beta^{1/3}h_0^{2/3}}{2} \left( \frac{3k^2 - l^2}{} \right)^{1/3} \right].
\]

Then (2.22) and (2.29) reduce to
\[
\frac{d\psi}{dT} + \eta\psi = -A_i,
\]
\[
\frac{dA_i}{dT} + \eta A_i - (\delta + \psi)A_i = 0,
\]
\[
\frac{dA_r}{dT} + \eta A_i + (\delta + \psi)A_r = 1,
\]
where
\[
\eta = \epsilon/\mu, \quad \delta = k\Delta/\mu.
\]

These equations agree with Hart's (1979) equations derived for the case of a mountain of infinite extent if 1) his parameters are set close to resonance and 2) his topographic amplitude \(S\) is considered to be small. Thus his detailed calculations concerning multiple equilibria apply directly to (4.3a,b,c) and the reader is referred to his paper for details of the discussion.

However, it is instructive to examine the question of multiple equilibria in the present model since its relative simplicity allows explicit representations of the solution not readily accessible in the earlier studies.

Steady solutions of (4.3a,b,c) are possible if
\[
A_i = -\eta\psi,
\]
\[
A_r = -\left(\delta + \psi\right)\psi
\]
from which (4.3c) yields a cubic equation for \(\psi\), i.e.,
\[
\eta^2\psi + (\delta + \psi)^2\psi = -1.
\]
Since by (4.1)
\[
U_1 = \frac{kh_0}{4l\epsilon\mu} (1 - \cos2ly)
\]
negative values of \(\psi\) correspond to a reduction of the zonal flow by the mountain drag in the steady state. Although the cubic could be solved directly for \(\psi(\delta, \eta)\) it is more revealing to reverse the procedure and solve for \(\delta\) as a function of \(\psi\) to obtain
\[
\delta = -\psi \pm \left( \frac{1}{\psi} - \eta^2 \right)^{1/2}.
\]
It is immediately clear that all steady solutions must have \(\psi < 0\) (i.e., a diminishment of the zonal flow) while at the same time \(\psi\) reaches a numerical maximum of \(\eta^2\) at \(\delta = -\eta^2\) for small \(\eta\), and \(\psi = 1 + \eta^2/3\) at \(\delta = 0\) so that the solution at the point of resonance remains finite. It is simple to sketch the solution for \(\psi(\delta)\) with the aid of (4.7) and the result is shown in Fig. 1 for the case \(\eta^2 = 0.1\). For all values of \(\delta\) to the right of point \(B\) in Fig. 1 three solutions exist. For small \(\eta\), as in the case of the figure, it is easy to show that on the upper branch \(\psi \sim -\delta - \delta^{-1/2}\), on the intermediate branch \(\psi \sim -\delta + \delta^{1/2}\), while the lowest branch has \(\psi \sim -\delta^{-2}\). The total degree of super-resonance, in the scaled variables of (4.3a,b,c), is simply \(\delta + \psi\) when the correction to the zonal flow is considered. Hence the upper branch is subresonant, the intermediate branch is slightly super-resonant, and the lowest branch has \(\psi = \delta = 0\) and is considerably super-resonant. It is intuitively clear, as a simple stability analysis can show, that the upper and lowest branches are stable (to small disturbances) while the intermediate branch is unstable. The instability of the intermediate branch is qualitatively similar to the topographic instability of Section 3. However, in that problem the basic state whose stability was considered is actually the solution on the lowest branch and instability required \(0 < \delta < 1\). A glance at Fig. 1 shows that, in fact, no equilibrium except the upper branch exists there in finite amplitude.

The point \(B\) in Fig. 1 marks the beginning of the \(\delta\) range of multiple equilibria. It is characterized by the condition that \(\delta\psi/\delta\delta \to \infty\) or equivalently that \(\delta\delta/\delta\psi \to 0\). The latter condition is easy to apply to (4.7) and yields
\[
\frac{\delta \delta}{\delta \psi} = -1 \pm \frac{1}{2} (-\psi^{-1} - \eta^2)^{1/2} \psi^{-2} = 0.
\]
For small $\eta$ the solution to (4.8) which corresponds to point $B$ in Fig. 1 is
\[ \psi \sim -0.62996 \]
which occurs at
\[ \delta = 1.8898. \tag{4.9} \]

Therefore multiple equilibria exist for all $\eta^2 > \delta > 1.889$ (for $\eta \ll 1$). Note that this is distinct from the stability threshold of the linear solution which, for $\eta \to 0$, required $0 \leq \delta \leq 1$. Further, the $\delta$ range for multiple equilibria becomes larger as $\eta \to 0$.

For sufficiently large values of $\eta$ no solution of (4.8) is possible and only one equilibrium solution for $\psi$ exists. The critical value of $\eta$ can easily be shown to be $\sqrt{3}/2$. For $\eta > \sqrt{3}/2$ only one equilibrium exists. Fig. 2 shows the steady state solution curves at $\eta = \sqrt{3}/2$ and $\eta = 1$ and it is apparent that the response curve at $\eta = \sqrt{3}/2$ is just barely single valued and has a vertical tangent at $\delta = 1.5$. Curves for smaller $\eta$ will contain multiple solutions as in Fig. 1. Higher values of $\eta$ will give rise to stable, single valued responses qualitatively similar to the linear, dissipative solution for which $\psi = -(\delta^2 + \eta^2)^{-1}$.

In this range of large $\eta$ the sole qualitative effect of nonlinearity is to slightly shift the peak of the response curve to $\delta > 0$.

Hart has shown by direct numerical calculation that for arbitrary initial conditions the solution will asymptotically be captured by one or the other of the stable branches of these equilibrium solutions.

In the interval for which multiple equilibria exist $\delta > 1.889$ the stable upper state, which has $\psi \approx -\delta - \delta^{-1/2}$, corresponds to a relatively weak zonal flow with a large wave amplitude $A \sim \delta^{1/2} + i\eta\delta$, while
the stable lower state, for which \( \psi = -1/\delta^2 \), has a stronger zonal flow and a relatively weak wave amplitude \( A = 1/\delta + i\eta/\delta^2 \). The former has been suggested as the prototypical "blocked" flow pattern. Fig. 1 reveals that if \( \delta \) is decreased slowly past the point \( B \), corresponding to a value of zonal velocity

\[
U = \frac{\beta}{a^2} \left[ 1 + 1.8898 \frac{k}{2} \frac{h_0^{2a}}{\beta^{2a}} \left( \frac{3k^2 - l^2}{a^2} \right)^{1/3} \right] \tag{4.10}
\]

the wave amplitude should suddenly increase by a factor \( \delta^{3/2} \) as the blocking state is entered. As \( U \) and \( \delta \) subsequently are then increased the flow remains in the blocking state until the point \( \delta = \eta^{-2} \) is reached, at which point

\[
U = \frac{\beta}{a^2} \left[ 1 + \frac{h_0^2}{8r^2} \frac{3k^2 - l^2}{a^2} \right] \tag{4.11}
\]

and the wave amplitude will plunge down to the unblocked state.

5. The baroclinic model

It will come as no surprise that the interaction dynamics between wave and topography are considerably more complicated in a stratified fluid. Nevertheless the barotropic analysis described above will be a useful guide in the analysis of the baroclinic problem.

For a two-layer model on the \( \beta \) plane, with the flow contained in a channel of width \( L \) the dimensionless quasi-geostrophic equations are

\[
\frac{\partial}{\partial t} \left[ \nabla^2 \psi_1 - F(\psi_1 - \psi_2) \right] + J[\psi_1, \nabla^2 \psi_1 - F(\psi_1 - \psi_2) + \beta y] = -r_1 \nabla^2 \psi_1 - r_1 \nabla^2 (\psi_1 - \psi_2) + \kappa F(\psi_2 - \psi_3) + Q_1, \tag{5.1a}
\]

\[
\frac{\partial}{\partial t} \left[ \nabla^2 \psi_2 - F(\psi_2 - \psi_1) \right] + J[\psi_2, \nabla^2 \psi_2 - F(\psi_2 - \psi_1) + \beta y + \eta_B] = -r_2 \nabla^2 \psi_2 - r_2 \nabla^2 (\psi_2 - \psi_1) + \kappa F(\psi_2 - \psi_1) + Q_2. \tag{5.1b}
\]

\( \psi_1 \) and \( \psi_2 \) are the nondimensional geostrophic streamfunctions in each layer (nondimensionalized by \( UL \) as in Section 2). The parameter

\[
F = f_0 L^2 / g \frac{\Delta \rho}{\rho} D,
\]

where \( L \) is the width of the channel, \( f_0 \) is the Coriolis parameter, \( g \) is gravity, \( D \) is the depth of each layer and \( \Delta \rho / \rho \) is the proportional density increase of the lower layer with respect to the upper layer. Variables defined for the upper and lower layers have subscripts 1 and 2, respectively. \( \eta_B \) is defined as in (2.3). The right-hand side of (5.1a, b) represents various dissipative and forcing terms. The term \( r_n \nabla^2 \psi_n \) \((n = 1, 2)\) represents the effect of Ekman layers on the upper and lower surfaces of the flow. If the upper layer is stress-free, \( r_1 = 0 \). The term \( r_i \nabla^2 (\psi_i - \psi_0) \) represents the effect of friction acting on the interface between the two layers while the final term, \( \kappa F(\psi_1 - \psi_2) \), is a dissipative term proportional to the interface height. In analogy with a continuous system this term may be thought of as a thermal damping term. As before, the term \( Q_n \) is an external potential vorticity source, which in the absence of bottom topography would produce a zonal flow with a streamfunction

\[
\Psi_n = -\int U_n dy \tag{5.2}
\]

which satisfies

\[
r_n \frac{d^2 \Psi_n}{dy^2} + (-1)^n [r_i \nabla^2 (\Psi_1 - \Psi_2)] + \kappa F(\Psi_1 - \Psi_2)] = Q_n. \tag{5.3}
\]

The total streamfield in the presence of a small wavelike topography

\[
\eta_B = h_0 W + \ast \tag{5.4}
\]

is

\[
\Psi_n = \Psi_n(y) + \varepsilon \phi_n. \tag{5.5}
\]

For simplicity, \( U_n \) will again be chosen to be independent of \( y \). Now, in the absence of dissipation, nonlinearity and topography, propagating wavelike solutions for \( \phi_n \) exist in the form

\[
\phi_n = A_n \frac{e^{i(kx-cy)}}{2} \sin ly + \ast, \tag{5.6}
\]

where \( c \) is given by

\[
c = \frac{U_1 + U_2}{2} - \frac{a^2 + F/ \beta}{a^2 + 2F/ a^2} \pm \frac{\sqrt{4F^2 \beta^2 - (U_1 - U_2)^2 a^4 ((4F^2 - a^4)^{1/2})}}{2 a^2 (a^2 + 2F)}, \tag{5.7}
\]

where \( a^2 = k^2 + l^2 \).

Consider the case which will be of interest in the present problem where \( U_2 \) is zero. Then it is a simple matter to show that stationary wave solutions with \( c = 0 \), of particular importance in the topographic resonance problem, will occur either when

\[
U_1 = \beta/a^2 \tag{5.8}
\]

or

\[
U_1 = \beta/F. \tag{5.9}
\]

The first case corresponds to a stationary Rossby wave confined entirely to the upper layer, i.e., for
which the wave amplitude in the lower layer vanishes. The second case corresponds to that value of the vertical shear for which the potential vorticity gradient of the lower layer vanishes. This latter value of $U_1$ corresponds to the minimum critical shear required for baroclinic instability in the absence of topography. The wave which is marginally stable when (5.9) is satisfied has a wavenumber $a^2 = 2\beta$. At that wavenumber the vertical shear given by (5.8) is $U_1 = \beta \sqrt{2} F < \beta F$. Hence, the wave defined by (5.8) in the wavenumber-vertical shear plane is always subcritical with respect to the classical baroclinic instability threshold as long as $a^2 > F$. Attention will be restricted to that range. If, as it turns out to be the case, topographic instability occurs whenever the resonance (or stationary wave) condition (5.8) is met, it can occur at values of the vertical shear which are considerably less than the standard baroclinic threshold especially for $a^2 \gg F$, i.e., for wavelengths considerably smaller than a deformation radius.

Therefore attention is focused on cases where

$$U_2 = 0,$$
$$U_1 = \frac{\beta}{a^2} + \Delta \equiv U_1 + \Delta. \quad (5.10a)$$

It turns out to be appropriate to consider the parameter ordering

$$h_0 = O(\epsilon),$$
$$\Delta = O(\epsilon^2),$$
$$r_n = O(r_i) = O(\kappa) = O(\epsilon^2). \quad (5.11)$$

Similarly, the time $t$, hitherto scaled on the ordinary advective time, is rescaled so that all variables are a function only of the "slow" time $T = \mu t$ where

$$\mu = O(\epsilon). \quad (5.12)$$

If these transformations and rescalings, along with (5.3) are substituted into (5.1a,b) we obtain

$$\mu \frac{\partial}{\partial T} \left[ \nabla^2 \phi_1 - F(\phi_1 - \phi_2) \right]$$
$$+ \frac{\partial \phi_1}{\partial x} (\beta + FU_1 + F\Delta) + \epsilon J(\phi_1, \nabla^2 \phi_1 + F\phi_2)$$
$$= -r_1 \nabla^2 \phi_1 - r_1 \nabla^2 (\phi_1 - \phi_2) + \kappa F(\phi_1 - \phi_2), \quad (5.13a)$$

$$\mu \frac{\partial}{\partial T} \left[ \nabla^2 \phi_2 - F(\phi_2 - \phi_1) \right] + \frac{\partial \phi_2}{\partial x} (\beta - FU_1 - F\Delta)$$
$$+ \epsilon J(\phi_1, \nabla^2 \phi_2 + F\phi_1) + J(\phi_2, \eta_B)$$
$$= -r_2 \nabla^2 \phi_2 - r_2 \nabla^2 (\phi_2 - \phi_1) + \kappa F(\phi_2 - \phi_1). \quad (5.13b)$$

As in the barotropic problem it is helpful to first derive an equation for the zonally averaged stream-function. Applying a zonal average to (5.13a, b) yields

$$\mu \frac{\partial}{\partial T} \left[ \nabla^2 \bar{\phi}_1 - F(\bar{\phi}_1 - \bar{\phi}_2) \right] + r_1 \frac{\partial^2 \bar{\phi}_1}{\partial y^2}$$
$$+ r_1 \frac{\partial^2}{\partial y^2} (\bar{\phi}_1 - \bar{\phi}_2) - \kappa F(\bar{\phi}_1 - \bar{\phi}_2)$$
$$= -\epsilon \frac{\partial}{\partial y} \left[ \frac{\partial \phi_1}{\partial x} (\nabla^2 \phi_1 + F\phi_2) \right], \quad (5.14a)$$

$$\mu \frac{\partial}{\partial T} \left[ \nabla^2 \bar{\phi}_2 - F(\bar{\phi}_2 - \bar{\phi}_1) \right] + r_2 \frac{\partial^2 \bar{\phi}_2}{\partial y^2}$$
$$+ r_2 \frac{\partial^2}{\partial y^2} (\bar{\phi}_2 - \bar{\phi}_1) - \kappa F(\bar{\phi}_2 - \bar{\phi}_1)$$
$$= -\epsilon \frac{\partial}{\partial y} \left[ \frac{\partial \phi_2}{\partial x} (\nabla^2 \phi_2 + F\phi_1 + \eta_B/\epsilon) \right]. \quad (5.14b)$$

The boundary conditions for $\bar{\phi}_n$ are

$$\frac{\partial \bar{\phi}_n}{\partial y} = 0 \quad \text{on} \quad y = 0, 1; \quad n = 1, 2, \quad (5.15)$$

while the wavelike part of $\phi_n$ satisfies

$$\frac{\partial \phi_n}{\partial x} = 0 \quad \text{on} \quad y = 0, 1; \quad n = 1, 2. \quad (5.16)$$

The fluctuation streamfunction $\phi_n$ is expanded in the asymptotic series

$$\phi_n = \phi_n^{(0)} + \epsilon \phi_n^{(1)} + \epsilon^2 \phi_n^{(2)} + \cdots \quad (5.17)$$

and the series is then substituted into (5.13a, b) to yield a sequence of problems for the $\phi_n^{(0)}$ after use is made of the ordering relations (5.11) and (5.12).

The O(1) problem obtained from (5.13b) is simply

$$\frac{\partial \phi_2^{(0)}}{\partial x} = 0. \quad (5.18)$$

Since the potential vorticity gradient in the lower layer is different from zero (indeed, it is positive as long as $\beta > FU_1$, i.e., as long as $a^2 > F$) the solution to (5.18) is simply

$$\phi_2^{(0)} = 0. \quad (5.19)$$

So, as in the linear solution the O(1) perturbation field is entirely limited to the upper layer streamfunction, $\phi_1^{(0)}$, which satisfies the O(1) portion of (5.13a), viz.,

$$U_1 \frac{\partial}{\partial x} \nabla^2 \phi_1^{(0)} + \beta \frac{\partial \phi_1^{(0)}}{\partial x} = 0 \quad (5.20)$$

or

$$\phi_1^{(0)} = A \frac{e^{kx}}{2} \sin y + * = AW + * \quad (5.21)$$
as long as
\[ U_1 = \frac{\beta}{a^2}. \]  

(5.22)

Thus the O(1) problem merely recovers the form of the linear, inviscid stationary wave which will be in resonance with the topography of the same wavenumber while nonlinear, dissipative dynamics will determine the amplitude.

At O(\(\epsilon\)), (5.13b) yields, again,
\[ (\beta - FU_1) \frac{\partial \phi^{(1)}_2}{\partial x} = 0. \]  

(5.23)

At this point it is necessary to anticipate the fact that the mean flow correction streamfunction will be O(\(\epsilon\)). Hence, an important nontrivial solution to (5.23) is
\[ \phi^{(1)}_2 = \Phi_2(y, T). \]  

(5.24)

The O(\(\epsilon\)) portion of (5.13a) yields
\[ U_1 \left[ \frac{\partial}{\partial x} \nabla^2 \phi^{(1)}_1 + \beta \frac{\partial \phi^{(1)}_1}{\partial x} \right] = 0. \]  

(5.25)

As in the barotropic problem, one solution to (5.25) is identical to the O(1) free wave and can always be included in our definition of \(A\) by renormalization. The important nontrivial solution to (5.25) is a correction to the upper layer zonal flow,
\[ \phi^{(1)}_1 = \Phi_1(y, T). \]  

(5.26)

The O(\(\epsilon^2\)) problem from (5.13b) becomes
\[ \left( \beta - FU_1 \right) \frac{\partial \phi^{(2)}_2}{\partial x} = -\frac{\mu}{\epsilon^2} F \frac{\partial \phi^{(0)}_1}{\partial T} + \frac{r_1}{\epsilon^2} \nabla^2 \phi^{(1)}_1 - \kappa \frac{\epsilon}{\epsilon^2} F \nabla^2 \phi^{(0)}_1 + \frac{\partial \Phi_2}{\partial y} \frac{\partial}{\partial x} \left[ \phi^{(0)}_1 + \frac{\epsilon}{\epsilon^2} \eta_B \right] \]  

(5.27)

where, we recall \(\eta_B = O(\epsilon)\). The fluctuation streamfunction in the lower layer, \(\phi^{(2)}_2\), is driven by the temporal and dissipative effects of the O(1) upper layer flow as well as by the zonal advection by the O(\(\epsilon\)) zonal velocity of the total thickness variation in the \(x\) direction. The thickness variation is due to the interface displacement of the O(1) upper layer wave and by the variable bottom topography.

From (5.14b) it follows that the lowest-order forcing of the mean flow in the lower layer is due to the term
\[ \frac{\partial}{\partial y} \left[ \frac{\partial \phi^{(2)}_2}{\partial x} \left( \nabla^2 \phi^{(0)}_2 + F \phi^{(0)}_1 + \frac{\eta_B}{\epsilon} \right) \right]. \]  

(5.28)

If (5.27) is used it follows that
\[ -\epsilon \frac{\partial \phi^{(2)}_2}{\partial x} \left[ \nabla^2 \phi^{(0)}_2 + F \phi^{(0)}_1 + \frac{\eta_B}{\epsilon} \right] \]
\[ = \epsilon^3 \left( \beta - FU_1 \right)^{-1} \left[ \frac{\mu}{\epsilon^2} F \frac{\partial}{\partial T} \phi^{(0)}_1 \right] \]
\[ - \frac{r_1}{\epsilon^2} \nabla^2 \phi^{(0)}_1 + k F \nabla^2 \phi^{(0)}_1 \]
\[ + \left( \beta - FU_1 \right)^{-1} \epsilon^3 \left[ \frac{\eta_B}{\epsilon} \left( \frac{\mu}{\epsilon^2} F \phi^{(0)}_1 \right) \frac{\partial}{\partial T} \phi^{(0)}_1 \right] \]
\[ - \frac{r_1}{\epsilon^2} \nabla^2 \phi^{(0)}_1 + k F \phi^{(0)}_1 \]  

(5.29)

Thus even though the potential vorticity flux in the lower layer is O(\(\epsilon^3\)) it may be evaluated solely in terms of the topography and the O(1) streamfunction.

Turning to the O(\(\epsilon^2\)) problem for the upper layer, the O(\(\epsilon^2\)) portion of (5.13a) yields
\[ U_1 \frac{\partial}{\partial x} \left[ \nabla^2 \phi^{(2)}_1 + F \phi^{(2)}_2 - F \phi^{(2)}_1 \right] = \frac{\partial \phi^{(2)}_1}{\partial x} \left( \beta + FU_1 \right) \]
\[ = -\frac{\mu}{\epsilon^2} \frac{\partial}{\partial T} \left[ \nabla^2 \phi^{(0)}_1 - F \phi^{(0)}_1 \right] - \frac{\Delta}{\epsilon^2} \frac{\partial}{\partial x} \nabla^2 \phi^{(0)}_1 \]
\[ + \frac{\partial \Phi_1}{\partial y} \frac{\partial}{\partial x} \left[ \nabla^2 \phi^{(0)}_1 \right] \]
\[ - \frac{\partial \phi^{(0)}_1}{\partial x} \frac{\partial}{\partial y} \left[ \nabla^2 \phi^{(0)}_1 - F \left( \Phi_1 - \Phi_2 \right) \right] \]
\[ - \frac{r_1}{\epsilon^2} \nabla^2 \phi^{(0)}_1 \]
\[ - \frac{r_1}{\epsilon^2} \nabla^2 \phi^{(0)}_1 + \frac{k F}{\epsilon^2} \phi^{(0)}_1 \]  

(5.30)

where, note, \(\phi^{(2)}_1\) is given by (5.27).

The \(x\)-averaged potential vorticity flux required to evaluate the mean flow equation for the upper layer involves both the Reynolds stress and heat flux contributions. [The Reynolds stress in the lower layer is zero to O(\(\epsilon^2\)).] Thus to O(\(\epsilon^2\))
\[ -\epsilon \frac{\partial \phi_1}{\partial x} \left( \nabla^2 \phi_1 + F \phi_2 \right) = -\epsilon \left[ \frac{\partial}{\partial x} \left( \phi^{(0)}_1 + \frac{\epsilon^2 \phi^{(2)}_2}{\epsilon} \right) \right] \]
\[ \times \left[ \nabla^2 \phi^{(0)}_1 + \frac{\epsilon^2 \nabla^2 \phi^{(2)}_1 + \epsilon^2 F \phi^{(2)}_1}{\epsilon} \right]. \]

(5.31a)

If use is made of the fact that \(\nabla^2 \phi^{(0)}_1 = -a^2 \phi^{(0)}_1\), this reduces to
\[
\begin{align*}
\phi_1^{(1)} & = \frac{\partial}{\partial x} \left( \nabla^2 \phi_1 + F \phi_2 \right) \\
& = +e^3 \left[ \phi_1^{(0)} \frac{\partial}{\partial x} \left( \nabla^2 \phi_1^{(2)} + a^2 \phi_1^{(2)} + F \phi_2^{(2)} \right) \right]. \quad (5.31b)
\end{align*}
\]

Now the left-hand side of (5.30) may be rewritten as
\[
\begin{align*}
U_1 \frac{\partial}{\partial x} \left[ \nabla^2 \phi_1^{(2)} + a^2 \phi_1^{(2)} + F \phi_2^{(2)} \right]
\end{align*}
\]

if (5.22) is used, so that the potential vorticity flux of the upper layer may also be calculated to the required order without the detailed calculation of \( \phi_1^{(1)} \), since by (5.30)

\[
\begin{align*}
\frac{\mu}{\epsilon^2} \frac{\partial}{\partial T} \left[ \frac{\partial^2 \Phi_1}{\partial y^2} - F(\Phi_2 - \Phi_1) \right] + \frac{r_1}{\epsilon^2} \frac{\partial^2 \Phi_2}{\partial y^2} + \frac{r_1}{\epsilon^2} \frac{\partial^2}{\partial y^2} (\Phi_2 - \Phi_1) - \frac{\kappa}{\epsilon^2} F(\Phi_1 - \Phi_2)
\end{align*}
\]

\[
= \frac{l}{4U_1} \left[ (a^2 + F) \frac{\mu}{\epsilon^2} \frac{d}{dT} |A|^2 + 2 \frac{(r_1 a^2 + \kappa F)}{\epsilon^2} \frac{d}{dT} \right] \sin 2ly, \quad (5.34a)
\]

\[
\begin{align*}
\frac{\mu}{\epsilon^2} \frac{\partial}{\partial T} \left[ \frac{\partial^2 \Phi_2}{\partial y^2} - F(\Phi_2 - \Phi_1) \right] + \frac{r_1}{\epsilon^2} \frac{\partial^2 \Phi_2}{\partial y^2} + \frac{r_1}{\epsilon^2} \frac{\partial^2}{\partial y^2} (\Phi_2 - \Phi_1) - \frac{\kappa F}{\epsilon^2} (\Phi_2 - \Phi_1)
\end{align*}
\]

\[
= \frac{lF}{4(\beta - FU_1)} \left[ \frac{\mu F}{\epsilon^2} \frac{d |A|^2}{dT} + 2 \left( \frac{r_1 a^2 + \kappa F}{\epsilon^2} \right) \frac{d |A|^2}{dT} \right] \sin 2ly
\]

\[
+ \frac{l}{4(\beta - FU_1)} \left[ \left( \frac{\mu F}{\epsilon^2} \frac{dA}{dT} + \frac{(r_1 a^2 + \kappa F)}{\epsilon} \right) \frac{h_0^*}{\epsilon} + * \right] \sin 2ly, \quad (5.34b)
\]

where \( h_0^* \) is the complex conjugate of \( h_0 \).

There are two important features of (5.34a, b) to note. First, topography enters directly into the forcing of the lower layer zonal flow and depends on the secular amplitude change and dissipation in the wave. Second, and more significantly, the Reynolds stress flux contribution to the potential vorticity balance is as important as the corresponding thickness flux. Both the truncated model and the models with infinite meridional extent artificially suppress the Reynolds stress. In Hart’s model this suppression is physical consistent while in the truncated spectral models the suppression is entirely an artifact of the truncation.

To determine the amplitude equation for \( A \) it is only necessary to return to (5.30). If (5.27) is used to evaluate \( \phi_1^{(1)} \) and the resulting inhomogeneous equation for \( \phi_1^{(2)} \) is swept of secular terms in the usual way, the amplitude equation for \( A \) is easily derived as

\[
\begin{align*}
\mu \frac{dA}{dT} + A \left[ \frac{r_1}{\epsilon} + \frac{(a^2 - F)}{a^2} + \frac{\kappa}{\epsilon^2} \frac{F}{a^2} \right] + \frac{ik}{\epsilon^2} \frac{\Delta (a^2 - F)}{a^2} \frac{h_0^*}{\epsilon} + \frac{ik}{\epsilon} \frac{b}{a^4} \frac{F}{2} \int_0^1 \Phi_2 \sin 2lydy
\end{align*}
\]

\[
+ \frac{a^2 (a^2 - F) \Phi_1 + a^2 F \Phi_2}{\partial y^2} \sin 2lydy = 0. \quad (5.35)
\]

The first three terms in (5.35) would alone describe the dynamics of a slowly decaying Rossby wave which is slowly propagating eastward when \( \Delta (a^2 - F) > 0 \). The fourth term represents the interaction of the lower layer zonal flow with the topography which alters the time evolution of the wave, while the final term in (5.35) represents the interac-
tion of the wave with the potential vorticity gradient of the altered zonal flow. At the same time (5.34a, b) describes how the evolving, dissipative wave field will alter the zonal flow.

An important difference between the baroclinic and barotropic problems is that the wave is not directly forced by a preexisting zonal flow. That is, the topographic interaction is entirely dependent on the ability of the wave field to transfer mean zonal momentum from the upper layer to the lower layer to commence the topographic interaction. As Charney and Strauss (1980) point out, one solution of the above set of equations is \( A = \Phi_n = 0 \) so that the flow field is the zonal flow determined by (5.3). They term this a Hadley circulation and the role of topography only enters as a result of the mutual growth of \( A \) and \( \Phi_n \) due to topographic effects. Topography must be destabilizing to allow \( A \) to grow, for as long as \( a^2 > F \) the flow is subcritical with regard to ordinary baroclinic instability.

### 6. Baroclinic topographic instability

To examine the stability of the original zonal flow in the presence of topography it is necessary to consider the linearized version of (5.34a, b) and (5.35). To keep matters as simple as possible, the non-dissipative problem \( r_1 = r_2 = r_1 = \kappa = 0 \) is considered first. Then in the limit of small amplitude \( A \) (5.34a, b) becomes

\[
\frac{\partial}{\partial T} \left[ \frac{\partial^2 \Phi_1}{\partial y^2} - F(\Phi_1 - \Phi_2) \right] = 0, \tag{6.1a}
\]

\[
\frac{\partial}{\partial T} \left[ \frac{\partial^2 \Phi_2}{\partial y^2} - F(\Phi_2 - \Phi_1) \right] = \frac{l}{2(\beta - FU_1)} \frac{h_o}{\epsilon} \frac{dA_r}{dT} \sin 2ly \tag{6.1b}
\]

where, without loss of generality \( h_o \) has been taken to a real constant while \( A_r \) denotes the real part of \( A \).

It follows directly from (6.1a, b) after an integration in \( T \), that

\[
\frac{\partial^2}{\partial y^2} (\Phi_1 + \Phi_2) = \frac{IF}{2(\beta - FU_1)} h_o \frac{A_r}{\epsilon} \sin 2ly, \tag{6.2a}
\]

\[
\frac{\partial^2}{\partial y^2} (\Phi_2 - \Phi_1) = \frac{IF}{2(\beta - FU_1)} h_o \frac{A_r}{\epsilon} \sin 2ly, \tag{6.2b}
\]

from which it follows that

\[
\Phi_2 - \Phi_1 = -\frac{h_o/eFA_r}{2(4l^2 + 2F)(\beta - FU_1)} \times \left[ \sin 2ly - \frac{2l}{(2F)^{1/2}} \sinh(2F)^{1/2}(y - \frac{1}{2}) \right], \tag{6.3b}
\]

so that

\[
2\Phi_2 = -\frac{h_o/eFA_r}{(\beta - FU_1)4l(4l^2 + 2F)} \left[ (4l^2 + F) \sin 2ly \right.
\]

\[
- \frac{2l}{(2F)^{1/2}} \sinh(2F)^{1/2}(y - \frac{1}{2}) \left. - \frac{4l^3}{(2F)^{1/2}} \sinh(2F)^{1/2}(y - \frac{1}{2}) \right], \tag{6.4}
\]

The linearized inviscid form of (5.35) is simply

\[
m \frac{dA}{dT} + ikA \left( a^2 - F \right) A
\]

\[
+ \frac{kl}{a^4} h_o F e^{2I} \int_{0}^{\infty} \Phi_2 \sin 2lydy = 0. \tag{6.5}
\]

If (6.4) is used in (6.5) and the resulting equation is separated into real and imaginary parts, we obtain

\[
\mu \frac{dA_r}{dT} - kA \left( a^2 - F \right) A_r = 0, \tag{6.6a}
\]

\[
\mu \frac{dA_i}{dT} + kA \left( a^2 - F \right) A_r = \frac{kF^2}{a^2 - 8a^4} h_o^2
\]

\[
\times \left[ \frac{8l^2 + 3F + 32l^4 \tanh(F/2)^{1/2}}{(4l^2 + 2F)(2F)^{1/2}} \right] A_r = 0. \tag{6.6b}
\]

Exponentially growing solutions for \( A_r \) and \( A_i \), proportional to \( e^{\sigma T} \), may be found where \( \sigma \) is given by

\[
\mu^2 \sigma^2 = \frac{k^2 h_o^2}{8U_1 a^6} \frac{F^2}{(4l^2 + 2F)} \left[ 8l^2 + 3F + \frac{16l^4}{4l^2 + 2F} \right] 
\]

\[
\times \left| \frac{\tanh(F/2)^{1/2}}{(F/2)^{1/2}} \right| \tag{6.7}
\]

Thus, as long as

\[
0 < U_1 - \beta/a^2 < \Delta_c, \tag{6.8}
\]

where

\[
\Delta_c = \frac{h_o^2 F^2}{8\beta(a^2 - F)^2} \left[ 8l^2 + 3F + \frac{16l^4}{(4l^2 + 2F)} \right] 
\]

\[
\times \left| \frac{\tanh(F/2)^{1/2}}{(F/2)^{1/2}} \right| \tag{6.9}
\]

the original zonal flow will become unstable to a wavelike disturbance. As mentioned previously the required vertical shear for \( a^2 > F \) is subcritical with respect to the threshold for ordinary baroclinic instability and, furthermore, the critical shear required for the topographic instability goes down as the scale of the disturbance (and topography) gets smaller.
Hence small-scale topography requires, relatively speaking, only very weak shears for the instability. As \( a^2 \) increases \( \Delta_c \) diminishes like \( a^{-4} \) so for large \( a^2 \), i.e., small scales with respect to a deformation radius, the interval in range in \( \mathcal{U}_1 \) for instability becomes increasingly narrow. In a qualitative way this agrees with the results of Charney and Strauss (1980) inssofar as super-resonant conditions are required for instability although they indicate a short-wave cutoff in the inviscid topographic instability in their stability curve, which is not found in (6.9). This may not be a deficiency of the spectral method per se, for in their discussion of the results they also claim instability whenever \( \Delta > 0 \), for \( \Delta \ll 1 \).

It follows directly from (6.9) that the range in \( \mathcal{U}_1 \) of topographic instability increases like \( h^2 \). The formula for \( \sigma \), Eq. (6.7), shows that although the critical shear required for topographic instability is reduced as \( a^{-2} \), the growth rate \( \sigma \) diminishes at the same rate so that small-scale instabilities will be very slowly growing.

The total energy in the wave field to lowest order is

\[
\varepsilon^2 \left( \int dxdy \left[ \frac{(\nabla \Phi_1)^2}{2} + F\Phi_1^2 \right] \right) = \varepsilon^2 \frac{(a^2 + F)}{8} |A|^2, \tag{6.10}
\]

and its rate of change is

\[
\mu \varepsilon^2 \frac{(a^2 + F)}{8} \frac{d |A|^2}{dT} = O(\varepsilon^4). \tag{6.11}
\]

The only energy transformation term which is as large as \( O(\varepsilon^4) \) is the transformation of zonal available potential energy by the eddy heat flux. The smallness of the motion in the lower layer renders the wave drag contribution to the energy balance of \( O(\varepsilon^8) \), while the Reynolds Stress, though large as the eddy heat flux, can transfer energy only from the horizontally sheared portion of the zonal flow, so that it, too, is \( O(\varepsilon^6) \).

When the flow is inviscid, (5.27) implies that the eddy heat flux, integrated over the y interval (0, 1) is simply

\[
\varepsilon^2 F \int_0^1 \phi_1 \frac{\partial \Phi_2}{\partial x} dy = \frac{Fe^4}{\beta - FU_1} \int_0^1 \phi_1^0 \frac{\partial (x_2^2)}{\partial x} dy
\]

\[
= \left[ -\mu \varepsilon^2 \frac{F^2}{8} \frac{d |A|^2}{dT} \right.
+ \varepsilon^4 \left( \int_0^1 \frac{F}{\beta - FU_1} \phi_1^0 \frac{\partial \eta_0}{\partial x} dy \right) \left( \beta - FU_1 \right)
\]

\[
= \left[ -\mu \varepsilon^2 \frac{F^2}{8} \frac{d |A|^2}{dT} \right.
- \varepsilon^4 \frac{kl}{2} \frac{h_0}{\varepsilon} A_1
\]

\[
\times \int_0^1 \Phi_2 \sin 2\theta dy \left/ \left( \beta - FU_1 \right) \right. \right. \tag{6.12}
\]

However, Eq. (6.5) implies that

\[
\mu \frac{d |A|^2}{dT} = -i \frac{kl}{a^4} h_0 h \varepsilon F \int_0^1 \Phi_2 \sin 2\theta, \tag{6.13}
\]

i.e., that the wave amplitude increases solely due to the interaction of the altered mean flow with the topography. When (6.13) is substituted into (6.12) and the identity \( (\beta - FU_1) = U_1(a^2 - F) \) is used, the eddy heat flux becomes

\[
\varepsilon^2 F \int_0^1 \phi_1 \frac{\partial}{\partial x} \phi_2 dy = \mu \varepsilon^2 \frac{(a^2 + F)}{8} \frac{d |A|^2}{dT}, \tag{6.14}
\]

so that the product of the vertical shear \( (U_1) \) and the eddy heat flux is precisely equal to the increase of available potential and kinetic energy in the wave field, i.e., the heat flux is downgradient. Since the basic flow is subcritical to ordinary baroclinic instability, Eq. (6.12) shows that only the effect of topography allows this potential energy transformation.

7. Equilibrium solutions (baroclinic)

Steady baroclinic wave solutions may be found by setting the time derivatives to zero in (5.34a, b) and (5.35). If (5.34a, b) are added together the dissipative terms proportional to the difference of \( \Phi_1 \) and \( \Phi_2 \) vanish leading to the simple result [after (2.8) is used],

\[
r_1 \Phi_1 + r_2 \Phi_2 = -\frac{a^2}{8} \left[ \frac{a^2}{8} \frac{A_1^2 + h_0 e^{-iA_1}}{\beta(a^2 - F)} \right]
\]

\[
\times \left( r_1 a^2 + \kappa F \right) \sin 2\theta - 2(1 - y^{2}), \tag{7.1}
\]

where, again, \( h_0 \) has been taken to be a real constant.

If \( r_1 = 0 \), that is, if the upper boundary is stress free, (7.1) determines \( \Phi_2 \). In particular, the mean zonal velocity in the lower layer is

\[
\overline{U}_2 = -\frac{\partial \Phi_2}{\partial y} = -\frac{a^2}{4} \left[ \frac{a^2}{8} \frac{A_1^2 + h_0 e^{-iA_1}}{\beta(a^2 - F)} \right]
\]

\[
\times \left( \frac{r_1 a^2 + \kappa F}{r_2} \right) (1 - \cos 2\theta). \tag{7.2}
\]

It is then possible to return to (5.34a) to solve for \( \Phi_1 \), i.e.,

\[
\Phi_1 = \Phi_2 - \frac{l}{2 \mathcal{U}_1} |A|^2 \left( \frac{r_1 a^2 + \kappa F}{r_1 A_1^2 + \kappa F} \right)
\]

\[
\times \left[ \sin 2\theta - \frac{2l}{\left( \kappa F r_1 \right)^{2/2}} \left( \frac{\sinh(\kappa F r_1)}{\cosh(\kappa F r_1)^{1/2}} \right) \right]. \tag{7.3}
\]

If (7.1) (with \( r_1 = 0 \)) and (7.3) are substituted into the steady form of (5.3) a nonlinear algebraic equation for \( A = A_r + iA_i \) is obtained. If that equation
is written in terms of its real and imaginary parts we obtain, after considerable algebra

\[ A_r \left[ \frac{\Delta \beta}{a^2} \left( \frac{a^2 - F}{a^2} \right) \frac{r_2}{(r_a a^2 + \kappa F)} - \frac{3}{8} \frac{h_0^2}{a^2(a^2 - F)} F \right] + \frac{\beta r_2}{ka^2} A_i + A_i h_0 \epsilon \left[ \frac{l^2}{2a^2} - \frac{3}{8} \frac{a^2}{a^2 - F} - (A_r^2 + A_i^2) \right] \times \frac{3}{8} \frac{h_0^2}{a^2 - F} - e^2 A_r (A_r^2 + A_i^2) Q = 0, \] (7.4a)

\[ A_i \left[ \frac{\Delta \beta}{a^2} \left( \frac{a^2 - F}{a^2} \right) \frac{r_2}{(r_a a^2 + \kappa F)} \right] - \frac{\beta r_2}{ka^2} A_r + A_i A_r h_0 \epsilon \left[ \frac{l^2}{2a^2} - \frac{3}{8} \frac{a^2}{a^2 - F} \right] - e^2 A_r (A_r^2 + A_i^2) Q = 0, \] (7.4b)

where

\[ Q(a, l, F, r_1, r_2, \kappa) = - \left[ \left( \frac{l^2}{2} - \frac{3}{8} \frac{a^4}{a^2 - F} + \frac{r_2(a^2 - F)}{(r_1 a^2 + \kappa F)} \left( \frac{l^2}{2a^2} - \frac{a^2}{4(l^2/a^2 - 1)} \right) \right) \right] \right] \times \frac{8 l^4 (a^2 + \kappa F r_1) \tanh(\frac{1}{2}(\kappa F r_1)^{1/2})}{\left( \frac{a^2}{4(l^2/a^2 - 1)} \right)^{1/2}}. \] (7.5)

Before examining the finite-amplitude solutions of (7.4a,b) it is instructive to determine what conditions on \( \Delta \) the linearized forms of (7.4a,b) imply. This is equivalent to determining the conditions for linear instability in the presence of dissipation, assuming that at the threshold of topographic instability the disturbance is completely steady, i.e., neither growing, decaying, nor oscillating. With that assumption the linearized version of (7.4a,b) implies that the marginal curve for topographic instability in the presence of dissipation is given by

\[ \Delta = \frac{3}{16} \frac{r_1 a^2 + \kappa F h_0^2 F}{r_2(r_a a^2 + \kappa F)} \beta \times \left\{ 1 \pm \left[ 1 - \frac{256 r_2^2 \beta (a^2 - F)^2}{9 h_0^4 F^2 k^2} \right]^{1/2} \right\} \] (7.6)

As in the inviscid theory (6.9), \( \Delta \) must be positive for topographic instability but there is no unique non-dissipative limit for (7.6). This simply reflects the fact that completely stationary solutions of (5.3a,b) depend vitally on dissipation and the nature of the solution will depend on the ratios \( r_1/r_2, r_i/r_2, \kappa/r_2 \). If all the dissipative coefficients are small the inviscid theory will apply, yielding (6.7), as long as \( \mu \sigma \) is greater than the largest of \( r_1, r_i, r_2, \kappa \). Right on the marginal curve where \( \sigma = 0 \) the effect of dissipation on the zonal flow is O(1) no matter how small the dissipation is, and this nonuniformity in dependence on dissipation as the neutral curve is approached is apparent in (7.6).

The root corresponding to the minus sign in (7.6) represents the lower stability boundary and the plus sign represents the upper boundary for \( \Delta \). For instability to occur at all, \( r_2 \) must be small enough, i.e.,

\[ a^2 < a_2^2 = F + \frac{3}{16} \frac{h_0^2 F}{\beta} \frac{k}{r_2} \] (7.7)

or equivalently,

\[ r_2 < \frac{3}{16} \frac{h_0^2}{\beta} F(k)(a^2 - F). \] (7.8)

This short-wave cutoff increases with decreasing bottom-layer friction, and in the limit \( r_2 \to 0 \), instability is again assured for all \( \Delta > 0 \).

The finite-amplitude solutions of (7.4a,b) are, generally speaking, difficult to obtain for arbitrary values of the dissipation parameters. However, in most cases of meteorological or oceanographic relevance these parameters are small.

If \( r_1, r_2, \) and \( \kappa \to 0 \), but with bounded ratios \( r_i/r_2, \kappa/r_2 \), a great simplification occurs. One can see immediately from (7.4b) that \( A_i \) is then \( O(r_2) \) with respect to \( A_r \). That is, the phase shift of the wave relative to the topography is small. Hence in that limit solutions of (7.4a,b) may be found in the form

\[ A_r = A_r^{(0)} + r_2 A_r^{(1)} + \cdots \]

\[ A_i = r_2 A_i^{(1)} + \cdots \]

so that \( A_r^{(0)} \) satisfies

\[ [e A_r^{(0)}] Q + e A_r^{(0)} h_0 \left[ \frac{3}{8} \frac{a^2 + F}{a^2 - F} - \frac{l^2}{2a^2} \right] - \left[ \frac{\beta \Delta (a^2 - F) r_2}{8 a^2 - F} \right] - \frac{3}{8} \frac{h_0^2 F}{a^2(a^2 - F)} = 0. \] (7.9)

In the limit \( r_2 \to 0 \) (for fixed \( r_i/r_2, \kappa/r_2 \)) the positivity of the last term in (7.9) is identical to the linear instability criterion given by (7.6). Ordinarily it is that condition that must be satisfied for the existence of finite-amplitude solutions, i.e., that the bifurcation point at which new, wavy steady solutions emerge is coincident with the instability threshold of the wave-free state. Were the middle term in (7.9) absent this would be the case here. However, it is clear from (7.9) that if \( A_r^{(0)} \) is negative solutions of (7.9) can exist for linearly stable values of \( \Delta \), i.e., \( \Delta < 0 \). In this regard the analysis here seems to disagree with the results of the truncated spectral model of Charney and Strauss (1980) who reported no wavy equilibrium solutions below the linear instability threshold.

At each parameter setting there are, if any, two solutions to (7.9), i.e.,
\[ \varepsilon A_r^{(0)} = \frac{h_0}{2Q} \left[ \frac{3}{8} \frac{a^2 + F}{(a^2 - F)} - \frac{l^2}{2a^2} \right] \]
\[ \pm \frac{h_0}{2Q} \left[ \frac{3}{8} \frac{(a^2 + F)}{(a^2 - F)} \right] \frac{l^2}{2a^2} - \frac{3}{2} \frac{QF}{a^2(a^2 - F)} \]
\[ + \frac{4\beta\Delta(a^2 - F)r_2 Q}{a^2(r_2a^2 + \kappa F)h_0^2} \right]^{1/2}. \] (7.10)

From (7.4b) and (7.9)
\[ r_2 A_r^{(0)} = \frac{8}{3} \frac{\beta A_r^{(0)}}{k} \frac{(a^2 - F)r_2}{h_0[h_0 + a^2 A_r^{(0)} \varepsilon]}. \] (7.11)

It is clear from (7.11) that the approximation scheme of (7.8) will be valid at all points except where either \( h_0 \) or \( h_0 + \varepsilon A_r^{(0)}a^2 \) vanishes. According to (7.2), at these latter points in parameter space the zonal flow induced in the lower layer vanishes and therefore the interaction with the topography is nil. I shall therefore avoid those points in the following discussion.

Fig. 3 shows the solution for \( A_r^{(0)} \) given by (7.10) for \( l = \pi, F = 5\pi^2, r_1 = \kappa = r_2 \) and for \( k = 3\pi \) and \( k = 5\pi \). In each case the bifurcation point \( \Delta_c \) at which the two equilibrium solutions first appear occurs for \( \Delta < 0 \), i.e., a value of \( \Delta \) which is subcritical with regard to the linear stability boundary. The depth of the subcriticality of the bifurcation is not numerically large in either case but is significantly different from zero and represents a qualitative difference to the results of the truncated expansion analysis of Charney and Strauss. Furthermore, for the case \( k = 3\pi \), the wave component in phase with the topography \( A_r^{(0)} \) is negative in both branches of the solution for \( \beta \Delta/h_0^2 \approx 1.15 \). Beyond that point, again in distinction to Charney and Strauss’ analysis, one branch has \( A_r^{(0)} > 0 \).

Using the results of Fig. 3 it is easy to show that \( U_z \) in (7.2) vanishes at \( \beta \Delta/h_0^2 = 0.1 \) and 1.15 for \( k = 3\pi \). For \( \beta \Delta/h_0^2 \) within this interval the two branches of the solution correspond to oppositely directed lower-layer zonal velocities. Outside this range both branches yield either lower-layer westerlies (\( \beta \Delta/h_0^2 < 0.1 \)) or easterlies (\( \beta \Delta/h_0^2 > 1.15 \)). The parameter values used to calculate the \( k = 3\pi \) curves in Fig. 3 are identical to Charney and Strauss (1980). However, it is difficult to make a precise comparison due to the different form of the basic zonal flow in the “Hadley” or wave-free state. Nevertheless, the results here again differ from Charney and Strauss’ insofar as they report oppositely directed zonal flows in the lower layer for each branch over their entire parameter range. It does appear that most of their reported range lies in the middle interval described above but this point remains unclear.

8. The effect of topography on baroclinic instability

The instabilities and bifurcations to finite-amplitude motion described above occur for shears that are subcritical with respect to the threshold of ordinary baroclinic instability, \( U_1 - U_2 = \beta/F \).

If \( U_2 = 0 \) and \( U_1 = \beta/F \), that is, at the minimum critical shear, a wave with wavenumber \( a^2 = 2^{1/3}F \) will be just marginally stable (in the absence of topography). Furthermore, its phase speed will vanish so that this wave will also be resonant with periodic topography with that wavenumber. To examine the role of topography at the threshold of
baroclinic instability it is necessary to rescale the long time so that \( \mu = O(\varepsilon) \) instead of \( O(\varepsilon^2) \). To simplify the following algebra, I consider only the inviscid problem. Then writing
\[
U_2 = 0, \quad U_1 = \frac{\beta}{F} + \Delta \quad (8.1)
\]
the equations of motion become
\[
\left[ \mu \frac{\partial}{\partial T} + \left( \frac{\beta}{F} + \Delta \right) \frac{\partial}{\partial x} \right] \times [\nabla^2 \phi_1 - F(\phi_1 - \phi_2)] + (2\beta + F\Delta) \times \frac{\partial \phi_1}{\partial x} + \varepsilon J[\phi_1, \nabla^2 \phi_1 + F(\phi_2 - \phi_1)] = 0 \quad (8.2a)
\]
\[
\mu \frac{\partial}{\partial T} \left( \nabla^2 \phi_2 - F(\phi_2 - \phi_1) \right) - F\Delta \frac{\partial \phi_2}{\partial x} + \varepsilon J[\phi_2, \nabla^2 \phi_2 + F(\phi_1 - \phi_2) + \eta_B/\varepsilon] = 0, \quad (8.2b)
\]
where, again,
\[
\eta_B = h_0 \frac{e^{ikx}}{2} \sin ly + * = h_0 W + * \quad (8.3)
\]
If \( \phi_1 \) and \( \phi_2 \) are written in the series
\[
\phi_n = \phi_n^{(0)} + \varepsilon \phi_n^{(1)} + \varepsilon^2 \phi_n^{(2)} + \cdots, \quad (8.4)
\]
then (8.2a) yields for the wavelike solution for \( \phi_n^{(0)} \),
\[
\phi_1^{(0)} = AW + *, \quad \phi_2^{(0)} = \gamma AW + *, \quad (8.5)
\]
where
\[
\gamma = \frac{a^2 - F}{F}, \quad a^2 = k^2 + l^2. \quad (8.6)
\]
The first nontrivial contribution for (8.2b) is
\[
\frac{\mu}{\varepsilon} \frac{\partial}{\partial T} \left\{ \nabla^2 \phi_2^{(0)} - F[\phi_1^{(0)} - \phi_2^{(0)}] \right\}
\]
\[
= -i \frac{k^2 \gamma l}{4\varepsilon} \left( A\phi_0^* - A^*\phi_0 \right) \sin 2ly. \quad (8.7)
\]
The right-hand side of (8.7) is independent of \( x \). In order that (8.7) be consistent we must insist that \( \phi_n^{(0)} \) is not phase shifted with respect to the topography. Hence, if \( h_0 \) is a real constant (without loss of generality), it follows that \( A \) must be a real constant, i.e., the \( O(1) \) wave is in phase with the topography. It then follows that to have \( A \neq 0 \) (8.7) further implies that \( \alpha^2 = 2^1/2F \) and places no further restriction on \( A \).

The \( O(\varepsilon) \) contribution from (8.2a) yields
\[
\phi_1^{(1)} = \Phi_1(y, T), \quad \phi_2^{(1)} = \Phi_2(y, T) + \frac{2F}{\mu} \frac{\partial A}{\partial T} + * \quad (8.8)
\]
If these results are used to calculate the \( O(\varepsilon^2) \) terms in (8.2a, b) and the resulting secular terms are removed as in Pedlosky (1970), equations for \( A, \Phi_1 \) and \( \Phi_2 \) result. The method is so similar to the work quoted above that only the results are given here, viz.,
\[
\mu^2 \frac{d^2 A}{dT^2} - \Delta \frac{k^2 \gamma^2}{2} \frac{\beta}{F} A - \varepsilon^2 \frac{k^2 \gamma^2}{2} \frac{U_c}{A} 2^1/2A 2
\]
\[
\times \int_0^1 \sin 2ly \left[ \frac{\partial^2 \Phi_2}{\partial y^2} - F(\Phi_2 - \Phi_1) \right] dy
\]
\[
= -2 \int_0^1 \Phi_2 \sin 2lydy h_0 e^{-k^2U_c/2F} \gamma l, \quad (8.9)
\]
\[
\frac{\partial}{\partial T} \left[ \frac{\partial^2 \Phi_2}{\partial y^2} - F(\Phi_2 - \Phi_1) \right] = -\frac{IF^2}{2\beta} \frac{dA}{dT} A^2 \sin 2ly
\]
\[
- \frac{IF h_0}{\beta} \frac{dA}{dT} \sin 2ly. \quad (8.10a)
\]
\[
\frac{\partial}{\partial T} \left[ \frac{\partial^2 \Phi_1}{\partial y^2} - F(\Phi_1 - \Phi_2) \right]
\]
\[
= \frac{IF^2}{2\beta} \frac{dA}{dT} A^2 \sin 2ly. \quad (8.10b)
\]
If \( h_0 = 0 \) (8.9) and (8.10a, b) reduce to the ordinary finite amplitude baroclinic instability problem, Pedlosky (1970).

Eqs. (8.10a, b) may easily be integrated to find \( \Phi_1 \) and \( \Phi_2 \) in terms of \( A \). If we insist that \( \Phi_1 \) and \( \Phi_2 \) are zero at the initial instant then it follows that \( A \) must satisfy
\[
\mu^2 \frac{d^2 A}{dT^2} - \frac{k^2}{2F} \beta \left[ \frac{\Delta}{\beta} \gamma^2 - \frac{h_0^2}{4\beta(4l^2 + 2F)} \left( 8l^2 + 3F \right) + \frac{16l^4}{(4l^2 + 2F) \tanh(F/2)^{1/2}} \right]
\]
\[
+ \varepsilon \frac{h_0}{2} \left[ k^2 \gamma^2 A[A - A(0)] + \frac{l^2}{4(4l^2 + 2F)} \left[ 1 + 8l^2 \tanh(F/2)^{1/2} \right] [A^2 - A(0)^2] + \frac{\varepsilon^2 k^2 \gamma^2 F}{4} A[A^2 - A(0)^2] \right]
\]
\[
= \frac{A(0)k^2}{8F(4l^2 + 2F)} \left( 8l^2 + 3F \right) + \frac{16l^4}{(4l^2 + 2F) \tanh(F/2)^{1/2}} \right]. \quad (8.11)
\]
The most striking feature of the amplitude equation (8.11) is apparent in the term linear in $A$. It is clear that topography will always reduce the growth rate (in agreement with the qualitative results of Charney and Strauss) and produce stability whenever

$$h^2 > \Delta \frac{4\gamma^2(4\ell^2 + 2F)}{(8\ell^2 + 3F) + \frac{16\ell^4}{4\ell^2 + 2F}(\tanh(F/2)^{1/2})} \tag{8.12}$$

where, recall, $\Delta = U_1 - \beta/F$. When the flow is linearly unstable the nonlinearity of the wave–mean flow interaction provides finite-amplitude stability but the presence of topography will render the resulting oscillation to be asymmetric about $A = 0$.

9. Discussion

The exploitation of the major character of resonance response, that is, that the solution at lowest order has the form of a free wave of known structure, allows the linear and finite-amplitude problem of topographic resonance to be studied analytically. Furthermore, restrictions or ad hoc simplifications required by earlier studies can be easily removed. By far the major advantage of the analytical study is the explicitness of dynamical relationships.

For example, the barotropic model studied here displays a sensitivity to the ratio of cross-stream to downstream wavenumber not apparent in truncated spectral models or models of infinite meridional extent. In particular, the possibility of subresonant topographic instability is apparently a new result of the analytical model. It also is clear from the analytical study that because of the asymmetry in the dynamics between positive and negative wave amplitudes due to the topography, the bifurcation points for finite-amplitude waves are not coincident with the stability threshold of the linear problem.

The baroclinic problem has special meteorological and oceanographic significance. Although topography has been shown to stabilize the standard mode of baroclinic instability, the resonant interaction of the free Rossby wave with the topography introduces a subcritical instability. This instability draws on the potential energy of the basic flow [as shown by (6.14)] but the results of the theory for the finite amplitude equilibria show the tendency for the steady-state amplitudes to scale with the amplitude of the topography with a relatively weak dependence on the degree of super-resonance. As remarked in the Introduction, this instability would seem to be of particular significance for large areas of the ocean where the flow is subcritical to baroclinic instability.

Several important questions remain to be examined. First, the time dependent behavior of the baroclinic amplitude equations must be studied. In the flat-bottom baroclinic stability problem a wide range of behavior has been noted (Pedlosky and Frenzen, 1980) including periodic limit cycle and chaotic behavior. It will be interesting to see to what degree the set (5.34a,b) and (5.35) share those qualitative features. In addition, the relative ease with which the analytical theory proceeds allows the consideration of more complex problems. In particular, it is a relatively simple matter to include topography which is not exactly periodic, i.e., which is a slowly varying function of $x$. This allows the consideration of spatially limited topographic ranges, and the resulting amplitude equations will be able to describe the evolution of the wave amplitude in the downstream direction for dissipative flows wherein potential vorticity is not conserved. These problems are under current study.

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