Wavenumber Selection in Nonlinear Baroclinic Instability

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ABSTRACT

Laboratory experiments with finite-amplitude baroclinic waves arising from instability of a two-layer $f$-plane shear flow are reported. They show that as the system becomes more and more supercritical, as measured by decreasing $E^{2}/R$, where $E$ is the Ekman number and $R$ the Rossby number associated with the driving, there are a succession of wavenumber transitions to lower and lower wavenumbers. At finite amplitude, the dominant wavenumber is considerably smaller than that predicted by linear stability theory. A simple weakly nonlinear model is constructed to interpret the laboratory results. It shows that because the longer growing waves do not extract energy as rapidly from the mean flow as the shorter ones, at finite amplitude, the preferred equilibrium states are dominated by the former. The theoretical calculation also indicates that at least near the neutral curve sideband harmonics do not substantially affect the equilibration process. In addition, a mechanism that may explain the observation of extremely long equilibration times is offered.

1. Introduction

There has been considerable interest in determining the dominant zonal scale for waves that arise from baroclinic instability of a zonal current. As a first approximation it is sometimes argued that the dominant zonal scale under supercritical conditions is close to that which gives the largest linear growth rate when small amplitude waves are superimposed on the mean flow. This assumption is expected to be rigorous only near the neutral curves but in some systems like Bénard convection and Taylor double cylinder flow it appears to work well at quite nonlinear parameter settings. For the more geophysically relevant baroclinic instabilities such a maximal growth rate assumption for choosing the zonal scale does not work very well. Numerical GCM computations by Gall et al. (1979) show that although linear calculations give a most rapid growth at wavenumber $n = 15$, the nonlinear regime is characterized by $n < 10$. A similar phenomena was observed by Hart (1973, 1980) in two-layer laboratory studies of baroclinic instability. Although the wavenumber at the neutral curve agreed well with linear theory, as the flow was forced more strongly a series of transitions to lower wavenumbers was observed. Thus at rather supercritical conditions the dominant wavenumber was a factor of 2 or 3 smaller than the critical one. A similar sort of behavior was also observed in the numerical ocean dynamics experiments of McWilliams et al. (1978).

It is the purpose of this paper to summarize the results of the two-layer experiments on this wave selection problem, including recent laboratory data. Then an interpretation of the phenomena is given, using a weakly nonlinear expansion of the basic quasi-geostrophic equations.

2. Experimental results

Two immiscible liquids are contained in a rotating cylinder of radius $L$. The quiescent depth of each layer is $H$. In addition to the basic rotation $\Omega$, relative fluid motion is induced by a disc in contact with the upper layer that rotates differentially at rate $\omega$. The top and bottom discs are parabolic and the basic rotation rate is chosen so that there is no depth variation in the absence of driving. In this manner $f$-plane dynamics are simulated in the laboratory. Measurements of the interface height are made at two azimuths (equal radii) by small capacitative level sensors. The two signals are digitized and cross correlated to obtain the wavelength in the steady wave regimes. The reader is referred to earlier papers (Hart, 1972, 1980) for further experimental details. A cross section of the apparatus is shown in Fig. 1.

When $\omega < \Omega$ the dynamics will be approximately quasi-geostrophic and are described by two external dimensionless parameters; the Froude number

$$F = \frac{4\Omega^2L^2}{g \Delta \rho H}$$

and a friction parameter

$$Q = (n\nu)^{1/3}/H\omega,$$
where $\nu$ is the fluid viscosity, and $\Delta \rho / \rho$ the fractional density difference across the interface. Experiments were done by fixing $F$ and slowly decreasing $Q$. At rather large $Q$ a stable steady axisymmetric regime with azimuthal velocity $v_1 = \frac{3}{4} \omega r$ and $v_2 = \frac{1}{4} \omega r$ exists. As the probes only see traveling interface height variations, this circulation leads to a steady output. At the neutral curve $Q_c(F)$ baroclinic waves of wavenumber $n$ and phase speed $\omega / 2$ appear. Fig. 2a shows that these waves typically equilibrate to a single frequency (and wavenumber) motion with a steady amplitude. Hart (1972) compared quasi-geostrophic linear instability theory in a cylinder with such experimental results. The good agreement between theory and experiment indicated that the instability was indeed the classic baroclinic one. As $Q$ is decreased further into the nonlinear regime the original wave grows in amplitude until at a second critical $Q_c$, there is a transition to a steady wave of lower wavenumber. This also is seen in frequency as shown in Fig. 2b for the transition from wavenumber 3 to 2. Decreasing $Q$ still further can either lead to more transitions or, if $Q$ becomes quite small, to vacillating or irregular motion at a lower wavenumber. Fig. 3 summarizes the transitions observed in earlier experiments (Hart, 1973, 1980) and in more recent ones. The prediction of linear theory for the wavenumber $n$ with the maximum growth rate is also given. This theoretical result is only weakly dependent on $Q$ (Hart, 1980), so that as $Q$ decreases the observed wavenumber moves further and further away from the linear result.

**Fig. 1.** Cross section of the laboratory two-layer experiment.

**Fig. 2.** Typical interface height-time records: (a) onset of instability and equilibration to steady waves, (b) transition from wavenumbers 3 to 2.
The basic equations are the two dimensionless nonlinear quasi-geostrophic vorticity equations with top and bottom Ekman layer friction as presented by Pedlosky (1970):

\[
\begin{align*}
\left[ \frac{\partial}{\partial t} + J(\phi_1) \right] \left[ \nabla^2 \phi_1 + F(\phi_2 - \phi_1) \right] - Q \nabla^2 \phi_1 &= 0, \quad (1) \\
\left[ \frac{\partial}{\partial t} + J(\phi_2) \right] \left[ \nabla^2 \phi_2 + F(\phi_1 - \phi_2) \right] - Q \nabla^2 \phi_2 &= 0, \quad (2)
\end{align*}
\]

where

\[
J(\phi) = \frac{\partial \phi}{\partial x} \frac{\partial}{\partial y} - \frac{\partial \phi}{\partial y} \frac{\partial}{\partial x}.
\]

These are solved with \( \phi(x,y,t) = 0 \) at \( y = 0, 1 \) for \( x \)-dependent parts of the streamfunction \( \phi \) and \( \partial^2 \phi/\partial y \partial t = 0 \) there for the zonally averaged streamfunction. Thus, \( x \) is the zonal and \( y \) is the meridional or radial independent variable. \( L \) becomes the channel width and length scale, \( U = \omega L \) the velocity scale and \( \omega^{-1} \) the time scale. We start with a basic rectilinear zonal flow\(^1\) in the \( x \) direction:

\[
u_1 = \frac{1}{2} , \quad \nu_2 = -\frac{1}{2} .
\]

Upon this is superimposed a disturbance field \( \psi(x,y,t) \). For the linear problem where all terms quadratic in \( \psi \) are neglected, solutions with form \( \sin \pi y e^{i\omega t} \) are possible. Neutral curves for each wavenumber \( n \) are given by

\[
Q_n = n(2F - l_n^2)^{1/2}/2n,
\]

with

\[
l_n^2 = n^2 + \pi^2.
\]

Consider these neutral curves in the \( F-Q \) plane. In the cylinder \( n \) is quantized by the periodicity condition so we only consider integer values for it. Thus \( n = 1 \) corresponds to one cycle per \( 2\pi \) channel widths (or per \( 2\pi \) radii in the cylindrical context). A schematic representation is shown in Fig. 4. We perturb off the \( Q_n \) curve defining a small quantity

\[
\Delta = Q_n - Q.
\]

It is assumed we are in the wave region so that \( \Delta \) is positive. At moderate to large \( F \), where \( Q \) is of order 1, the \( Q_n \) and \( Q_{n-1} \) neutral curves are very close together. That is, the quantity

\[
\Delta_w = Q_n - Q_{n-1} = Q_n \left[ 1 - \frac{l_n(n-1)(2F - l_{n-1}^2)^{1/2}}{l_{n-1}n(2F - l_n^2)^{1/2}} \right] > 0
\]

\(^1\) This basic state is chosen so the neutral waves will have zero phase speed. This corresponds to a basic rotation of \( \Omega + \omega/2 \) of the sensor coordinate frame, and at small Rossby number, is essentially equal to the cylinder rotation rate \( \Omega \).
is a small number with respect to 1. In general, for a given F, we choose n such that $Q_n$ is the largest possible. Then $Q_n$ represents the boundary between axisymmetric flow and the wave-regime, while $Q_{n-1}$ gives the next boundary in the wave regime where linear theory predicts exponential growth of wavenumber $n - 1$. It is seen that $\Delta_w$ is a fixed parameter for a given constant F experiment. At larger values of F, and hence of n, $\Delta_w$ is smaller, but of course vanishes where the two neutral curves cross.

The linear growth rates for the n and $n - 1$ waves are respectively

$$\text{inc}_n = l^n_2 Q_n \Delta (l^n_2 + F),$$

$$i(n - 1)\text{c}_{n-1} = \text{inc}_n \frac{l_{n-1}^2 (l_{n-1}^2 + F)(1 - \Delta_w/\Delta)}{l_n^2 (l_n^2 + F)}$$

to order $\Delta^2$. It is clear that at any F above the point where the two curves cross, the n wave has the higher linear growth rate.

The nonlinear analysis follows Pedlosky (1970). Since $Q$ is order 1, the linear growth rates are order $\Delta$ and a new slowtime variable $T = \Delta t$ is defined. The fluctuating streamfunction is expanded in increasing powers of $\Delta^{1/2}$, with the lowest order problem reducing to the linear neutral one. That is, we write

$$\psi_1 = \Delta^{1/2}[A(T)e^{inx} + B(T)e^{i(n-1)x}] \sin n \pi y + \Delta \psi_1^{(1)} + \Delta^{3/2} \psi_1^{(2)} \cdots, \quad (3)$$

$$\psi_2 = \Delta^{1/2}[A(T)\gamma e^{inx} + B(T)\gamma e^{i(n-1)x}] \sin n \pi y + \Delta \psi_2^{(1)} + \Delta^{3/2} \psi_2^{(2)} \cdots, \quad (4)$$

where

$$\gamma_k = \frac{l_k^2 - F}{F} - i(2F - l_k^2)^{1/2} \frac{l_k}{F}$$

for $k = n$ or $k = n - 1$. These expansions are substituted into the basic equations (1)–(2) and the successive quasi-linear problems solved with the intent of obtaining a pair of ordinary differential equations governing $A(T)$ and $B(T)$.

At order $\Delta$, the interaction of each fundamental wave with itself generates the correction $\tilde{\psi}(y,t)$ to the zonal flow. In addition the wave-wave interactions between n and $n - 1$ generate a pair of sidebands of wavenumber 1 and $2n - 1$. Thus the form of the order $\Delta$ streamfunction is

$$\psi_1^{(1)} = ge^{ix} + h e^{i(n-1)x} + \tilde{\psi}_1,$$

$$\psi_2^{(1)} = ge^{ix} + h e^{i(n-1)x} - \tilde{\psi}_1.$$

The coefficients g, h and $\tilde{\psi}$ are functions of y, A and B, and are given in Appendix A.

At order $\Delta^{3/2}$ removal of terms that project in a certain way (see Appendix A) on the fundamentals yields the desired amplitude equations. These equations are

$$\dot{A} = A - b_1 A (|A|^2 + d_1 |B|^2), \quad (7)$$

$$\dot{B} = a B - b_2 B (|B|^2 + d_2 |A|^2) \quad (8)$$

where

$$\frac{\partial}{\partial \text{inc}_n} = \frac{\partial}{\partial T},$$

and

$$a = \frac{l_{n-1}^2 (l_{n-1}^2 + F)(1 - \Delta_w/\Delta)}{l_n^2 (l_n^2 + F)}.$$
interaction \( n + (n - 1) \xrightarrow{\text{st}} 1 \), then \( 1 + (n - 1) \xrightarrow{\text{st}} n \), etc. The numerical values of \( d_1 \) and \( d_2 \) are contained implicitly in the values of \( d_1 \) and \( d_2 \) in Table 1. We note here that they are real.

Let us first discuss the character of solutions to (7)–(8). These equations are similar to those derived by Drazin (1972) for the nonlinear Eady problem with two waves, but his conclusions do not cover all the possible steady states found here. The coefficients \( b_1 \) and \( b_2 \) are positive, while \( a \) can be of either sign (negative meaning linear decay of the \( n - 1 \) wave).

There are several steady solutions that may exist depending on the numerical values of the coefficients. These solutions may or may not be stable, a question that is easily answered by perturbing Eqns. (7) and (8) about the steady solutions. By setting the time derivatives equal to zero we find steady solutions:

(I) \[ \dot{A} = \dot{B} = 0 \] for all parameters.

(II) \[ \dot{A} = (1/b_1)^{1/2}, \quad \dot{B} = 0 \] for all parameters.

(III) \[ \dot{A} = 0, \quad \dot{B} = (a/b_2)^{1/2} \] for \( a > 0, (\Delta > \Delta_e) \).

(IV) \[ \dot{B} = (1/b_1)^{1/2} \frac{S - d_2}{1 - d_1 d_2}, \quad \dot{A} = (1/b_1)^{1/2} \frac{1 - d_1 S}{1 - d_1 d_2}, \] provided \( d_2 < S < 1/d_1 \) or \( 1/d_1 < S < d_2 \), with

\[ S = \frac{ab_1}{b_2}. \] (16)

Solution (I) is always unstable—at least the \( n \)-wave grows initially. Stability of the other modes depend on \( S \), \( d_1 \) and \( d_2 \). The one-wave \((n)\) steady state (II) is stable if \( S < d_2 \). The single \( n - 1 \) wave steady state is stable if \( S > 1/d_1 \). Finally, the mixed-mode solution is stable if \( d_2 < 1/d_1 \). Thus there are two qualitatively different regime diagrams, depending on whether \( d_2 > 1/d_1 \) or vice versa. These are shown in Fig. 5. Because in the experiment \( S \) goes from \(-\infty \) at \( \Delta = 0 \), to a finite positive value as \( \Delta \) becomes large, in either case one would first expect to observe the single \( n \)-wave. If \( d_2 > 1/d_1 \) and background noise is sufficiently small, then the \( n \)-wave will persist until \( S = d_2 \) at which point it will lose stability to the single \( n - 1 \) wave. Note that this situation displays hysteresis. Once \( S \) becomes greater than \( d_2 \), decreasing it will not lead back to the \( n \)-wave state until \( S \) is less than \( 1/d_1 \). On the other hand, if \( 1/d_1 > d_2 \) as \( S \) increases the transition to the \( n - 1 \) wave state takes place via the stable, mixed wave regime. In this instance there is no hysteresis. Numerical solution of (7)–(8) verifies this qualitative interpretation.

There are thus two key issues to be addressed. Is \( d_1 d_2 \) greater, less than, or equal to one, and is the maximum value of \( S \) greater than the maximum of \( d_2 \) and \( 1/d_1 \)?

Referring back to our detailed model, it is interesting that without the wave-wave interaction \( d_1 d_2 = 1 \), it is only the sidebands that lead to either mixed-wave states or hysteresis. Table 1 gives the maximum value of \( S \) as \( \Delta_e/\Delta \rightarrow 0 \), the values of \( d_2 \) and \( 1/d_1 \) including wave-wave interactions, their values \( d_0 \) and \( 1/d_0 \) with only the self-interaction mean correction, and the two single-state equilibration amplitudes, again for \( \Delta_e/\Delta \rightarrow 0 \). \( F_e \) and \( Q_e \) given the point where the \( n \) and \( n - 1 \) curves cross. These are the values used to evaluate the coefficients. Values for intermediate \( F \) can reliably be obtained by linear interpolation if they are needed. The \( n = 2 \) pair is not included because for it \( Q \) is substantially less than 1 and our time scaling does not apply. In addition, it is self-resonant in the sense that one of the sidebands is a member of the fundamental set. Experimentally (Hart, 1973) one does not observe a simple transition between two steady states for these wavenumbers, rather, complex wavenumber vacillations occur.

Several conclusions can be made from this table and our preceding discussion:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( F_{\text{critical}} )</th>
<th>( Q_{\text{critical}} )</th>
<th>( S_{\text{max}} )</th>
<th>( d_{0^{-1}} )</th>
<th>( d_{1^{-1}} )</th>
<th>( d_{2} )</th>
<th>( (b_{1^{-1}})^{1/2} )</th>
<th>( (b_{2^{-1}})^{1/2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>13.252</td>
<td>0.9558</td>
<td>1.966</td>
<td>1.361</td>
<td>1.301</td>
<td>1.373</td>
<td>0.0420</td>
<td>0.0589</td>
</tr>
<tr>
<td>4</td>
<td>24.73</td>
<td>1.912</td>
<td>1.621</td>
<td>1.371</td>
<td>1.311</td>
<td>1.369</td>
<td>0.0321</td>
<td>0.0411</td>
</tr>
<tr>
<td>5</td>
<td>45.73</td>
<td>3.186</td>
<td>1.478</td>
<td>1.348</td>
<td>1.311</td>
<td>1.345</td>
<td>0.0212</td>
<td>0.0258</td>
</tr>
<tr>
<td>6</td>
<td>81.11</td>
<td>4.779</td>
<td>1.392</td>
<td>1.315</td>
<td>1.295</td>
<td>1.317</td>
<td>0.0140</td>
<td>0.0165</td>
</tr>
<tr>
<td>7</td>
<td>136.9</td>
<td>6.69</td>
<td>1.331</td>
<td>1.283</td>
<td>1.273</td>
<td>1.284</td>
<td>0.00948</td>
<td>0.0109</td>
</tr>
<tr>
<td>8</td>
<td>220.6</td>
<td>8.92</td>
<td>1.287</td>
<td>1.254</td>
<td>1.249</td>
<td>1.255</td>
<td>0.00636</td>
<td>0.00753</td>
</tr>
<tr>
<td>9</td>
<td>340.5</td>
<td>11.5</td>
<td>1.253</td>
<td>1.230</td>
<td>1.227</td>
<td>1.230</td>
<td>0.00479</td>
<td>0.00536</td>
</tr>
<tr>
<td>10</td>
<td>506.5</td>
<td>14.33</td>
<td>1.225</td>
<td>1.209</td>
<td>1.207</td>
<td>1.209</td>
<td>0.00355</td>
<td>0.00393</td>
</tr>
</tbody>
</table>
1) Since $S_{\text{max}} > \max (d_2, 1/d_1)$ for all $n$, one expects a transition to lower wavenumber for all $F$.

2) In general, $d_2 > 1/d_1$ indicating that hysteresis will occur in these transitions. However only for the lowest wavenumbers are these parameters perhaps separated significantly enough for it to be observed.

3) Wave-wave interactions have an almost negligible influence on the equilibration process. This reflects the weak generation of the sidebands 1 and $2n - 1$ that are distant from the unstable parameter space region. In particular, we find $d_1 \ll d_0$ and $d_2 \ll d_0^{-1}$.

It is apparent that the wave-mean interaction is the crucial one in the transition process. Each wave grows by the linear mechanism. This growth reflects an excess of energy flowing out of the basic zonal shear over that dissipated by bottom friction for the waves. The nonlinear self-interaction of a growing wave produces a correction to the mean zonal shear that in the finite-amplitude state reduces the available potential energy until there is a nonlinear equilibration. This saturated state comes about when the wave energy loss is just balanced by a gain from the nonlinearly modified basic state. Now the self-interaction is proportional to the wave vorticity, so the $n - 1$ wave produces a weaker mean flow correction per unit amplitude, as can be seen in Eq. (A1). Thus even though the linear growth rate is smaller for the $n - 1$ wave, than for the $n$ wave, it can attain a larger saturation amplitude. We have shown that only the single wave state with the maximum possible amplitude is stable. Thus, the appropriate wavenumber selection principal is to choose precisely this wave. Fig. 6 shows how an initial spec-

**Fig. 5.** Diagrams outlining the qualitative behavior of the amplitudes $A$ and $B$ as parametric functions of $S$, $d_1$, and $d_2$.

**Fig. 6.** Typical time integration of the amplitude equations: $F = 15$, $Q = 1.15$, $\Delta w/\Delta = 0.2$. 
trum with two unstable waves evolves toward the single wave end state. Wavenumber $3$ grows fast initially, but as wavenumber $2$ increases the amplitude $A$ diminishes until the single wave state with $B \neq 0, A = 0$ is attained. The condition derived above for linear stability of this particular end state, $S > d_2$, just assures that when $A$ reaches its maximum (equilibrated) value, $B$ will still be able to grow, albeit very slowly if $S \approx d_2$. That is, $A$ will not have reduced the mean shear to the point where $B$ cannot grow at all. As $B$ grows the mean shear is reduced further and $A$ decreases, allowing $B$ to continue its climb to dominance. Inspection of Table 1 shows that above $n = 4$, $S_{\text{max}} - d_2$ is a small quantity. In the presence of $A = \bar{A}$, the growth rate of $B$ scales with $\omega \Delta (S - d_2)$ so the equilibration times can be substantial.

So far we have considered only two waves $n$ and $n - 1$. If more than these are present in the initial spectrum, or if $\Delta$ is large enough so that the $n - 2, n - 3, \ldots$ neutral curves are crossed it is natural to ask what selection process will occur. If $F$ is large the sidebands excited by the $n, n - 1, n - 2$ set, say, will still be rather distant from the fundamentals in wavenumber space and the above calculations suggest that the wave-wave interactions may still be small. The weakly nonlinear theory that includes a large set of waves with coupling only through the mean-field correction gives the same qualitative picture as the two-wave problem. As $\Delta$ is increased there are a succession of transitions to single wave states with lower wavenumber. The details of this model are presented in Appendix B. However, as it has not been shown by direct calculation when the many-wave sidebands can be neglected, the conclusions given there should perhaps be regarded as tentative.

In general, the theoretical results are suggestive of the experimental observations. The transitions to single-wave states at lower wavenumbers, the presence of hysteresis at low $n$ and the absence of mixed-wave states for small $\Delta$ is in qualitative agreement with experiment. The above theory applies at order one $Q$ and fairly large wavenumber. Pedlosky (1981) has studied the nonlinear equilibration problem at the opposite parameter extreme, namely, very small $Q$, low $F$ and low $n$. His amplitude equations are more complicated in that they are second order, but the basic nonlinear mechanism in this limit turns out to be almost identical. For small $n$ (recall that in an annulus configuration $n$ can be less than 1) the sideband interactions scale out of the problem leaving nonlinear wave coupling through the mean field correction alone. Pedlosky's system can evolve to a finite-amplitude steady solution involving a single wave of wavenumber lower than that predicted by linear theory, but which has the maximum stand alone amplitude. In addition, he finds vacillating single-wave solutions of lower wavenumber as well. These latter solutions are typical of what happens in the two-layer experiments at small $Q$ with $n = 2$ (Hart, 1973). However, neither the Pedlosky model or this one seems to give permanent frequency vacillating solutions involving two or more waves. Perhaps these motions, observed at modest supercriticality and $Q$, with $n = 3$ or 2, are fundamentally dependent on direct wave-wave coupling.

To further test these ideas with respect to the experiments cylindrical geometry needs to be included. Unless one is ready to neglect the wave-wave interactions a priori, it is probably easier to integrate a spectral expansion of (1)-(2) numerically, than to work out all the interaction coefficients by hand.

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Appendix A

Calculation of the Amplitude Equation Coefficients

There are two parts to the solution of the order $\Delta$ problem. The first involves the self-interaction of both of the fundamental waves to generate in sum a correction to the zonal flow in each layer. This interaction is essentially worked out by Pedlosky (1970) for one wave, and the two-wave problem is a simple extension of his result. Thus, the mean flow correction is

$$\dot{\psi}_1 = \frac{F(\sin 2\pi y - 1)}{8Q^2}\times [n \gamma_n |A|^2 + (n - 1) \gamma_{n-1} |B|^2]. \quad (A1)$$

A more difficult problem is the determination of the sidebands. They are obtained by solving the order $\Delta$ potential vorticity equations [with $\omega_i$ denoting the potential vorticity $\nabla^2 \psi_i + F(-1)^i(\psi_1 - \psi_2)$ in each layer]

$$\omega_1 = -2F\psi_1 + \frac{2Q_i \bar{\psi}_1}{ik} - \frac{2J_{1c}}{ik}, \quad (A2)$$

$$\omega_2 = -2F\psi_2 - \frac{2Q_i \bar{\psi}_2}{ik} + \frac{2J_{2c}}{ik}, \quad (A3)$$

with $k = 1$ or $k = 2n - 1$. The forcing Jacobians are

$$J_{11} = -Q\pi \left(\frac{l_{1o}}{n} - \frac{l_{1m}}{m}\right)(2n - 1)AB^* \sin 2\pi ye^{ix},$$

$$J_{21} = -\gamma_n \gamma_m J_{11},$$

for the $e^{ix}$ sideband and

$$J_{1p} = -Q\pi \left(\frac{l_{1o}}{n} - \frac{l_{1m}}{m}\right)AB \sin 2\pi ye^{ipx},$$

$$J_{2p} = -\gamma_n \gamma_m J_{1p},$$
for the \( e^{i(2n-1)x} \) sideband, where \( m = n - 1 \) and \( p = 2n - 1 \). Note that the forcing of the wavenumber one harmonic is more prominent because it involves the sum of the weighted total wavenumber squared instead of the difference, and is also larger by a factor \((2n-1)^2\). Solving (A2)–(A3) we get the harmonic functions

\[
g = \frac{-2i[(l_i^2 - F + 2il_i^2Q)J_{11} - FJ_{12}]}{l_i^2(l_i^2 - F) + 4l_i^4Q^2}, \quad (A4)
\]

\[
\hat{g} = \frac{-2i[(F - l_i^2 + 2il_i^2Q)J_{12} + FJ_{11}]}{l_i^2(l_i^2 - F) + 4l_i^4Q^2}, \quad (A5)
\]

\[
h = \frac{-2i/p[(l_i^2 - F + 2il_i^2Q/p)J_{1p} - FJ_{12}]}{l_p^2(l_p^2 - 2F) + 4l_p^4Q^2/p^2}, \quad (A6)
\]

\[
\hat{h} = \frac{-2i/p[(F - l_p^2 + 2il_p^2Q/p)J_{12} + FJ_{1p}]}{l_p^2(l_p^2 - F) + 4l_p^4Q^2/p^2}. \quad (A7)
\]

Proceeding to the \( O(\Delta^{3/2}) \) problem, it is noticed that certain of the inhomogeneous terms have either \( e^{in\pi x} \) or \( e^{i(2n-1)x} \) dependence and \( y \) dependence that has nonzero projection onto the basic eigenfunction \( \sin \pi y \). Denoting such forcing by \( F_{1k} \) and \( F_{1k} \) for the two layers, where \( k = n \) or \( n - 1 \), it can be shown that a necessary and sufficient condition for solvability of the forced boundary value problem is that

\[
\int_0^1 \sin \pi y (F_{1n} - \gamma_n^2F_{2n}) dy = 0
\]

and

\[
\int_0^1 \sin \pi y (F_{1n-1} - \gamma_{n-1}^2F_{2n-1}) dy = 0.
\]

Application of these two conditions yields the amplitude equations and formulas for the coefficients. The straightforward terms to calculate involve the slow-time rate of change of the fundamental, the linear growth rate, and the wave-mean-correction interaction. These are similar to Pedlosky (1970). However, it is apparent that Jacobian interactions between the sidebands and the fundamentals also will give rise to resonance. It turns out that the interactions with the \( e^{i(2n-1)x} \) wave are negligible (down by two to three orders of magnitude) in comparison with the \( e^{ix} \) wave. Not only is the forcing larger for the latter, the response [Eq. (A4)] vs. (A6)] is much larger as well because for the moderate to large \( n \) we are considering \( l_i^2 \approx l_j^2 \) and \( p \gg 1 \). After much tedious but straightforward algebra we find

\[
d_i = \frac{Qp_i^2(l_i^2 - l_m^2/m)}{2b_i(l_i^2 + F)} \left( 2 + \frac{4Q(l_i^2 - l_m^2/m)}{l_i^2(l_i^2 - 2F) + 4l_i^4Q^2} \right)
\]

\[
\times \left[ 2l_i^2Q - i \left( \frac{(l_i^2 - F)(1 + \gamma_m^2) + 2F\gamma_m \text{Re}(\gamma_m)}{1 - \gamma_m^2} \right) \right]
\]

and

\[
d_j = \frac{Qp_j^2(l_j^2 - l_m^2/m)}{2b_j(l_j^2 + F)} \left( 2 + \frac{4Q(l_j^2 - l_m^2/m)}{l_j^2(l_j^2 - 2F) + 4l_j^4Q^2} \right)
\]

\[
\times \left[ 2l_j^2Q - i \left( \frac{(l_j^2 - F)(1 + \gamma_m^2) + 2F\gamma_m \text{Re}(\gamma_m)}{1 - \gamma_m^2} \right) \right]
\]

Numerical evaluation of these formulas indicate that both coefficients are real and, as described in the text, that their values are small with respect to the coefficients for the wave-mean interactions.

**APPENDIX B**

The \( N \)-Wave Problem

If \( F \) is very large it is possible to cross the stability boundary for more than one lower wavenumber even when \( \Delta \) is small. This is because in this region of parameter space the neutral curves of the higher wavenumbers are very close together. What happens when more than two waves are considered can be explained by a simple extension to the model presented in the body of the paper. There it was shown that for moderate to large \( F \) (and hence \( n \)) the distant sideband response was too weak to have a significant effect on equilibration. As a wider spectrum of unstable waves is considered, starting with the linearly most unstable one, the sidebands become less and less distant. For example, the fundamental set \( n = 10, 9, 8, 7 \) excites \( n = 1, 2, 3 \) and \( n = 15-19 \). These are still probably distant enough to have negligible influence on the equilibration dynamics of the fundamentals. Clearly as the fundamental set expands, and/or as \( F \) takes on smaller values, eventually a point is reached where a self-resonant interaction (e.g., \( n = 5, 4, 3, 2 \) excites 1, 2, 3) can arise, and this should be considered explicitly in any theory. If we omit these cases and concentrate on the large \( F \)-large \( Q \) problem, the nonlinear equilibration of \( N \) waves is probably controlled by coupling through the mean flow alone.

Without including any sideband interactions, the amplitude equations for \( N \) waves, analogous to (7) and (8), can be written

\[
\dot{A}_j = a_j A_j (1 - \sum_k g_{jk} A_k^2).
\]

for the normalized amplitude \( A_j = (b_j/a_j)^{1/2} A_j \) of the \( j \)-th wave. In this equation \( j \) ranges from \( n \) down to \( n - N + 1 \) and the sum goes over the same range. The coefficient \( a_j \) is just the growth rate (linear) of the \( j \)-th wave relative to that for the most unstable \( n \)-wave. Clearly, \( a_n = 1 \) and from Eq. (9) we have

\[
a_j = \frac{a_j}{a_n} = \frac{l_j^2(l_j^2 + F) \left( 1 - \frac{\Delta_{\text{red}}}{\Delta} \right)}{l_n^2(l_n^2 + F)}
\]
where

\[ \Delta_{w_j} = Q_n - Q_j. \]

The matrix \( g_{jk} \) gives the ratio of the squared equilibration amplitudes of the \( k \) wave to the \( j \) wave, times the relative effect on the mean flow of \( k \) with respect to \( j \) per unit amplitude. From (10)–(12) and (15) we get

\[ g_{jk} = \frac{j a_k N_j^2 I_m(\gamma_k)}{k a_j N_k^2 I_m(\gamma_j)}. \]

Note that \( g_{jj} = 1, \ g_{jk} = 1/g_{kj} \), and \( g_{jm}/g_{km} = g_{jk} \). \( N \) is chosen so that all waves that are linearly unstable for a given \( \Delta \) are included. Thus \( a_j > 0 \). It is also easily shown that \( g_{jk} > 0 \).

Steady solutions to (B1) are of two types, either single wave states, or mixed. Single waves are always possible for any particular \( j = M \) and are given by

\[ \begin{align*}
&\tilde{A}_M = 1, \quad n \geq M \geq n - N + 1 \\
&\tilde{A}_j = 0, \quad j \neq M
\end{align*} \tag{B2} \]

Mixed wave steady states must satisfy

\[ \sum_k g_{jk} A_k^2 = 1. \]

In order for this set of algebraic equations to have a solution it is necessary that

\[ \det |g_{jk}| \neq 0. \]

Using the general properties of \( g_{jk} \) listed above it can be shown that this determinant is identically zero. As with the two-wave problem, if sidebands are not included there are no mixed wave steady states.

Consider the linear stability of (B2). Linearizing (B1) about this solution leads to a set of simple stability equations for the perturbations \( \tilde{A}_j' \). These are

\[ \tilde{A}_j' = a_j A_j' (1 - g_{jm}), \quad j \neq M \tag{B3} \]

and

\[ \tilde{A}_M' = -2 A_M'. \]

Thus the \( M \)-wave steady state will be stable if \( g_{jm} \geq 1 \) for all \( j \neq M \). This condition is mutually exclusive in the sense that only one single wave state can be stable at the same point in parameter space. It also can be shown that there always is one stable single wave state for any \( \Delta \). Thus this theory predicts a series of transitions to lower and lower wavenumber as \( \Delta \) is increased.

The existence of limit cycle solutions or perhaps even aperiodic solutions is possible. However the fact that the eigenvalues of (B3) are all real suggests this is probably not the case, and numerical integrations of (B1) for \( N = 3 \) and \( N = 4 \) always evolve to a steady state in a manner similar to that shown in Fig. 6.

REFERENCES


