Symmetric Stress Tensor Formulation of Horizontal Momentum Diffusion in Global Models of Atmospheric Circulation

ERICH BECKER

Leibniz-Institut für Atmosphärenphysik an der Universität Rostock e.V., Kuhlungsborn, Germany

(Manuscript received 14 April 1999, in final form 30 May 2000)

ABSTRACT

In climate and weather forecast models, small-scale turbulence in the free atmosphere is usually parameterized by horizontal diffusion of horizontal momentum. This study proposes a formulation that is based on a symmetric stress tensor. The advantage over conventional methods is twofold. First, the Eulerian law of angular momentum conservation is fulfilled. Second, a self-consistent formulation of the momentum and thermodynamic equations of motion becomes possible due to incorporation of the local frictional heating rate, that is, the proper dissipation.

The importance of these issues is demonstrated by numerical experiments performed with a simple general circulation model. For example, the new scheme precisely accounts for the irreversible increase of total potential energy during the decay of a baroclinic life cycle. Also the stress generated by horizontal momentum diffusion is found to be significant in the angular momentum budget of multiple life cycle experiments.

1. Introduction

Pushed by the rapid increase in computer facilities, numerical models of the general circulation of the atmosphere (hereafter GCMs) have reached considerably advanced states (e.g., Washington and Parkinson 1986; Schmitz 1991). Nevertheless, principal difficulties in the design of weather forecast and climate models remain, mainly because the physical processes involved vary in their typical time- and spatial scales by many orders of magnitude. Therefore, to appropriately simulate weather and climate models, fast subscale processes must be parameterized as functions of the large-scale flow. Since present day GCMs employ a huge body of parameterization schemes, there is ample room for ambiguity in the formulation of the dynamics. The present study addresses the question whether horizontal diffusion schemes, which are commonly used to model turbulent friction in the free atmosphere, are compatible with elementary fluid mechanical constraints.

Convenient models of the planetary boundary layer are based on Prandtl’s mixing length concept. Accordingly, vertical and horizontal turbulent velocities are assumed to be of the same order of magnitude. As a result of the shallow-atmosphere approximation for the large-scale flow, divergence of vertical turbulent momentum flux is retained in the equations of motion while horizontal turbulent fluxes give no contributions at all (e.g., Pichler 1986). A priori, these assumptions for the turbulent motion should also be appropriate in the free atmosphere. However, one encounters a serious physical inconsistency associated with the dynamics of synoptic and planetary waves. In these wave systems of the atmosphere, enstrophy is transferred into smaller scales until it is dissipated due to small-scale turbulence (Pedlosky 1987). Hence, turbulent friction in the free atmosphere is scale-selective of the horizontal motion. In practice, this mechanism is accounted for by empirically adding scale-selective damping terms to the equations of motion. An explicit horizontal diffusion (i.e., an even power of the horizontal nabla operator applied to the horizontal wind vector) is most common and works sufficiently well (Smagorinsky 1963). Order and magnitude of horizontal diffusion is adapted to the spatial resolution of the numerical model in order to suppress truncation errors. The impact of different orders and magnitudes of horizontal diffusion has been analyzed, for instance, by Barnes and Young (1992), Laursen and Eliassen (1989), and Alexeev et al. (1996).

However, it is a first principle of fluid mechanics that friction must be formulated as divergence of a symmetric tensor. Otherwise the Eulerian law of angular momentum conservation for finite control volumes is not fulfilled (e.g., Serrin 1959; Szabó 1977; Lindzen 1990). Since the Reynolds stress tensor is symmetric by definition, the symmetry constraint applies to molecular friction and turbulent friction as well. In this study it will be shown that conventional horizontal diffusion
corresponds to a nonsymmetric stress tensor. As an alternative, a symmetric stress tensor for horizontal momentum diffusion is proposed that is similar to a previous formulation of Smagorinsky (1993). The symmetric formulation leads to simple modifications of the conventional friction forces. It furthermore yields the dissipation to be included as diabatic heating in the thermodynamic equation of motion. Generally, friction schemes that are not based on a symmetric stress tensor give rise to both spurious angular momentum and heat sources.

The outline of this study is as follows. In section 2 we will briefly recapitulate the governing equations of motion underlying a simple general circulation model (SGCM). Section 3a introduces a symmetric stress tensor formulation for second-order horizontal diffusion. High-order diffusion schemes are discussed in section 3b. Section 4 presents SGCM experiments to illustrate the advantages of a symmetric horizontal diffusion scheme. Section 5 gives some concluding remarks.

2. Simple framework of a GCM

The large-scale atmospheric flow is governed by the primitive equations that may be written as [see Phillips (1973) for instance]

\[
\begin{align*}
\partial_t \mathbf{v} &= \mathbf{v} \times \mathbf{e}_z (f + \mathbf{e}_z \cdot (\nabla \times \mathbf{v})) - \nabla \frac{\mathbf{v}^2}{2} \\
&\quad - \mathbf{w} \partial_z \mathbf{v} - \nabla^2 \frac{\mathbf{v}_p}{\rho} + \mathbf{Z} + \mathbf{H} \\
\partial_t T + (\mathbf{v}_s \cdot \nabla) T &= \frac{\omega}{c_p \rho} + Q + (Z_T + H_T) \\
&\quad + \frac{1}{c_p} (d(Z) + d(H)) \\
\partial_t \rho &= -g \rho \\
\partial_t \rho + \nabla \cdot (\rho \mathbf{v}_s) &= 0.
\end{align*}
\]

Here, \( \mathbf{v}_s = \mathbf{v} + \mathbf{e}_z \) and \( \nabla \mathbf{v}_s = \nabla + \mathbf{e}_z \partial_z \) are the three-dimensional velocity field and nabla operator, while \( \mathbf{v} \) and \( \nabla \) consist of horizontal components only. The symbol \( T \) is temperature, \( \mathbf{e}_z \) is the unit vector in vertical direction, \( z \) denotes the height above sea level, and \( f \) is the Coriolis parameter. The atmosphere is considered as an ideal gas. Friction forces and diabatic heating terms will be specified below. The notation in (2.1)–(2.4) is otherwise standard.

The general circulation is driven by radiative plus latent heating, which is represented by \( Q \) in (2.2). Turbulent boundary layer mixing is parameterized in terms of vertical diffusion of horizontal wind and potential temperature \( \Theta \):

\[
\begin{pmatrix}
\mathbf{Z} \\
Z_T
\end{pmatrix} = \frac{1}{\rho} \partial_z \left\{ \frac{\rho \nu \partial_z \mathbf{v}}{\rho(P_r \partial_z \mathbf{v})} \right\},
\]

with \( P_r \) denoting a globally constant Prandtl number. Appropriate dynamic boundary conditions

\[
\left( \rho \nu \partial_z \mathbf{v} \right)_{\text{surface}} = C \rho_p \mathbf{v}_p \quad \text{and}
\]

\[
\left( \rho \frac{\nu}{P_r} \partial_z \Theta \right)_{\text{surface}} = \frac{C}{P_r} \rho_s (\Theta_p - \Theta_s)
\]

balance the lower-boundary wind stress with the wind at the top of the Prandtl layer (indicated by the index \( p \)) and the lower-boundary sensible heat flux with the corresponding temperature deviation from a prescribed surface temperature \( \Theta_s \). The top of the Prandtl layer is usually identified with the lowermost full model layer. The viscosity and surface coefficients \( \nu \) and \( C \) are defined as functions of the large-scale flow via a chosen turbulence model (e.g., Holtslag and Boville 1993).

Horizontal diffusion is represented by \( \mathbf{H} \) and \( H_T \). A common ansatz for \( H_T \) is

\[
H_T := (-)^{\mu} a^{2n}(K/P_r) \nabla^{2n+2} T, \quad \mu = 0, 1, 2, \ldots,
\]

where \( K \) is a kinematic viscosity and \( a \) denotes the radius of the earth. Horizontal diffusion of wind is usually incorporated as
which is distinguished from alternative formulations by the index "conv." Let $L = 10^3, \ldots, 10^4$ km and $h = 10$ km denote the typical horizontal and vertical length scales of the large-scale atmospheric flow. Then, in addition to

\[ |w|/|v| \sim |\nabla v|/|\partial_z v| \sim h/L \ll 1 \quad \text{and} \quad r = a + z \approx a, \quad (2.9) \]

it is assumed that

\[ \mathbf{H}_{\text{conv}} = (-)^m \zeta^2 K \nabla^{2m+2} \mathbf{v}, \quad m = 0, 1, 2, \ldots, \quad (2.8) \]

Frictional heating (dissipation) is represented by $d(Z)$ and $d(H)$ in the thermodynamic equation (2.2). These terms ensure a physically self-consistent formulation of the equations of motion. Note that dissipation is well defined only if the stress tensor is symmetric. As concerns vertical diffusion, it is straightforward to show that the symmetry constraint generally applies. Let us indicate the tensor product by an open circle such that $(\mathbf{a} \cdot \mathbf{b}) \mathbf{c} = \mathbf{a} (\mathbf{b} \cdot \mathbf{c})$ and $(\mathbf{a} \cdot \mathbf{b}) \mathbf{c} = (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$ for arbitrary vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Then, owing to (2.9), $Z$ can be written as

\[ |\nabla \rho| \ll \rho/L \quad \text{and} \quad |\nabla K| \ll K/L. \quad (2.10) \]
\[ Z = \frac{1}{\rho} \nabla \cdot \Sigma Z \]

with \( \Sigma z := \nu \rho (e_z \cdot \partial \cdot v + \partial \cdot v \cdot e_z) \). \hfill (2.11)

In the appendix it is shown that the corresponding dissipation yields (see also Smagorinsky 1993)
\[ d(Z) = \nu (\partial \cdot v)^2. \hfill (2.12) \]

3. Conventional versus symmetric formulation of horizontal diffusion

In the following section, a symmetric horizontal diffusion tensor, the associated friction force, and dissipation are derived for the low-order case \([\mu = 0 \text{ in } (2.8)]\). The presentation provides the main formulas while mathematical details can be found in the appendix. Higher orders of horizontal diffusion will be discussed in section 3b.

It is appropriate to retain (2.10) as working hypothesis only. The scale approximations (2.9) are assumed to be applied to expressions of friction force or frictional heating in order to achieve consistency with the notation of the previous section. Accordingly, the horizontal nabla operator is redefined as
\[ \nabla = e_x \partial_x + e_y \partial_y = e_x \frac{\partial_x}{r \cos \phi} + e_y \frac{\partial_y}{r}. \hfill (3.1) \]

The vectors \( e_x \) and \( e_y \) are unit vectors in zonal and meridional direction while \( \lambda \) and \( \phi \) denote geographical longitude and latitude.

\( \text{a. Second-order horizontal diffusion} \)

First of all we note that the explicit form of conventional horizontal diffusion
\[ H_{\text{conv}} = K \nabla^2 v = Ke \left( \nabla^2 v - \frac{u}{r^2 \cos^2 \phi} - \frac{2 \tan \phi}{r} \partial_v u \right) + Ke \left( \nabla^2 v - \frac{v}{r^2 \cos^2 \phi} + \frac{2 \tan \phi}{r} \partial_y u \right) - 2 Ke \frac{D}{r} \]

has a vertical component that is proportional to the horizontal divergence \( D = \nabla \cdot v \) and can be neglected by means of the hydrostatic approximation. The \( u \) and \( v \) are zonal and meridional velocity components.

Since (3.2) can be written as \( K \nabla (\nabla \cdot v) \), it is suggestive to formulate a symmetric stress tensor for horizontal momentum diffusion in the following way:
\[ \Sigma_{\text{sym}} := K \rho [ \nabla \cdot v ] + K \rho [ \nabla \cdot v ]^T. \hfill (3.3) \]

In (3.3), the transposed tensor is indicated by the superscript \( T \). With \( \xi = e_z \cdot (\nabla \times v) \) denoting the horizontal vorticity, the explicit form of (3.3) yields
\[ \Sigma_{\text{sym}} = \rho K (2 e_x \cdot e_x (D - \partial_z u) + e_y \cdot e_y (\xi + 2 \partial_y u) - e_x \cdot e_y u + e_y \cdot e_x u/r + e_z \cdot e_x (\xi + 2 \partial_z u) + 2 e_x \cdot e_y u + e_y \cdot e_y u/r - e_x \cdot e_z u/r), \]

and the resulting friction force is
\[ H_{\text{sym}} = \frac{1}{\rho} \nabla \cdot \Sigma_{\text{sym}} = -K \left\{ \nabla \cdot v + \nabla D - \frac{D}{r} \right\} - \frac{1}{\rho} \partial_v (K \rho v). \hfill (3.5) \]

From (3.4) we see that \( \Sigma_{\text{sym}} \) imposes tangential stress on horizontal surfaces and thus requires dynamic boundary conditions. The corresponding tensor components imply upward flux of zonal (meridional) momentum for \( u > 0 \) (\( v > 0 \)), and vice versa. Hence, upper-level winds can strongly be amplified. This is also obvious from the last term on the rhs of (3.5) in connection with the exponential vertical decay of density \( \rho \). Therefore, the ansatz (3.3) is not appropriate to describe scale-selective damping of the horizontal motion. One must introduce some modification such that stress on horizontal surfaces vanishes. This can be achieved, letting
\[ \Sigma := \rho (K \nabla \cdot v + K \nabla \cdot v)^T = K \rho [ \nabla \cdot v ] + K \rho [ \nabla \cdot v ]^T = 0, \]

The new friction force
\[ H = \frac{1}{\rho} \nabla \cdot \Sigma = K \left\{ \nabla \cdot v + \nabla D + 2 v/r^2 \right\} \]

differs from (3.2) by two terms. The \( \nabla D \) is well known from molecular friction. The term \( 2 v/r^2 \) results from anisotropic viscosity \( (K \gg \nu) \) in connection with spherical geometry. It ensures that any arbitrary superrotation is not damped (see section 3c). The tensors (3.3) and (3.6) become equivalent in Cartesian coordinates.

An additional problem of empirical friction parameterizations is concerned with the laws of thermodynamics. In fact, friction can change the total energy of a finite fluid volume via the boundary stress only. To fulfill this constraint, one must generally include the frictional heating in the thermodynamic equation of motion (e.g., Serrin 1959; Lindzen 1990). Let \( \Sigma_{\text{tot}} \) denote the total stress tensor. Then, the frictional heating per unit mass is given by \( \rho \cdot [\Sigma_{\text{tot}} \cdot \nabla] \cdot v \). It is evident that turbulent friction describes an irreversible process. Hence, in addition to symmetry of the stress tensor, we have to demand that the frictional heating (dissipation) must not be negative. As anticipated by (2.11) and (2.12), vertical diffusion is generally in accordance with both constraints. The dissipation due to \( \Sigma \) yields
\[ d(\mathbf{H}) = K \left[ 2(D - \partial_x v)^2 + 2(\partial_y v)^2 + (\xi + 2\partial_y u)^2 + \frac{2Dw}{r} \right] \tag{3.8} \]

and is greater equal zero by definition with regard to (2.9). Thus the tensor (3.6) describes an irreversible process.

**b. Horizontal diffusion of higher orders**

For high-resolution GCMs it is common to employ (2.7) and (2.8) with \( \mu \geq 1 \). This is done to restrict the damping to the smallest resolved scales while leaving the planetary-scale flow almost unaffected. Our consideration from the previous section can easily be formulated for arbitrary \( \mu \geq 0 \).

Let us substitute the horizontal velocity field according to

\[ \mathbf{v} \rightarrow \mathbf{v}_\mu = u_\mu \mathbf{e}_x + v_\mu \mathbf{e}_y := (-\mu r^2 \nabla^2 u_\mu, \nu). \tag{3.9} \]

Using the abbreviations \( D_\mu := \nabla \cdot \mathbf{v}_\mu \) and \( \xi_\mu = \mathbf{e}_x \cdot (\nabla \times \mathbf{v}_\mu) \), the symmetric tensor (3.6) is generalized according to

\[
\Sigma_\mu := K\rho[(\nabla + \mathbf{e}_x) \cdot \mathbf{v}_\mu + \nabla D_\mu + 2\mathbf{v}_\mu / r^2] = K\rho[2\mathbf{e}_x \cdot \mathbf{e}_y (D_\mu - \partial_x u_\mu) + 2\mathbf{e}_x \cdot \mathbf{e}_y \partial_x u_\mu + 2\mathbf{e}_x \cdot \mathbf{e}_y \partial_y u_\mu] + 2\mathbf{e}_x \cdot \mathbf{e}_y (\xi_\mu + 2\partial_y u_\mu) + 2\mathbf{e}_x \cdot \mathbf{e}_y \partial_x u_\mu, \tag{3.10} \]

and the corresponding friction force becomes

\[ \mathbf{H}_\mu = \frac{1}{\rho} \nabla \Sigma_\mu = K(\nabla^2 \mathbf{v}_\mu + \nabla D_\mu + 2\mathbf{v}_\mu / r^2). \tag{3.11} \]

A widely used method in the formulation of a GCM is to expand vorticity \( \xi \) and divergence \( D \) into a series of spherical harmonics. Let \( n \) and \( m \) denote the total both symmetric and zonal wavenumbers. Then in spectral space \( \{ \xi_{nm}, D_{nm} \} \) and owing to (2.9), the conventional formulation (2.8) and the symmetric formulation (3.11) give rise to the following damping terms:

**conventional**

\[
\begin{align*}
\partial_t \xi_{nm} &= \cdots - \frac{K}{\alpha^2} [n(n + 1)]^{1+\mu} \xi_{nm} \\
\partial_t D_{nm} &= \cdots - \frac{K}{\alpha^2} [n(n + 1)]^{1+\mu} D_{nm}
\end{align*}
\tag{3.12}
\]

**symmetric**

\[
\begin{align*}
\partial_t \xi_{nm} &= \cdots - \frac{K}{\alpha^2} [n(n + 1)]^{1+\mu} - 2[n(n + 1)]^\mu \xi_{nm} \\
\partial_t D_{nm} &= \cdots - \frac{K}{\alpha^2} [2(n(n + 1)]^{1+\mu} - 2[n(n + 1)]^\mu D_{nm}
\end{align*}
\tag{3.13}
\]

With regard to vorticity, the correction term \(-2[n(n + 1)]^\mu\) in (3.13) imposes a stronger scale selectivity in comparison to the conventional formula (3.12); that is, the damping is strongly reduced at low wavenumbers and it is unchanged for \( n \rightarrow \infty \). The symmetric formulation furthermore determines an exact relationship between damping of vorticity and divergence, whereas the conventional method has often been modified by empirically introducing different kinematic viscosities for \( \xi \) and \( D \).

The frictional heating per unit mass associated with (3.10) yields

\[ d(\mathbf{H}) = K \left[ 2(D_\mu - \partial_x v_\mu)(D - \partial_x v) + 2(\partial_y v_\mu)(\partial_y v) + (\xi_\mu + 2\partial_y u_\mu)(\xi + 2\partial_y u) + \frac{2Dw}{r} \right], \tag{3.14} \]

which leads to (3.8) in case of \( \mu = 0 \). For \( \mu \geq 1 \), (3.14) is not positive definite. Of course, any biharmonic or higher order horizontal diffusion (either symmetric or not) removes kinetic energy from the flow. But this does not necessarily ensure the increase of entropy due to frictional heating. A physically reasonable formulation of horizontal diffusion that satisfies the constraints of both symmetric stress and positive definite dissipation is given by (3.11) and (3.14) for \( \mu = 0 \) only.

**c. Superrotation**

The superrotation is an elementary example to test the properties of horizontal diffusion. Let us consider a
superrotation with the maximum zonal velocity being a function of height \( z \):

\[
\psi_z := -u_z(z)r \sin \phi, \quad v_1 := u_z(z) \cos \phi e_x. \quad (3.15)
\]

[For \( u_z(z) = \text{constant} \), (3.15) describes to an ordinary superrotation; \( u_z(z) \propto r/a \) yields a solid body rotation.]

Using methods of analytic geometry, we can compute corresponding streamfunctions and velocity fields of superrotations with their axes lying in the equatorial plane and pointing from \( \lambda = 180^\circ \) to \( \lambda = 0 \),

\[
\psi_z := -u_z(z)r \cos \phi \cos \lambda, \quad v_z := -u_z(z)(\sin \phi \cos \lambda e_x - \cos \lambda e_z). \quad (3.16)
\]

or from \( \lambda = 270^\circ \) to \( \lambda = 90^\circ \),

\[
\psi_z := -u_z(z)r \cos \phi \sin \lambda, \quad v_z := -u_z(z)(\sin \phi \sin \lambda e_x + \cos \lambda e_z). \quad (3.17)
\]

Any superposition of (3.15)–(3.17) must be unaffected by horizontal diffusion because otherwise the angular momentum of the flow could change without a balancing zonal drag. Now it is readily shown that

\[
\nabla^2 v_1 = -2v_1/r^2, \quad \nabla^2 v_2 = -2v_2/r^2 \quad \text{and} \quad \nabla^2 v_3 = -2v_3/r^2. \quad (3.18)
\]

This proves that the conventional friction force (2.8) cannot in any way be derived from a symmetric stress tensor because every superrotation is uniformly damped out. In contrast, the symmetric formulation (3.7) gives
no contribution as it should be. It is furthermore evident that the dissipation (3.8) vanishes for any superposition of \( v_1, v_2, \) and \( v_3 \) since we have \( D = \partial_\nu = 0 \) and \( \xi = -2\partial\mu \) in each case (3.15)–(3.17).

4. SGCM experiments

In this section, several SGCM experiments are considered to explore the consequences of employing nonsymmetric instead of symmetric horizontal diffusion. The present SGCM has already been used in previous studies (e.g., Becker et al. 1997). The numerical methods consist of representation by spherical harmonics and finite differences in vertical direction (Hoskins and Simmons 1975; Simmons and Burridge 1981). The spatial resolution is spectral truncation at total wavenumber 29 and 32 hybrid layers representing equal mass portions. Application of horizontal diffusion is restricted to the low-order case \( m = 0 \), and for simplicity, diffusion of heat is generally neglected (1/\( \rho_c = 0 \)). In each example (sections 4a–c), two runs have been performed using either the symmetric horizontal diffusion scheme (3.6)–(3.8) (case I) or the conventional scheme (3.2) (case II). In case III, the friction force (3.7) due to the symmetric tensor (3.6) is applied but the dissipation (3.8) is neglected. A globally constant kinematic viscosity of \( K = 2.5 \times 10^3 \) m\(^2\) s\(^{-1}\) is used. Such a value is typical for low-resolution climate models (e.g., McAvaney et al. 1978) and corresponds to a damping time constant of about 2.2 days for wavenumber 29 of relative vorticity.

a. Thermally balanced zonal flow

A thermally balanced zonal flow is a stationary solution of the primitive equations without any friction and diabatic processes. It consists of a temperature field \( T_z(\phi, p) \) and a corresponding zonal wind \( u_z(\phi, p) \) such that \( T_z \) and \( u_z \) obey thermal wind balance (see Fig. 1). Even though the present example of a thermally balanced flow is baroclinically unstable, it is conserved by the SGCM as long as wave perturbations are absent.

The situation is different if horizontal diffusion is applied while surface drag (vertical diffusion) and external diabatic heating \( Q \) are still neglected. Starting with \( (T_z, u_z) \) as initial condition, the 2D version of the SGCM has been integrated for 500 days according to cases I, II, and III. Figure 2 shows the relative temperature changes, the zonal winds, and meridional mass streamfunctions at day 500 for cases I and II. (It is reasonable to calculate a momentary mass streamfunction since the time evolution of the system is very slow and nonoscillatory.) Both simulations appear qualitatively equivalent. We observe similar temperature deviations relative to the initial condition (Figs. 2a,b). These changes reflect adiabatic heating and cooling associated with the meridional circulations (Figs. 2e,f) that in turn are induced by deviations from thermal wind
balance. At day 500, both zonal wind profiles (Figs. 2c,d) show that the subtropical jets are smoothed out while equatorial westerlies are enhanced; that is, for \( t \to \infty \), horizontal diffusion selects something like the superrotation component of the initial wind profile \( u_E \). Close to the surface, the zonal winds change sign. Despite these similarities, quantitative differences between both simulations are obvious even from the representation in Fig. 2.

The advantage of symmetric over nonsymmetric horizontal diffusion can clearly be demonstrated by considering global energy and angular momentum:

\[
TP := \int_{z=0}^{\infty} dz \rho c(T + gz) = \int_{z=0}^{\infty} dz \rho c_T \tag{4.1}
\]

\[
TK := \int_{z=0}^{\infty} dz \frac{v^2}{2} \tag{4.2}
\]

\[
L_r := \int_{z=0}^{\infty} dz p\bar{u} a \cos \phi \tag{4.3}
\]

\[
L_0 := \int_{z=0}^{\infty} dz \rho \Omega a^2 \cos^2 \phi \tag{4.4}
\]

In (4.1)–(4.4) the horizontal average is indicated by brackets. Here, \( TP, TK, L_r, \) and \( L_0 \) are total potential energy, total kinetic energy, relative angular momentum, and absolute angular momentum of the atmosphere. Total energy \( TP + TK \) and total angular momentum \( L_r + L_0 \) must be conservative quantities in our simulations.

Figure 3 shows (4.1)–(4.4) as well as total energy and total angular momentum as functions of time for cases I–III. In \( \Delta TP \) or \( \Delta L_0 \) the total potential energy or the absolute angular momentum of the initial state has been subtracted. In comparison with the conventional horizontal diffusion (case II, short-dashed lines in Fig. 3), the symmetric formulation (case I, solid lines) generates qualitatively different temporal evolutions of \( \Delta TP, TK, \) and \( L_r \). Note that in case I the increase of TK from about day 100 on does not indicate violation of irreversibility but rather reflects a transfer from available potential energy into kinetic energy via the action of the meridional circulation. Total energy and total angular momentum are well conserved if the horizontal momentum diffusion (3.7) is used together with its associated dissipation (3.8). If (3.7) is retained and (3.8) is neglected (case III, long-dashed lines in Fig. 3), the time evolutions of TK, \( L_r, \Delta L_0, \) and \( \Delta L_0 + L_r \) show no visible deviations from case I (solid lines). However, total en-
ergy is no more conserved. This result emphasizes the importance of accounting for the frictional heating. The conventional formula (3.2) gives rise to spurious angular momentum and energy losses. At day 500, these losses are of the same order of magnitude as the kinetic energy and relative angular momentum of the initial state.

b. Baroclinic life cycle

The thermally balanced flow of Fig. 1 has been perturbed by the most unstable normal mode of zonal wavenumber 6. Then, the evolution of a baroclinic life cycle has been simulated with the 3D version of the SGCM using different horizontal diffusion schemes according to cases I, II, and III.

Figure 4 shows available potential energy due to the zonal-mean flow ($A_Z$) and the eddies ($A_E$) as defined by Lorenz (1955) together with the corresponding kinetic energies $K_Z$ and $K_E$. The well-known growth and decay of a baroclinic wave (Simmons and Hoskins 1978) is evident from each simulation. Differences are quantitatively negligible when considering $A_Z$, $A_E$, and $K_E$ (Figs. 4a,c,d). The behavior of $K_Z$ (Fig. 4b) is at least qualitatively the same in each case I–III.

Also the zonally averaged temperature and zonal wind at day 40 look similar. Figures 5a,c show corresponding results for case I only. However, computing the differences case II minus case I reveals significant anomalies due to the differences in the horizontal diffusion schemes. For example, in the lower troposphere, the temperature signal exceeds $\pm 6$ K (Fig. 5b). Anomalies of the zonal-mean zonal wind amount to about 1–2 m s$^{-1}$ (Fig. 5d).

The time evolutions of (4.1)–(4.4) are presented in
Fig. 6. Total energy and total angular momentum are approximately conserved in case I only (solid lines). (There are weak spurious changes in $\Delta L_o + L_r$ and $\Delta TP + TK$ between day 10 and day 15, which may be attributed to aliasing errors during the strong life cycle maximum.) Total kinetic energy as well as relative and absolute angular momentum behave qualitatively similar in each case I–III. There is a continuous spurious loss of total angular momentum in case II (short–dashed lines). The worst shortcoming of conventional horizontal diffusion consists of a substantial loss of total energy. Within a few days during the life cycle maximum, this loss is of the same magnitude as the overall kinetic energy of the model atmosphere. From the third life cycle experiment (case III, long-dashed lines in Fig. 6) it is evident that the dissipation (3.8) is essential for conservation of total energy.

This result may be interpreted as follows. During the phase of baroclinic instability (days 0–12), total kinetic energy grows at the cost of total potential energy. Since the accompanying loss of available potential energy is almost the same, the gain of kinetic energy can be considered as a reversible process. In other words, enstrophy cascade and thus horizontal diffusion are not very
effective. From day 14 on, kinetic energy is lost while available potential energy does not increase again. Hence, the decay phase of a baroclinic life cycle is strongly irreversible due to enstrophy transfer into small scales and the action of horizontal momentum diffusion. However, there must be some diabatic heating to balance the frictional loss of kinetic energy. Only a symmetric horizontal diffusion scheme combined with the proper dissipation accounts for the corresponding irreversible increase of total potential energy.

\[Q := \frac{T - T_E}{16 \text{ days}} + Q_c,\]  

as well as vertical diffusion and surface drag \((\nu = 2.5 \text{ m}^2\text{s}^{-1} \text{ and } C = 0.005 \text{ m} \text{s}^{-1})\). The \(T_E\) is like in the

c. Multiple life cycles

Let us consider the climatology of the SGCM if the model setup is completed by thermal forcing according to the angular momentum budget \([\text{Eq. (4.6), unit } 10^{18} \text{ N m}]\) and energy budget \([\text{Eq. (4.7), unit } 10^{15} \text{ J s}^{-1}]\) in the multiple life cycle experiments according to case I (solid lines), case II (short-dashed lines), and case III (long-dashed lines). (a) Northward angular momentum flux. (b) Surface drag plus drag due to the symmetric horizontal diffusion tensor (cases I and III only). (c) Northward flux of total enthalpy. (d) Integrated diabatic heating \((4.5)\) plus work due to surface drag.
previous examples up to a slight asymmetry with respect to the equator. The $Q_c$ is a zonally symmetric and temporally constant heating rate to mimic tropical cumulus convection in an otherwise dry model (e.g., Becker et al. 1997; Hou 1998). Horizontal diffusion is applied according to cases I–III as defined above. To assess the overall role of frictional heating, (2.12) is applied in case I but neglected in cases II and III. The SGCM has been integrated for 900 days using the initial condition of the life cycle experiment. (Only zonal wavenumbers 0, 6, 12, 18, and 24 are involved.) Time averaging has been performed on the basis of the last 720 days.

Climatological temperature, zonal wind and meridional mass streamfunction are shown in Figs. 7a,c,e for case I. To illustrate the model response to nonsymmetric horizontal diffusion and neglect of dissipation, Figs. 7c,d,f show the corresponding differences between cases II and I. The anomalies are considerably weak. Nevertheless they remain almost the same if the model is integrated for another 720 days, indicating their statistical significance.

A systematic bias in the model climatology can also be deduced from globally integrated properties. As an example, Fig. 8 shows total potential energy $\Delta TP$ and total kinetic energy $TK$ (defined as in section 4a) for cases I and II. For conventional horizontal diffusion, both energies are systematically weaker than for the new scheme (3.6)–(3.8) in connection with (2.12). The overall bias in total potential energy is of the same order of magnitude as the total kinetic energy.

We next apply the Eulerian law of angular momentum conservation to a control volume that ranges from the bottom to the top of the model atmosphere and from the South Pole to some latitude $\phi$ farther north. Indicating time averaging by the $\langle \rangle$ symbol, the Eulerian law states that

$$
\langle (a \cos \phi)^2 \rangle \int_0^{2\pi} d\lambda \int_{z=0}^{\infty} dz \; \rho u x = \langle -a^2 \rangle \int_0^{2\pi} d\lambda \int_{-\pi/2}^{\pi/2} d\phi \int_{z=0}^{\infty} dz \rho v \cdot (\nabla \phi)
$$

$$
+ \langle (a \cos \phi)^2 \rangle \int_0^{2\pi} d\lambda \int_{z=0}^{\infty} dz \; \rho v \cdot (\nabla \phi)
$$

The left-hand side (lhs) of (4.7) represents the northward flux of total energy at latitude $\phi$. This flux must be balanced by the integrated diabatic heating (4.5) plus the work done by friction at the boundaries of the control volume. Since $v$ is not specified for $z \to 0$, the frictional work due to vertical diffusion is approximated by multiplying the boundary wind stress with the horizontal velocity.

The rhs of (4.7) shows total potential energy $\Delta TP$ and total kinetic energy $TK$ (defined as in section 4a) for cases I and II. For conventional horizontal diffusion, both energies are systematically weaker than for the new scheme (3.6)–(3.8) in connection with (2.12). The overall bias in total potential energy is of the same order of magnitude as the total kinetic energy.

We next apply the Eulerian law of angular momentum conservation to a control volume that ranges from the bottom to the top of the model atmosphere and from the South Pole to some latitude $\phi$ farther north. Indicating time averaging by the $\langle \rangle$ symbol, the Eulerian law states that

$$
\langle (a \cos \phi)^2 \rangle \int_0^{2\pi} d\lambda \int_{z=0}^{\infty} dz \; \rho u x = \langle -a^2 \rangle \int_0^{2\pi} d\lambda \int_{-\pi/2}^{\pi/2} d\phi \int_{z=0}^{\infty} dz \rho v \cdot (\nabla \phi)
$$

In other words, the northward flux of zonal angular momentum at latitude $\phi$ (lhs) is balanced by the stress acting at the boundaries of the control volume (rhs). The first term on the rhs of (4.6) is the well-known surface drag. The second term is the drag at the northward boundary of the control volume due to (3.6). This term is well-defined in case of a symmetric stress tensor only. Figures 9a and 9b show the left- and right-hand sides of (4.6) for the three multiple life cycle experiments (cases I–III). The angular momentum flux is almost the same in each case. A reasonable balance by

There are several important points to note:

1. The frictional drag is achieved with a symmetric stress tensor only (either case I or III).

2. It is also worthwhile to consider energy conservation analogously to (4.6):

$$
\langle a \cos \phi \int_0^{2\pi} d\lambda \int_{z=0}^{\infty} dz \rho v (c_p T + g z + v^2/2) \rangle
$$

$$
= \langle a^2 \int_0^{2\pi} d\lambda \int_{-\pi/2}^{\pi/2} d\phi \int_{z=0}^{\infty} dz \rho v \phi \rangle
$$

$$
+ \langle -a^2 \int_0^{2\pi} d\lambda \int_{-\pi/2}^{\pi/2} d\phi \rho v \phi \rangle \cdot \langle \Sigma \phi \rangle
$$

The lhs of (4.7) represents the northward flux of total enthalpy at latitude $\phi$. This flux must be balanced by the integrated diabatic heating (4.5) plus the work done by friction at the boundaries of the control volume. Since $v$ is not specified for $z \to 0$, the frictional work due to vertical diffusion is approximated by multiplying the boundary wind stress with the horizontal velocity.

5. Concluding remarks

The central importance of a symmetric stress tensor corresponds to the Eulerian law of angular momentum conservation for arbitrary control volumes. In addition, energy conservation does crucially depend on the parameterization of frictional processes as well. For a symmetric stress tensor of horizontal diffusion, the local frictional heating rate (dissipation) is well defined and can easily be accounted for. The general formula (3.14) reveals that only in case of nabla-square symmetric horizontal diffusion, the dissipation is positive definite. Therefore, the physical meaning of higher orders is open to question even if the stress tensor is symmetric.

The physical consistency of a GCM is improved if turbulent friction is based on a symmetric stress tensor. This has been demonstrated by idealized SGCM experiments for the examples of a thermally balanced zonal flow, a baroclinic life cycle, and multiple life cycles. In case of vanishing external torques and heat sources,
conventional horizontal diffusion causes significant losses of total angular momentum and total energy. While the first occurs due to asymmetry of the stress tensor, the latter is due to the lack of frictional heating. Analogous shortcomings can be found in multiple life cycle experiments when evaluating the axiomatic forms of angular momentum and energy conservation for finite control volumes.

A review on nonlinear parameterizations of turbulent viscosities by Smagorinsky (1993) also provides a symmetric stress tensor formulation of second-order horizontal momentum diffusion. It differs from the present result (3.6) by the trace of the stress tensor. After deducing the final formulae for friction force and dissipation with regard to (2.9) and (2.10), Smagorinsky’s formulation has been tested in the present SGCM experiments. Corresponding results have not been shown since they are almost identical to those obtained using the proposed method. Therefore, both Smagorinsky’s ansatz and the present formulation appear equivalently suitable for application in GCMs.

The conventional biharmonic horizontal diffusion has often been modified in GCM applications such as to conserve a superrotation or to achieve a realistic energy spectrum in longtime simulations. Nevertheless, spurious torques and heat sinks have hardly been removed from the equations of motion since emphasis has been spent on the formulation of the friction force rather than the stress tensor. In this context it is worth mentioning that also the frictional heating due to vertical momentum diffusion is usually neglected in the design of GCMs.

In this study we have investigated the properties of explicit horizontal diffusion because it is the usual ansatz to account for scale-selective damping in a spectral GCM. Gridpoint models do often involve filter methods instead of an explicit diffusion. Therefore, the compatibility of a particular filter algorithm with the constraints of a symmetric stress tensor, energy conservation, and irreversibility deserves to be investigated.

The importance of accounting for first principles in the design of a climate or weather forecast model is evident. Present SGCM experiments suggest that replacing conventional turbulent friction by the proposed formulation or that of Smagorinsky (1993) has weak, but nevertheless significant, effects on the climatology of a GCM. It is thus worthwhile to ask whether delicate properties of comprehensive climate models, like internal variability or the response to a perturbation in the external forcing, are possibly affected by the intrinsic torques and cooling rates associated with conventional formulations of turbulent friction.

Acknowledgments. Encouraging comments and suggestions by G. Schmitz and one anonymous reviewer have greatly improved the manuscript. Also the valuable report of a second anonymous reviewer is gratefully acknowledged.

APPENDIX

Mathematical Details

This appendix provides some mathematical guidance to follow the presentation of section 3.

In spherical coordinates the derivatives of the unit vectors yield

\[ \begin{align*}
\partial_r e_x &= \frac{\tan \phi}{r} e_x - e_z, \\
\partial_r e_y &= \frac{\tan \phi}{r} e_z, \\
\partial_r e_z &= \frac{e_x}{r},
\end{align*} \]

(A.1)

Using this notation and indicating the tensor product by an open circle, it is readily shown that

\[ \nabla \circ \mathbf{v} = (\mathbf{e}_x \partial_r + \mathbf{e}_z \partial_\phi) \circ (u \mathbf{e}_r + v \mathbf{e}_\phi) = \begin{align*}
&= \mathbf{e}_x \circ \mathbf{e}_r (D - \partial_r v) + \mathbf{e}_z \circ \mathbf{e}_r (\xi + \partial_r u) \\
&\quad - \mathbf{e}_z \circ \mathbf{e}_z u/r + \mathbf{e}_x \circ \mathbf{e}_r \partial_r u + \mathbf{e}_y \circ \mathbf{e}_r \partial_r v \\
&\quad - \mathbf{e}_r \circ \mathbf{e}_r v/r.
\end{align*} \]

(A.2)

The transposed tensor yields

\[ \{ \nabla \circ \mathbf{v} \}^T = \begin{bmatrix}
\mathbf{e}_r \circ \mathbf{e}_r (D - \partial_r v) + \mathbf{e}_z \circ \mathbf{e}_r \partial_r u \\
\mathbf{e}_x \circ \mathbf{e}_z (\xi + \partial_r u) + \mathbf{e}_y \circ \mathbf{e}_r \partial_r v \\
\mathbf{e}_z \circ \mathbf{e}_z u/r - \mathbf{e}_r \circ \mathbf{e}_r v/r
\end{bmatrix}. \]

(A.3)

Combination of (A.2) and (A.3) gives rise to the components of the symmetric tensor (3.5). The divergence of (A.2) is

\[ \nabla \cdot \{ \nabla \circ \mathbf{v} \} = (\mathbf{e}_x \circ \partial_r + \mathbf{e}_z \circ \partial_\phi) \circ [\nabla \circ \mathbf{v}] = (\nabla \cdot \mathbf{v} + \mathbf{e}_r \cdot \nabla \mathbf{v}) = \nabla^2 \mathbf{v}. \]

(A.4)

To compute the divergence of (A.3),

\[ \nabla \cdot \{ \nabla \circ \mathbf{v} \}^T = (\mathbf{e}_x \circ \partial_r + \mathbf{e}_z \circ \partial_\phi + \mathbf{e}_r \circ \partial_\rho) \circ \begin{bmatrix}
\{ \mathbf{e}_x \circ \mathbf{e}_r (D - \partial_r v) + \mathbf{e}_z \circ \mathbf{e}_r \partial_r u \\
\mathbf{e}_x \circ \mathbf{e}_z (\xi + \partial_r u) + \mathbf{e}_y \circ \mathbf{e}_r \partial_r v \\
\mathbf{e}_z \circ \mathbf{e}_z u/r - \mathbf{e}_r \circ \mathbf{e}_r v/r
\end{bmatrix}, \]

we have to take the derivatives (A.1) as well as \( \partial_\rho \mathbf{v} = \partial_r \mathbf{v} - (\xi \partial_\rho + \partial_\rho \partial_\phi) \partial_r \) explicitly into account. After several computational steps we arrive at

\[ \nabla \cdot \{ \nabla \circ \mathbf{v} \}^T = \nabla D - \frac{1}{r} \partial_r (\mathbf{v} + \mathbf{D} \cdot \mathbf{e}_r). \]

(A.5)

The friction force (3.5) follows from combination of (A.4), (A.5), and (2.10):
\[
\frac{1}{\rho} \nabla_s (\rho K (\nabla \cdot \mathbf{v} + (\nabla \cdot \mathbf{v})^T)) \\
= K \nabla_s ((\nabla \cdot \mathbf{v} + (\nabla \cdot \mathbf{v})^T) \\
+ \frac{1}{\rho} ((\nabla \cdot \mathbf{v} + (\nabla \cdot \mathbf{v})^T) \nabla_s (\rho K)) \\
= \frac{K}{\rho} \left( \nabla^2 \mathbf{v} + \nabla D - \frac{1}{r} (\partial_r \mathbf{v} + D \mathbf{e}_r) \right) - \mathbf{v} \partial_r (\rho K). \quad (A.6)
\]

It is straightforward to show that
\[
\nabla_s (\mathbf{e}_r / r \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{e}_r / r) = \frac{2}{r^2} \mathbf{v} + \frac{D}{r} \mathbf{e}_z + \frac{\partial_r \mathbf{v}}{r}. \quad (A.7)
\]

Thus, combination of (A.4), (A.5), and (A.7) yields
\[
\nabla_s ((\nabla \cdot \mathbf{v} + (\nabla \cdot \mathbf{v})^T) \\
= \nabla^2 \mathbf{v} + \nabla D + 2 \mathbf{v} / r^2. \quad (A.8)
\]

Since the tensor (3.6) contains no \( \mathbf{e}_z \cdot \mathbf{e}_z \) or \( \mathbf{e}_z \cdot \mathbf{e}_r \) components, the friction force (3.7) follows immediately with regard to the scale approximation (2.10). The dissipation associated with vertical diffusion is
\[
d(\mathbf{Z}) = \frac{1}{\rho} (\Sigma \mathbf{e}_z \cdot \nabla_s) \cdot \mathbf{v}_3 \\
= \nu (\mathbf{e}_z \cdot \partial_r \mathbf{v} \cdot \mathbf{n}) + \partial_r \mathbf{v}_3 \cdot \mathbf{v}_3. \quad (A.9)
\]

which yields (2.12) according to (2.9). The dissipation associated with the horizontal diffusion tensor (3.10) is evaluated analogously:
\[
d(\mathbf{H}_\mu) = \frac{1}{\rho} (\Sigma \mathbf{e}_r \cdot \nabla_s) \cdot \mathbf{v}_3 \\
= K \{ (2D - \partial_r \mathbf{u}_r \mathbf{e}_r \partial_r + (\xi_\mu + 2 \partial_r u_\mu) \mathbf{e}_z \partial_r \\
+ (\xi_\mu + 2 \partial_r u_\mu) \mathbf{e}_z \partial_r + 2 \partial_r v_\mu \mathbf{e}_r \mathbf{e}_z \} \cdot \mathbf{v}_3. \quad (A.10)
\]

Owing to (A.1), the final formula (3.14) is obtained after a few steps.

REFERENCES


