A Model of Inertial Oscillations with Deformation Frontogenesis

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(Manuscript received 30 September 1996, in final form 15 April 1997)

ABSTRACT

A model of inertial oscillations that may occur with, and be modulated by, deformation frontogenesis is formulated. The deformation parameter is $a \sim 10^{-5}$ s$^{-1}$ and the Coriolis parameter is $f \sim 10^{-4}$ s$^{-1}$. This timescale separation, distinguished by the ratio $a/f \sim 10^{-1}$, provides the basis for application of a two-timescale analysis that separates the frontal evolution from the inertial frequency oscillations. To lowest order, the inertial oscillations do not influence frontogenesis, described by the classical Hoskins and Bretherton model. The frontal evolution, characterized by the alongfront geostrophic wind, does, however, provide an amplitude modulation of the inertial wind oscillation and of the temperature that also undergoes an oscillation at the inertial frequency. Parameter values are chosen to illustrate frontal contraction and translation characteristics that can distort the wind hodograph from circular motion. Ground-level temperature traces also exhibit unusual attributes, such as an initial temperature increase with a cold frontal passage, that can be associated with the relative phase of the oscillation compared to the leading edge of the front. Lack of adequate observations for verification purposes and neglect of the boundary layer provide two important limitations.

1. Introduction

The geostrophic momentum model of frontogenesis introduced by Hoskins and Bretherton (1972) has remained a mainstay of frontal theory for the past quarter century. Numerous research papers have emphasized various aspects of the theoretical model, and illuminating reviews of both the theoretical and observational foundation of the subject may be found, for example, in Pedlosky (1987) and Bluestein (1993).

Ostdiek and Blumen (1995, 1997) recently compared the deformation model of frontogenesis, constrained by the geostrophic momentum approximation, to a well-observed (at least at the surface) frontogenetical situation that occurred during the Stormscale Operational and Research Meteorology-Fronts Experiment Systems Test (STORM-FEST) on the evening of 20–21 February 1992. Satisfactory agreement between theory and observation was attained except in the lowest 400 m of the atmosphere. Boundary layer friction and the widespread presence of horizontal wind oscillations at the inertial frequency $f$, appropriate at approximately 40°N, made the theoretical prediction moot. Ostdiek and Blumen (1997) established that the frictional contribution to the low-level wind field was 1) essentially decoupled from the evolving inertial oscillations and 2) could be modeled by a steady solution to the Ekman–Taylor boundary layer solution using observed and estimated parameter values. No attempt was made to provide a model of the inertial oscillations that, at least in the one case observed, appear to be related to the presence of a cold front but do not appear to play a significant role in its evolution. The purpose of the present study is to quantify the relationship between frontogenesis and inertial oscillations by means of a model that is subject to observational verification.

This is not the first study to recognize the presence of inertial oscillations in association with frontogenesis. Orlanski et al. (1985), for example, provided a simple model of inertial oscillations (much simpler than the one proposed here) to explain the horizontal velocity distribution observed in the Williams (1967) numerical frontogenesis experiment. Inertial oscillations in a basic deformation flow were also explored by S. Garner (unpublished manuscript) but the results were not compared to either observations or to a model simulation.

Inertial oscillations are not the only type of disturbance that should be associated with frontogenesis. The emission of inertia–gravity waves has been examined, for example, by Ley and Peltier (1978) and by Snyder et al. (1993). Wave generation near frontal collapse is associated with imbalances between the mass and momentum fields in the region of the front. Although the fronts examined by Ostdiek and Blumen (1997) were not near a state of collapse, the mechanism of wave generation by imbalance should still exist. This process has usually been examined in the more general context of the geostrophic adjustment process. As noted for example by Blumen (1972), an unbalanced disturbance of limited vertical extent will initiate a broad spectrum of waves, which will disperse at a rate determined by their

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respective horizontal and vertical group velocities. The rotationally dominated waves will disperse at a relatively slow rate, remaining close to the source and oscillating at the inertial frequency. This process could account for the presence of inertial oscillations during the frontal events documented by Ostdiek and Blumen (1997), but no observational evidence has been presented to provide a verification.

The present development takes advantage of the timescale separation between the relatively low-frequency frontal evolution, characterized by the deformation parameter \( \alpha \sim 10^{-5} \text{s}^{-1} \), and the relatively high-frequency oscillations, characterized by the Coriolis parameter \( f \), inertial frequency \( f \sim 10^{-4} \text{s}^{-1} \). Two sets of equations are derived in section 2: deformation frontogenesis is described by the balanced model that incorporates the geostrophic momentum approximation, while the unbalanced high-frequency motions are determined by a set of linear equations whose nonconstant coefficients are evaluated from the balanced model solutions. The inertial oscillation solution is derived in section 3. In contrast to the usual solution for inertial oscillations, the present solution is coupled to the frontal evolution, exhibiting spatial variability and temporal evolution according to the two timescales \((f, \alpha)\). The present development is only carried out to first-order in the non-dimensional expansion parameter \( \alpha f \sim 10^{-1} \), and, to this order, the inertial oscillations do not affect frontal evolution.

Frontal evolution is described by the Davies and Müller (1988) deformation model, as in Ostdiek and Blumen (1995), and various characteristics of the inertial oscillations are displayed in section 4. The spatial distribution of upper-air data on 20–21 February 1995 were not sufficient to provide observational verification of the theoretical predictions. Further, the frontal gradient was relatively weak during this observational period, and its effect on the observed inertial oscillations is questionable. Some final remarks in section 5 suggest some extensions and other applications of the present theoretical development.

2. Model

The model development in this section follows that of Hoskins and Bretherton (1972, hereafter HB) and Pedlosky (1987, § 8.4) with one important exception: the scale analysis will take account of unbalanced motion that is filtered out of the HB model equations by means of the geostrophic momentum approximation. Not all of the unbalanced \( \text{flow}, \) \( \text{flow} \) retained under the geostrophic momentum approximation, and \( \text{flow} \) filtered from the class of allowable solutions by means of the geostrophic momentum approximation. The \( \theta \) field is partitioned according to

\[
\theta = \theta_0 + (N^2 \theta_s/g) z + \theta_u + \theta_v,
\]

where \( \theta_0 \) is a constant surface value, the basic state \( \bar{\theta} \) varies linearly with \( z \), \( \theta_s = \theta_0 \) is a constant reference value, and \( (\theta_u, \theta_v) \) are associated with the balanced and nonsteady \( \text{flow} \), as above. The constant Brunt–Väisälä frequency is \( N \). For fluid motions constrained to relatively shallow depths, the Boussinesq approximation, \( \theta/\theta_0 \), may be replaced by \( -\rho/\rho_0 \).

It will be assumed that a front is aligned parallel to the \( y \) axis, which is coincident with the deformation axis. The deformation flow is

\[
u_x = -\alpha x, \quad v_y = \alpha y.
\]

where \( \alpha \) is assumed to be constant. All deviations from the deformation flow are independent of \( y \), \( \partial/\partial y = 0 \). All variables, with the exception of those designated by \( \ast \) in (1) and (2), are scaled as in HB and in Pedlosky.

The length scales are \( l \) the cross-front scale; \( L \) the alongfront scale; \( D \) the motion scale in the vertical; and \( H \) the characteristic density scale height. The cross-front velocity scale is \( U \), the alongfront velocity scales with \( V \), and the vertical velocity \( W \) scales with \( U/D \). The designation of a timescale will be deferred for the moment. This choice of scales is characterized by

\[
\frac{U}{V} = \frac{L}{l} = \frac{\alpha}{f} \sim 10^{-1}
\]

with

\[
\frac{V}{fl} = 1, \quad \frac{U}{fl} = \frac{\alpha}{f} \sim 10^{-1}.
\]

The timescale \( f^{-1} \), which characterizes inertial oscillations, is approximately an order of magnitude smaller than the advective timescale \( \alpha^{-1} = l/U = L/V \), which is associated with frontal evolution. Choice of a timescale \( t \sim l/U \) leads to the development of the geostrophic momentum approximation. It will be demonstrated that the choice \( t \sim f^{-1} \) retains inertial oscillations that are filtered from the class of allowable solutions by means of the geostrophic momentum approximation.

The remaining scaling relationships that are required to cast the governing equations into a nondimensional set are...
Following Pedlosky, the deformation radius \( L_D = ND/f \) is chosen as an appropriate scale to define \( l \) at the initial time. Further, the density-scale height is

\[
H = \left( -\frac{1}{\rho_s} \frac{\partial \rho}{\partial z} \right)^{-1},
\]

so that the Brunt–Väisälä frequency may be defined as

\[
\nu = \frac{g}{H} N = \left( g/H \right)^{1/2}.
\]

The potential temperature may then be scaled as

\[
\theta_s \sim \frac{D}{H} \theta_0 \sim \frac{f^2 L^2}{gD} \theta_v, \quad \theta_v \sim fUl.
\]

where \( \theta_v \) is defined by (2). The ratio of these latter terms is, from (5), \( \theta_0/\theta_v \sim Ul/f - \alpha f/\rho_0 \).

The present analysis is concerned with surface-based fronts and motions constrained to the lower troposphere, so that \( D/H \ll 1 \). The relevant equations of motion, hydrostatic balance, continuity, and conservation of \( g/\theta \) \( \theta_v \) expressed in nondimensional form are

\[
\frac{a}{f} \left[ \frac{\partial}{\partial t} + \frac{\alpha}{f}(-x + u_a + u_a) \frac{\partial}{\partial x} + \frac{\alpha}{f} (w_a + w_a) \frac{\partial}{\partial z} - \frac{a}{f} \right] (u_a + u_a) - \frac{a}{f} v_a - v_a + \frac{\partial \pi_a}{\partial x} = 0,
\]

(6)

\[
\left[ \frac{\partial}{\partial t} + \frac{\alpha}{f}(-x + u_a + u_a) \frac{\partial}{\partial x} + \frac{\alpha}{f} (w_a + w_a) \frac{\partial}{\partial z} - \frac{a}{f} \right] v_a + \frac{\alpha}{f} v_a = 0,
\]

(7)

\[
0 = -\frac{\partial \pi_a}{\partial z} - \theta_s, \quad 0 = -\frac{\partial \pi_a}{\partial z} - \theta_v,
\]

(8)

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0 \quad \text{and} \quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,
\]

(9)

\[
\left[ \frac{\partial}{\partial t} + \frac{\alpha}{f}(-x + u_a + u_a) \frac{\partial}{\partial x} + \frac{\alpha}{f} (w_a + w_a) \frac{\partial}{\partial z} \right] 1 + \theta_s + \frac{\alpha}{f} \theta_v = 0.
\]

(10)

where \( \pi = \rho/\rho_0 \).

In the notation of (1), the subscripted variables \((g, a)\) are associated with the geostrophic momentum approximation. This fact may be demonstrated by further analysis. The parameter \( \alpha f \sim 10^{-1} \), as noted above, represents a ratio of timescales. Following Cole (1968), the solution of (6)–(10) is to be represented by an expansion of each variable of the form

\[
r = r_0(x, z, t_0, \tilde{t}) + \frac{\alpha}{f} r_1(x, z, t_0, \tilde{t}) + O \left( \frac{\alpha}{f} \right)^2 + \cdots,
\]

(11)

where \((\tilde{t}, t_0)\) represent “slow” and “fast” times defined as

\[
\tilde{t} = \frac{\alpha}{f} t, \quad t_0 = t + O \left( \frac{\alpha}{f} \right)^2.
\]

(12)

The time derivative may now be expressed as

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \frac{\partial}{\partial \tilde{t}} + O \left( \frac{\alpha}{f} \right)^2.
\]

(13)

Insertion of (13) into (6), (7), and (10) and expansion by means of (11) yields the zero-order balance

\[
N = \left( g/H \right)^{1/2}.
\]

and the first-order set,

\[
\frac{\partial u}{\partial t} - v_a + \frac{\partial \pi_a}{\partial x} = 0,
\]

(15)

\[
\frac{\partial v}{\partial t} + v_a + \frac{\partial \pi_a}{\partial x} + w_a \frac{\partial v}{\partial z} = 0,
\]

(16)

\[
0 = -\frac{\partial \pi}{\partial z} + \theta_s, \quad 0 = -\frac{\partial \pi}{\partial z} + \theta_v,
\]

(17)

\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,
\]

(18)

\[
\frac{\partial \theta}{\partial t} + u_a + \frac{\partial \theta}{\partial x} + w_a \left( 1 + \frac{\partial \theta}{\partial z} \right) = -\frac{\partial \theta}{\partial \tilde{t}} + \frac{\partial \theta}{\partial \tilde{t}} + \left( \frac{\partial \theta}{\partial \tilde{t}} \right)^2.
\]

(19)
The scaling has been carried out to isolate two distinct characteristic flows, at least to order \( \alpha f \). The first is frontal in nature that evolves in accord with a “slow” advective timescale \( \ell \). Superposed on frontogenesis are disturbances that may oscillate and/or propagate in association with a “fast” time \( t^* \), and may be modulated as well by the slow time evolution. The separation may be extracted from the set (14)–(19) by noting that the expressions on the right-hand sides of (16) and (19) are independent of \( t^* \). The terms on the left side will grow in amplitude to \( O(\ell f/\alpha) \) in a finite time unless the resonance producing terms on the right-hand sides of (16) and (19) are set equal to zero.

Two sets of equations emerge. Frontogenesis is described by

\[
\begin{align*}
  \frac{\partial}{\partial t} + (u_\alpha - x) \frac{\partial}{\partial x} + w_\alpha \frac{\partial}{\partial z} + 1 \nu_x + u_\alpha &= 0 \\
  \frac{\partial u_x}{\partial x} + \frac{\partial w_x}{\partial z} &= 0 \\
  \frac{\partial \theta_x}{\partial z} + (u_\alpha - x) \frac{\partial \theta_x}{\partial x} + w_\alpha \left( 1 + \frac{\partial \theta_x}{\partial z} \right) &= 0.
\end{align*}
\]

These are the HB deformation frontogenesis equations presented, for example, by Pedlosky (1987, §8d). Further, the thermal wind relation

\[
\frac{\partial \nu_x}{\partial z} = \frac{\partial \theta_x}{\partial x}
\]

follows from (20a,d).
Unbalanced motions are solutions of the set:

$$ \frac{\partial u_*}{\partial t_*} - v_* + \frac{\partial \pi_*}{\partial x} = 0 \quad (22a) $$

$$ \frac{\partial v_*}{\partial t_*} + u_* \left( 1 + \frac{\partial u_*}{\partial x} \right) + w_\ast \frac{\partial v_*}{\partial z} = 0 \quad (22b) $$

$$ \frac{\partial u_*}{\partial x} + \frac{\partial w_*}{\partial z} = 0 \quad (22c) $$

$$ 0 = -\frac{\partial \pi_*}{\partial z} + \theta_* \quad (22d) $$

$$ \frac{\partial \theta_*}{\partial t_*} + u_* \frac{\partial \theta_*}{\partial z} + w_\ast \left( 1 + \frac{\partial \theta_*}{\partial z} \right) = 0 \quad (22e) $$

where (21) has been used to introduce $\frac{\partial v_*}{\partial z}$ into (22e). This set of equations $[(22a)-(22e)]$ is essentially the same as the set that is used to examine symmetric instability (e.g., Emanuel 1982). The only differences from Emanuel’s model equations are the hydrostatic assumption and the retention of $\frac{\partial \theta_*}{\partial z}$ in addition to the static stability $N^2$ in (22e). Since the set of equations (22a)–(22e) is characterized by non-constant coefficients, determined from the solution of the frontogenesis equations (20a)–(20e), a general solution is not readily available. An integral expression for the frequency and a rederivation of the condition that is sufficient for the existence of stable solutions appears in the appendix.

There does exist, however, a subset of solutions that oscillate at the inertial frequency $f$. These solutions will be exposed in the following section.

### 3. Inertial oscillations

The total solution for the velocity field is expressed by (1), where the deformation flow is given by (3); the flow $\mathbf{u}_\ast, \mathbf{u}_\ast$ is determined by (20a)–(20e), while the class of solutions that exhibit inertial oscillations will be derived here. It will be convenient to expose these motions in *dimensional notation* and to drop the sub-
FIG. 3. Axes of maximum relative vorticity \( \partial \phi / \partial x \) at 9 h (solid), 15 h (dotted), and 17.45 h (dashed). (a) Front characterized by \( \theta_g = 2.568.5 \text{ km} \). Arrows delineate the positions of \( u = 0 \) at 9 h, 15 h, and 17.45 h. (b) \( \theta_g = 0 \). Arrows delineate \( u = 0 \) and positions 50 and 100 km east of this stationary position.

The inertial oscillations are to be derived under the assumption that \( w = 0 \). The set (22a)–(22e) reduces to

\[
\begin{align*}
\frac{\partial u}{\partial t} - fu + \frac{\partial \pi}{\partial x} &= 0, & (23a) \\
\frac{\partial \nu}{\partial t} + fu \left( 1 + \frac{1}{f} \frac{\partial \nu}{\partial x} \right) &= 0, & (23b) \\
\frac{\partial}{\partial t} \left( \frac{\partial \pi}{\partial z} \right) + fu \frac{\partial \nu}{\partial z} &= 0, & (23c)
\end{align*}
\]

with \( \partial u / \partial x = 0 \). It is straightforward to show that solutions with time-dependence \( \exp(ift) \) require \( \partial u / \partial z = 0 \). As a consequence, (23c) may be replaced by

\[
\frac{\partial \pi}{\partial t} + fuv_z = 0. & (24)
\]

Then (23a), (23b), and (24) yield

\[
\left( \frac{\partial}{\partial t} - if \right) \left( u + \frac{\partial \phi}{\partial x} + i \nu \right) = 0, & (25)
\]

where \( \phi = \pi / f \). This set of model equations is now isomorphic to the shallow-water equations used by Blumen and Washington (1969) in their examination of inertial oscillations. In the present context \( \pi \) plays the role of the free-surface displacement, but, most impor-
tantly, \( \nu \) is also a function of the vertical coordinate \( z \) as are \( (u, \pi) \), while \( u \) is independent of both \( (x, z) \).

The solution of (25) is

\[
v - \frac{\partial \phi}{\partial x} + i \nu = C (\cos ft + i \sin ft), & (26)
\]

where the complex constant \( C \) is determined by the initial conditions. There is, however, a time-independent solution \( v_\infty \) that may be determined from (23b) expressed as

\[
\frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial x} \right) + \frac{v}{1 + f^{-1} \partial v / \partial x} = 0. & (27)
\]

Two integrations of (27) yield

\[
v_\infty - v_i = C' \left( 1 + \frac{1}{f} \frac{\partial \nu}{\partial x} \right). & (28)
\]

where \( v_i = v \) at \( t = 0 \). The constant \( C' \) is determined by the requirements \( (v_\infty, v_i) \to 0 \) as \( |x| \to \infty \). The constant is \( C' = -v \). The geostrophic velocity and streamfunction are

\[
v_\infty = \frac{\partial \phi}{\partial x}, \quad \psi_\infty = -\frac{v_i v_\infty}{f}. & (29)
\]

The designation \( \infty \) has been used here to distinguish the present origin of this geostrophic flow (27) from that associated with the set (20a)–(20e).

The time-dependent solution (26) is determined by use of the initial conditions \( (u_i, v_i) \), which provide

\[
u = A \sin (ft + \lambda), \quad \nu = A \left( 1 + \frac{1}{f} \frac{\partial \nu}{\partial x} \right) \cos (ft + \lambda), & (30)
\]

where

\[
A = (u_i^2 + v_i^2)^{1/2} = \text{const}, \quad \lambda = \tan^{-1}(u_i/v_i) = \text{const}. & (31)
\]

Further, the expressions for \( \phi = \pi / f \) and \( g \theta / \theta_0 \) are

\[
\phi = A \frac{v_i}{f} \cos (ft + \lambda) , \quad \frac{g \theta}{\theta_0} = A \frac{\partial v_\infty}{\partial z} \cos (ft + \lambda). & (32)
\]

Finally, using (29) the time-independent potential temperature is

\[
\frac{g \theta}{\theta_0} = -v_i \frac{\partial v_\infty}{\partial z}. & (33)
\]

Specification of \( \partial \phi / \partial x \), instead of the wind field, will give similar expressions.
4. Inertial oscillations during frontogenesis

a. Deformation front

Evaluation of the inertial oscillation solutions, (30)–(32), requires either a model solution or an observational specification of frontogenesis, in particular, the specification of \( v(x, z, t) \). Properties of the inertial oscillations will be established by use of the deformation frontogenesis solutions provided by Davies and Müller (1988), which are given by

\[
\begin{align*}
\nu_x &= -V_t \left( \frac{Xe^{\alpha t} - D}{\left( Xe^{\alpha t} - D \right)^2 + \left( L_1 + \frac{N}{f}Ze^{\alpha t} \right)^2} \right) x, \\
\theta_z &= \frac{\Delta \theta_t}{L_1 \left( L_1 + \frac{N}{f}Ze^{\alpha t} \right)^2} \left( Xe^{\alpha t} - D \right)^2 + \left( L_1 + \frac{N}{f}Ze^{\alpha t} \right)^2.
\end{align*}
\]

The parameter \( \Delta \theta_t \) is a characteristic cross-stream potential temperature change, \( V_t = g \Delta \theta_t / \theta_0 \), is a characteristic alongfront velocity scale, denoted as \( V \) in section 2; \( L_1 \) is a cross-stream length scale, \( D \) represents an initial \((t = 0)\) displacement of the \( \theta_z \) field relative to the deformation axis, and \( (X, Z) \) are geostrophic coordinates defined as

\[
X = x + \frac{v_x}{f}, \quad Z = z.
\]

Ostdiek and Blumen (1995, 1997) have established that (34) and (35) provide a not unreasonable representation of surface-based deformation frontogenesis. Davies and Müller have also displayed the ageostrophic flow \((u_a, w_a)\) that corresponds to (34) and (35), but this forced response is not needed for present purposes.

In order to illustrate the inertial oscillation response during frontogenesis, the following parameter values, approximately those used by Ostdiek and Blumen (1995) to model a real frontogenesis event, will be adopted: \( \Delta \theta_t = 10 \) K, \( V_t = 16.85 \) m s\(^{-1}\), \( g = 9.8 \) m s\(^{-2}\), \( N = 2 \times 10^{-3} \) s\(^{-1}\), \( f = 10^{-4} \) s\(^{-1}\), \( \alpha = 2 \times 10^{-5} \) s\(^{-1}\), \( \theta_0 = 291 \) K, \( \theta_0 = 280 \) K, \( L_1 = 400 \times 10^3 \) m, and \( D \) that displace the surface position of the thermal axis \( \theta_x = 0 \) both west and east of the deformation axis \((x = 0)\) will be specified. These parameter values provide an inertial period of 17.45 h and a critical time \( (\partial \nu_x/\partial x = \infty) \) of 18 h. This choice permits a full inertial period to occur during a period of significant frontogenesis, but before the relative vorticity becomes infinite.

The principal features of the modification of the inertial oscillations (30) and (32) by frontal evolution, expressed by (34), may be demonstrated by two values of \( D \). The value \( D = -568.5 \times 10^3 \) m places the surface position of the thermal axis 400 km west of the deformation axis at \( t = 0 \), and the value \( D = 0 \) places the surface position of the thermal axis at 168.5 km east of the deformation axis. The significance of those choices is that in the former case the thermal axis moves eastward to the point where \( \partial \nu_x / \partial x = \infty \), a distance of 434 km in 18 h. In the latter case, the thermal axis remains stationary at \( x = 168.5 \) km, but surface position of the vorticity axis starts at \( x = 358 \) km and retrogresses until \( \partial \nu_x / \partial x = \infty \) at \( x = 189.6 \) km; that is, the thermal axis \( \theta_x = 0 \) remains to the west of the maximum gradient of \( \nu_x \).

Cross sections of both the along front wind and the temperature field given by (2) and (35), but without the inertial oscillations (30) and (32), are displayed in Figs. 1 and 2. The times 9 and 17.45 h are chosen to display the frontal contraction in both cases and the translation of the moving front. In the latter case, two-thirds of the translation occurs half-way through the period, but the average speed for the whole period is approximately 24 km h\(^{-1}\).

b. Wind hodographs of the inertial oscillation

Principal features of frontogenetical modification of inertial oscillations and of the surface temperature trace will be examined. The steady solutions (29) and (33) that are independent of the fast time will not be considered. The inertial oscillation of the wind field will be exhibited by examination of wind hodographs at the locations of \( \theta_x = 0 \) at \( t = 9, 15, \) and 17.45 h for the front that translates from west to east (see Fig. 1). These positions are located in Fig. 3a. The initial values used in (31) are \((u_x, u_y) = (4, 4) \) m s\(^{-1}\), providing \( A = 5.66 \) m s\(^{-1}\) and \( \lambda = \pi/4 \). The relative vorticity \( \partial \nu_x / \partial x \) decays rapidly with height (see Figs. 1 and 2) so that most of the distortion from a circular hodograph occurs in the lowest 500 m.

The hodograph presented in Figs. 4 and 5 are chosen to illustrate a range of possible responses that occur when the relative vorticity axis \( |\partial \nu_x / \partial x|_{\max} \) passes a fixed point. The anticyclonic turning of the wind is affected by both the translation of the vorticity axis and by the increase of the relative vorticity with time—that is, frontogenesis. The passage of the vorticity axis is evident at all locations in Fig. 4. Frontogenesis is responsible for the relative increases in the wind magnitudes, going from top to bottom in the panels. In some cases the magnitude of the velocity at the end of the inertial period is less than its initial value. This latter feature is particularly noticeable at \( x = -128 \) km (both elevations) and \( x = -24 \) km (ground level).

Wind hodographs associated with the stationary thermal axis, \( \theta_x = 0 \), situated at \( x = 168.5 \) km at \( z = 0 \), are shown in Fig. 5. The positions chosen for display, \( x = 168.5, 218.5 \), and 268.5 km, are at \( \theta_x = 0 \) and at 50 km and 100 km eastward. The relative vorticity axis moves westward past the latter two locations, reaching...
Fig. 4. Hodographs of the inertial component of the wind at locations shown in Fig. 3a. Times are shown on the curves. The left panel corresponds to \( z = 0 \); the right panel to \( z = 250 \) m. The wind reaches 25 m s\(^{-1}\) at \( t = 15 \) h at \( x = 8 \) km and \( z = 0 \).

Features similar to those associated with Fig. 4 are evident in Fig. 5. Frontogenesis, expressed by an increase in the magnitude of the relative vorticity is a dominant feature of the ground-level hodograph at \( x = 218.5 \) km. Interestingly, this frontogenetical feature is more evident above the surface at \( x = 168.5 \), which is explained by the characteristic bulge in the vorticity axis in Fig. 3b (dashed line).

Principal characteristics of the inertial oscillation response to a propagating front undergoing frontogenesis are illustrated in these wind hodographs. There are, however, limitations in the application of the present
model to frontal conditions near the critical time when the relative vorticity becomes infinite—that is, frontal collapse. The amplitudes of the inertial oscillations do not remain relatively small compared to the characteristic flow velocities associated with the frontal evolution, and terms of order \((a/f)^2\) containing spatial gradients of the balanced flow may not be negligible.

c. Temperature traces

Temperature traces at ground level, \(z = 0\), will be examined using the same parameter values and locations presented above with one exception. The initial amplitudes \((u, v)\) will be presented by (a) \(u_i = v_i\), and (b) \(u_i, -v_i\) where \(|u_i| = 4\) m s\(^{-1}\). Case a is as above with \(\lambda = \)
Fig. 6. Temperature traces at the surface \((z = 0)\) at the locations shown in Fig. 3a. Frontogenesis temperature \(\theta_g\) (thin solid), inertial oscillation temperature \(\theta_i\) (dashed), and their sum (thick solid). Top: case (a) \(u_i = u_i\). Bottom: case (b) \(u_i = -u_i\), where \(|u_i| = 4\) m s\(^{-1}\).

\(\pi/4\). Case b provides \(\lambda = -\pi/4\). The temperature field \(\theta_g + \theta_i\) where \((\theta_g, \theta_i)\) are provided by \((35)\) and \((32)\), respectively, are displayed and interpreted. Only the part of \(\theta\) associated with the inertial oscillation is considered. The purpose here is to emphasize only the impact of the inertial oscillations \((32)\) on the more slowly evolving temperature changes associated with frontogenesis. Figures 6 and 7 are the temperature counterparts of Figs. 4 and 5 with case a above and case b below for comparison.

The phase difference between cases a and b is \(\pi/2\), and this difference has a significant effect on the temperature change associated with frontal passage. As the front approaches, moving eastward in Fig. 6, \(\theta_g\) declines monotonically from warm to cold air. The inertial responses, on the other hand, are oscillatory. Its phase, relative to the frontal temperature change, and its magnitude determined by \(|\theta_g|/\theta_g\), from the thermal wind relation \((21)\), will produce unexpected temperature traces associated with frontal passages. The traces at \(x = -24\) km provide a case in point. There is an initial temperature increase associated with frontal passage for case a, but the inertial oscillation for case b initially reinforces the temperature drop provided by frontal passage.

The temperature changes associated with \(\theta_g\) in Fig. 7 reflect frontal contraction, not the passage of a front in the temperature field. Superposed inertial oscillations, more or less, determine the dominant features of the temperature traces associated with the stationary thermal axis, situated at \(x = 168.5\) km.

5. Final remarks

A theoretical framework has been developed to examine frontal modification of inertial wind and temperature oscillations. The wind field is modified by the horizontal shear of the alongfront wind, the relative vorticity \(\zeta\) in the two-dimensional model. Values that are typically associated with fronts, \(\zeta \sim f \sim 10^{-4}\) s\(^{-1}\), could be expected to produce characteristic signatures as depicted in Figs. 4 and 5. Temperature oscillations at the inertial period are also a by-product of the present model. These oscillations, superposed on frontally produced temperature changes, are also modified by frontal contraction and translation. It is the evolution of vertical shear of the alongfront wind, or equivalently the cross-front horizontal temperature gradient, that provides an amplitude modulation of the inertial temperature oscillation \((32)\).

Inertial oscillations have previously been documented, principally in nonfrontal events in the Great Plains.
Fig. 7. Same as Fig. 6 except for locations shown in Fig. 3b.

(Bonner 1968), and more recently in association with a frontal event in the same locale by Ostdiek and Blumen (1995, 1997). The latter event, observed during 20–21 February 1992, was relatively weak, producing surface values of relative vorticity of order \( \zeta \approx 2 \times 10^{-5} \, \text{s}^{-1} \). Moreover, the upper-air data were too sparse to determine adequate values of parameters that could be used for verification purposes. Data from more recent observations are presently being examined for this purpose.

The evidence for inertial oscillations in the presence of frontal gradients does not appear to be in question. In the event examined by Ostdiek and Blumen (1997), evidence for inertial oscillations, decoupled from boundary layer frictional influence, could even be detected in some 10-m level data as well as through a much deeper layer. Unresolved issues are related to the relative influence of inertial oscillations on the rate of frontogenesis, and on the role of the boundary layer on low-level frontogenesis with or without the presence of inertial oscillations.

Acknowledgments. Support for this investigation was provided by the National Science Foundation under Grants ATM-930311 and ATM-9627792, and by the Air Force Office of Scientific Research under Grant F49620-95-1-0141. Computational assistance and data analysis by Julie Lundquist is gratefully acknowledged.

APPENDIX

Inertial Stability Condition

Equations (20a,b,d,e) yield

\[
\frac{\partial^2}{\partial t^2} \left( \frac{\partial u_\theta}{\partial z} \right) + \frac{\partial}{\partial z} \left( u_\theta \left( 1 + \frac{\partial v_\theta}{\partial x} \right) + w_\theta \frac{\partial v_\theta}{\partial z} \right)
\]

\[
- \frac{\partial}{\partial x} \left( u_\phi \frac{\partial v_\phi}{\partial z} + w_\phi \left( 1 + \frac{\partial \theta_\phi}{\partial z} \right) \right) = 0. \quad (A1)
\]

Solutions of the form \( u_\phi = \tilde{u}(x, z) \exp i \sigma t \), and similarly for \( w_\phi \), are introduced into (A1). A streamfunction \( \psi \), defined by

\[
u_\phi = -\frac{\partial \psi}{\partial x}, \quad w_\phi = \frac{\partial \psi}{\partial z}, \quad (A2)
\]

is also introduced into (A1) to yield

\[
\sigma^2 \left( \frac{\partial^2 \psi}{\partial z^2} \right) + \frac{\partial}{\partial z} \left( -\frac{\partial \psi}{\partial z} \left( 1 + \frac{\partial v_\psi}{\partial x} \right) + \frac{\partial \psi \partial v_\psi}{\partial x \partial z} \right)
\]

\[
- \frac{\partial}{\partial x} \left( \frac{\partial \psi \partial v_\psi}{\partial z} + \frac{\partial \psi}{\partial z} \left( 1 + \frac{\partial \theta_\psi}{\partial z} \right) \right) = 0. \quad (A3)
\]
The disturbance streamfunction vanishes at the top and bottom boundaries, and \((u, w)\) are assumed to vanish as \(|x| \to \infty\). Multiplication of (A3) by \(\psi^*\), the complex conjugate of \(\psi\), and integration over the flow domain \(A\) yields

\[
\sigma^2 \int_A |\tilde{\psi}|^2 \, dx \, dz = \int_A \left\{ \left( 1 + \frac{\partial u}{\partial x} \right) |\tilde{\psi}|^2 + \left( 1 + \frac{\partial \theta}{\partial z} \right) |\tilde{\psi}|^2 \right. \\
\left. + \frac{\partial \psi}{\partial \zeta} [\tilde{u}^* \tilde{\psi} + \tilde{\psi}^* \tilde{u}] \right\} \, dx \, dz \quad (A4)
\]

Introduction of the inequality

\[
\sigma |\tilde{\psi}|^2 + b |\tilde{\psi}|^2 \geq \sqrt{ab} (\tilde{u}^* \tilde{\psi} + \tilde{\psi}^* \tilde{u}), \quad (A5)
\]

where \((a, b)\) are arbitrary coefficients provides the sufficient condition for stable solutions (\(\sigma\) real), expressed in dimensional units as

\[
f + \frac{\partial \psi}{\partial \zeta} \left( \frac{g}{\theta} \frac{\partial \theta}{\partial \zeta} \right) \geq 1. \quad (A6)
\]

Hoskins (1974) has noted that (A6) is the condition that the potential vorticity (in this case the potential vorticity associated with geostrophic momentum approximation) be positive to ensure stability. This latter constraint is also satisfied by deformation frontogenesis solutions to be used in the present study (e.g., Davies and Müller 1988).

REFERENCES


