Dynamics of Singular Vectors in the Semi-Infinite Eady Model: Nonzero $\beta$ but Zero Mean PV Gradient

H. de Vries and J. D. Opsteegh

Institute for Marine and Atmospheric Research Utrecht, Utrecht University, Utrecht, Netherlands

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ABSTRACT

A nonmodal approach based on the potential vorticity (PV) perspective is used to compute the singular vector (SV) that optimizes the growth of kinetic energy at the surface for the $\beta$-plane Eady model without an upper rigid lid. The basic-state buoyancy frequency and zonal wind profile are chosen such that the basic-state PV gradient is zero.

If the $f$-plane approximation is made, the SV growth at the surface is dominated by resonance, resulting from the advection of basic-state potential temperature (PT) by the interior PV anomalies. This resonance generates a PT anomaly at the surface. The PV unshielding and PV–PT unshielding contribute less to the final kinetic energy at the surface.

The general conclusion of the present paper is that surface cyclogenesis (of the 48-h SV) is stronger if $\beta$ is included. Three cases have been considered. In the first case, the vertical shear of the basic state is modified in order to retain the zero basic-state PV gradient. The increased shear enhances SV growth significantly first because of a lowering of the resonant level (enhanced resonance), and second because of a more rapid PV unshielding process. Resonance is the most important contribution at optimization time. In the second case, the buoyancy frequency of the basic state is modified. The surface cyclogenesis is stronger than in the absence of $\beta$ but less strong than if the shear is modified. It is shown that the effect of the modified buoyancy frequency profile is that PV unshielding occurs more efficiently. The contribution from resonance to the SV growth remains almost the same. Finally, the SV is calculated for a more realistic buoyancy frequency profile based on observations. In this experiment the increased value of the surface buoyancy frequency reduces the SV growth significantly as compared to the case in which the surface buoyancy frequency takes a standard value. All growth mechanisms are affected by this change in the surface buoyancy frequency.

1. Introduction

It is well known that transient interactions between perturbations may temporarily lead to significant disturbance growth (Orr 1907). Farrell realized that this transient growth mechanism is important for incipient cyclogenesis (Farrell 1982, 1984, 1989). If initial disturbances are configured properly, finite-time growth rates will in general be several times larger than growth rates obtained from standard normal-mode instability theory. It is for this reason that researchers started to compute so-called optimal perturbations or singular vectors (SVs). The optimal perturbation constitutes that initial perturbation of unit amplitude (in some norm) that linearly amplifies most rapidly for a given time interval and according to a second norm, which may differ from the norm used at initial time. For instance, the norm can be total energy (TE), kinetic energy (KE), or potential enstrophy. At optimization time, limited-area regional norms can also be considered in order to localize the SV. One of these regional norms is the surface contribution to the kinetic energy (SKE). The various choices of the norm lead to as many different initial structures and subsequent life cycles of the optimal perturbation.

One of the potential difficulties of SV dynamics is that the SV significantly changes its vertical structure during the time evolution. In terms of the traditional normal-mode view (Pedlosky 1987), the cause for these structural changes is that the SV is composed of a large number of normal modes. The normal modes, which
may be growing, neutral, or even decaying, propagate with different propagation speeds. The nonmodal perturbation, which is a complex superposition of the normal modes, will therefore be structurally unstable. As a result of exponential instability, linear resonance (if present in the system), and unshielding of the individual normal modes, the SV attains a large amplitude at optimization time.

To reveal and understand the mechanisms underlying the rapid finite-time baroclinic growth of the SV, it is useful to study optimal perturbation evolution in models containing “just enough” complexity for the problem of interest. For incipient midlatitude surface cyclogenesis studied in this paper, the models introduced by Eady (1949) and Charney (1947) have proven to be valuable tools. Eady (1949) and Charney (1947) focused mainly on the growth of the discrete normal modes. Davies and Bishop (1994) subsequently showed that the discrete normal modes can be viewed as a pair of interacting boundary PT anomalies called edge waves (Gill 1982).

After the pioneering normal-mode studies by Eady and Charney, Pedlosky (1964) however, realized that the discrete (un)stable normal modes obtained by Eady and Charney are not the complete story. Only by including an infinite number of so-called continuum modes (CMs) can the inviscid linear dynamics of an arbitrary initial perturbation be correctly described (Pedlosky 1964). The CM of the Eady model is associated with potential vorticity (PV) at one specific interior level and a certain potential temperature (PT) at the boundaries. To maintain a time-independent vertical structure, the sign and amplitude of the interior PV and boundary PT have a fixed relation. As a result of this relation, CMs with nonzero PV near the steering level of one of the edge waves have a very large PT at the boundaries. Whenever the phase speed of the CM coalesces with the phase speed of one of the neutral edge waves, a linear resonance occurs and the streamfunction of this edge wave grows linearly in time (Thorncroft and Hoskins 1990; Chang 1992). Linear (as opposed to exponential) growth occurs because the interior PV cannot amplify in the Eady model.

Without the upper rigid lid and without a gradient of the basic-state PV, this resonant CM is the only unstable (although not exponentially) normal-mode structure. Despite this possibility to generate sustained perturbation growth, the resonant CM has not received much attention in the existing literature on SVs and explosive surface cyclogenesis. Analytical SV studies mostly concentrate on the discrete normal modes, and ignore the contribution from the CM (Rotunno and Fantini 1989; Fischer 1998). In numerical studies, implementation of the resonant CM has not been considered (Mukougawa and Ikeda 1994; Farrell 1989; Morgan and Chen 2002; Morgan 2001; Kim and Morgan 2002).

It is possible however, to both remove the aforementioned singular behavior of the near-resonant CMs and stay much closer to the PV perspective introduced by Hoskins et al. (1985). De Vries and Opsteegh (2005, hereafter DVO) discussed an approach to calculate the SV that is slightly different from the standard approach followed in the literature (Farrell 1984, 1989; Borges and Hartmann 1992; Mukougawa and Ikeda 1994; Morgan 2001; Morgan and Chen 2002; Kim and Morgan 2002). Instead of a basis of normal modes, a basis of nonmodal functions [called PV building blocks (PVBs)] is used to compute the SV. Each PVB is formed initially by a delta function spike of PV at one interior level and zero PT at the surface. As a result of the existing meridional gradient in the basic-state PT, perturbation PT is created at the surface as time increases.

The use of PVBs has three main advantages. First, it is easy to include the resonant CM. This is done by including the PVB with PV at the resonant level. The second advantage is that the singular behavior of the near-resonant CMs is removed in a way that is both simple and physically transparent. Finally, the nonmodal basis allows the SV problem to be attacked in a “bottom-up” way. One may start with a basis formed by one PVB and consider the surface cyclogenesis event resulting from optimally positioning this PVB. A second PVB is introduced to study the impact from PV unshielding and finally a continuous distribution of PVBs can be used to get insight into the dynamics of the most general SV perturbation.

In DVO, the SV analysis has been presented for the $f$-plane Eady model without the upper rigid lid. The results in DVO confirm results of existing studies of the shortwave SVs in the Eady model with upper rigid lid (Morgan 2001; Morgan and Chen 2002). Initially, PV unshielding is most important for the interior of the flow and a vertically aligned tower of PV is formed at optimization time. The PV–PT unshielding is maximized beyond optimization time. Finally, the long time amplitude of the SV is dominated by the contribution from the boundary PT, thereby confirming Morgan’s three-stage scenario for cyclogenesis (Morgan 2001). At the surface, however, the contributions from the PT (caused by resonance) dominate the SKE already at optimization time. The PV–PT unshielding contributes negatively all the way toward the optimization time. In a recent paper addressing a similar problem for the $f$-plane Eady model with an upper rigid lid, Dirren and Davies (2004) find that the short time (<2 days) growth
of the surface PT is favored by a low-level PVB, whereas the long time (>2 days) growth of the surface anomaly is significantly enhanced for a PVB residing at the midlevel (the steering level for the unstable normal modes). This sensitivity of the growth of the surface PT to the vertical position of the PVB increases with increasing wavenumber (Dirren and Davies 2004). In contrast, DVO observe that the PVB optimizing SKE in the Eady model without rigid lid approaches the steering level from above for increasing the optimization time. The main reason for the discrepancy is the choice of initial and final optimization norms. Dirren and Davies (2004) consider a PVB of unit PV, whereas DVO consider a PVB with a unit of initial SKE. Furthermore, Dirren and Davies (2004) optimize the amplitude of the PT anomaly, whereas DVO optimize the SKE. We refer to DVO for a more complete comparison between other existing studies.

In the present paper the same approach and techniques are used as in DVO to perform the SV analysis for a class of modified Eady models introduced by Lindzen (1994). In Lindzen (1994) the standard f-plane Eady model is augmented with a nonzero value of the planetary vorticity gradient (β-plane approximation). However, the gradient of the basic-state PV is kept at zero by modifying the basic-state velocity and buoyancy frequency profile. The argument for studying these adjusted basic states is that the basic-state meridional PV gradients in the troposphere are very small, possibly indistinguishable from zero (Lindzen 1994). Lindzen (1994) retained the upper rigid lid and investigated the stability of the normal modes. In the present paper the upper rigid lid has been removed. The resulting basic state should therefore be considered as a prototype “underworld” only. This underworld is characterized by zero gradients in the basic-state PV and we examine the models instability to SV perturbations. A next step would be to include a “midlevelworld” and an “upperworld.” In this view the midlevelworld represents the tropopause region with concentrated PV gradients of the order of several times β. The upperworld is a stratosphere-like domain characterized by PV gradients that are typically of the order of β. These characteristics roughly match with observations. Harnik and Lindzen (1998) have performed the stability analysis for such a basic state. The SV analysis for such a more complex basic state has not been performed and will be addressed in a forthcoming paper.

The present paper is organized as follows. In section 2 the model and its basic state are introduced. In section 3 we present nonmodal solutions to the system and section 4 discusses the nonmodal basis to be used for the SV calculations. In section 5 we discuss the different growth mechanisms that occur in the evolution of the SV. Sections 6–8 discuss, similar to DVO, the optimal perturbation evolution for an increasing number of PVBs. Finally, section 9 summarizes the main points of the text and two appendixes outline mathematical details.

2. Model and basic flow

a. Dynamics

Linear, inviscid quasigeostrophic dynamics of perturbations evolving on a basic state [specified by a velocity profile \( \bar{U}(y, z) \) and the buoyancy frequency, \( N^2(z) \)] is described by the PV equation:

\[
\left( \frac{\partial}{\partial t} + \bar{U} \frac{\partial}{\partial x} \right) q + \nu \frac{\partial q}{\partial y} = 0, \tag{1}
\]

where \( q = q(x, y, z, t) \) is the perturbation PV, \( \nu = \partial \bar{\psi} / \partial x \bar{\xi} \) is the meridional velocity, and \( \bar{\xi} \) is the basic-state PV. Perturbation and basic-state PV are related to the streamfunction \( \bar{\psi} \) and the properties of the basic state by

\[
q = \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{S^2} \frac{\partial}{\partial z} \left( \frac{1}{N^2} \frac{\partial}{\partial z} \right) \right] \bar{\psi}, \tag{2}
\]

\[
\frac{\partial \bar{q}}{\partial y} = \beta \left[ \frac{\partial^2}{\partial y^2} + \frac{1}{S^2} \frac{\partial}{\partial z} \left( \frac{1}{N^2} \frac{\partial}{\partial z} \right) \right] \bar{U}. \tag{3}
\]

Variables have been nondimensionalized using the scalings given in Table 1 (Pedlosky 1987). The Coriolis parameter \( f_c \) is approximated by \( f_c = f + \beta y \) where \( f = f(y, d) \), and \( \beta = (\partial f_c / \partial y)(d) \) where \( d \) is the reference latitude \( (d = 45^\circ \text{N}) \). In the present study the reference stratification parameter \( S^2 = N_c^2 H^2/(f_c^2 L^2) \) is kept at unity. Here \( N_c^2 \sim 10^{-4} \text{ s}^{-2} \) is a reference value for the buoyancy frequency and \( f_c \sim 10^{-4} \text{ s}^{-1} \) is the lowest-order approximation of the Coriolis parameter at midlatitudes. At midlatitudes \( \beta \) takes the dimensional value of \( 1.6 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1} \) (\( \beta = 0.5 \) in nondimensional

<table>
<thead>
<tr>
<th>Variable</th>
<th>Scaling</th>
<th>Dimensional value</th>
</tr>
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<tbody>
<tr>
<td>((x, y))</td>
<td>( L \sim HN_z J_n f_c )</td>
<td>1000 km</td>
</tr>
<tr>
<td>( z )</td>
<td>( H )</td>
<td>10 km</td>
</tr>
<tr>
<td>((u, v))</td>
<td>( U )</td>
<td>30 m s(^{-1})</td>
</tr>
<tr>
<td>( \psi )</td>
<td>( LU )</td>
<td>( 3 \times 10^7 \text{ m}^2 \text{ s}^{-1} )</td>
</tr>
<tr>
<td>( t )</td>
<td>( L/U )</td>
<td>9.26 h</td>
</tr>
<tr>
<td>( q )</td>
<td>( U/L )</td>
<td>( 3 \times 10^{-8} \text{ s}^{-1} )</td>
</tr>
<tr>
<td>( \theta )</td>
<td>( ULf_c )</td>
<td>9.18 K</td>
</tr>
<tr>
<td>( \theta )</td>
<td>( L/(\bar{\xi} + \theta) )</td>
<td>( 9.18 \text{ K} )</td>
</tr>
<tr>
<td>( N^2 )</td>
<td>( N_c^2 )</td>
<td>( 10^{-4} \text{ s}^{-2} )</td>
</tr>
<tr>
<td>( f )</td>
<td>( f_c )</td>
<td>( 10^{-4} \text{ s}^{-1} )</td>
</tr>
<tr>
<td>( \beta )</td>
<td>( U/L )</td>
<td>( 3 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1} )</td>
</tr>
</tbody>
</table>
units). To get more insight in the general effect of $\beta$ we will take $\beta \sim O(1)$.

Equation (2) requires boundary conditions at two levels in the vertical to determine the streamfunction. At the earth’s surface the condition that the vertical velocity equals zero leads to

$$
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \frac{\partial \psi}{\partial z} - \frac{\partial U}{\partial z} \frac{\partial \psi}{\partial x} = 0 \quad (z = 0),
$$

where $\partial \psi / \partial z = \theta$ defines the PT.\(^1\) As an upper-level boundary condition it is required that the perturbation streamfunction vanishes at infinite height. Furthermore, the meridional velocity vanishes at the channel walls at $y = \pm Y$. With a $y$-independent velocity profile of the basic state the problem remains separable. We write

$$\psi(x, y, z, t) = \tilde{\psi}(x, z, t) \cos y,$$

where the meridional wavenumber $l = (n + \frac{1}{2}) \pi / Y$ with $n = (0, 1, 2, \ldots)$. A two-dimensional problem results if the channel walls are located at infinity. The tilde in $\tilde{\psi}$ is omitted here.

b. Basic state

As discussed in Lindzen (1994) the basic state is modified as compared to the conventional $f$-plane Eady model such that $\sigma_1 / \omega$ remains zero. Following Lindzen (1994) we introduce a parameter $0 \leq \alpha \leq 1$ controlling the adjustment process:

$$\frac{N^2(z)}{N_0^2} = \frac{1}{1 + abz}, \quad \frac{1}{m_0} \frac{\partial U(z)}{\partial z} = \frac{1 + bz}{1 + abz}.
$$

with $b = \beta N_0^2 / m_0$ and $m_0 = \tilde{U}_0 \left| \partial \tilde{U} / \partial z = 0 \right| = 1$ a reference value corresponding to $3 \text{ m s}^{-1} \text{ km}^{-1}$. Here $N_0 = N(z = 0) = 1$ is a convenient choice to compare the results with the existing literature. For $\alpha = 0$, all adjustment is in the basic-state velocity, while for $\alpha = 1$ all adjustment is in the buoyancy frequency. For intermediate values, both $U$ and $N^2$ are modified. We have varied $\alpha$ and $\beta$ to get insight in the general effects the adjustment process has on the structure and the evolution of SV perturbations. We consider the following three cases:

- **R case:** ($\beta = 0$). This is the reference experiment with the original semi-infinite version of the Eady model. The value of $\alpha$ is arbitrary.
- **U case:** ($\alpha = 0, \beta$). Unless otherwise specified, we will take $\beta = 0.5 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$.
- **N case:** ($\alpha = 1, \beta$). As in the U case, we will take $\beta = 0.5 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$ unless otherwise specified.

To properly interpret the forthcoming results in terms of the differences of the basic states for nonzero $\beta$, we have not considered the potential synergistic effects of taking $0 < \alpha < 1$ in detail in the present paper.

Note that in the present semi-infinite model, we have chosen the wind shear at the surface, $m_0 = 1$, to be independent of $\beta$. This differs from Lindzen (1994) who for the bounded Eady model fixed the surface and tropopause zonal wind speed (and thereby fixing the mean tropospheric zonal wind shear). For nonzero $\beta$ and $\alpha \neq 1$ (notably the U case), it is clear that the surface shear will decrease for nonzero $\beta$ in Lindzen’s approach. The results to be obtained in the present paper cannot, therefore, be directly related to the results in Lindzen (1994).

3. Initialization

The model is initialized with certain types of perturbations. In analogy with Morgan and Chen (2002), the streamfunction $\psi$ of a perturbation is decomposed as $\psi = \psi_0 + \psi_{pv}$. In this decomposition, which will be called the PV perspective, the two components $\psi_0$ and $\psi_{pu}$ satisfy the relations:

$$\left[ \frac{\partial}{\partial z} \left( \frac{1}{N_0^2} \frac{\partial \psi_0}{\partial z} \right) - \mu^2 \right] \psi_0 = 0, \quad \left[ \frac{\partial}{\partial z} \left( \frac{1}{N^2} \frac{\partial \psi_{pu}}{\partial z} \right) - \mu^2 \right] \psi_{pu} = 0, \quad (z = 0). \quad (6)
$$

where $\mu = S(k^2 + F)^{1/2} = 1/\mu_0$ defines the inverse Rossby height of the $\beta = 0$ problem. Initial perturbations are grouped in two classes. Members of the first class have zero interior PV and can be completely described in terms of boundary PT anomalies. Such perturbations are the so-called edge waves [e.g., solutions to Eqs. (6)]. Elements of the second group have non-zero PV at specific levels in the interior and zero boundary PT [e.g., solutions to Eqs. (7)]. As in DVO, these are called the PVBs. Together, the edge waves and the PVBs can be used generate any distribution of interior PV and boundary PT.

a. Boundary PT perturbations: Edge waves

The edge wave is a zero-PV PT perturbation with $\partial \psi / \partial z(x, z = 0, t = 0) = T_0 \exp(i k x)$. Given the basic state [specified by ($\alpha, \beta$) in Eq. (5)], the edge wave contains modified Bessel functions $K_\alpha(y(z))$ and

---

1 A nondimensional PT perturbation $\theta = 1$ corresponds with a dimensional value of $(ULF g^{-1} H^{-1}) \theta_u = 9.18 \text{ K}$, where $\Theta_u = 300 \text{ K}$ is a reference PT and $g = 9.81 \text{ m s}^{-2}$ the gravity constant (see Table 1).
$K_i [\gamma(z)]$ (see Abramowitz and Stegun 1970 and Lindzen 1994),

$$
\psi(x, z, 0) = \phi_0(x, z, 0) e^{i(kx - c_{ew} t)} \rightarrow \psi(x, z, t)
$$

where $c_{ew}$ is the phase speed of the edge wave and

$$
\phi_{ew}(z) = -(T_0/\mu) [K_0 [\gamma(z)]/K_1 (\gamma_o)] , \text{ with } \gamma(z) = \gamma_0 \sqrt{1 + abz}, \gamma_0 = 2\mu N_0/(ab). \text{ The phase speed of the edge wave is calculated from the rigid-lid condition in Eq. (4) to be }
$$

$$
c_{ew} = U_0 + \frac{m_0}{\mu N_0} \frac{K_0 (\gamma_0)}{K_1 (\gamma_0)}.
$$

If $\alpha$ is zero, both phase speed and structure of the edge wave are independent of $B$; that is, $c_{ew} = U_0 + (m_0/\mu N_0)$ and $\phi(z) \sim \exp(-\mu N_0 z)$. The structure of the edge wave only depends on $B$ if $\alpha$ is nonzero, because for nonzero $\alpha$ the buoyancy frequency decreases slowly with height. As a result, surface PT perturbations penetrate to higher levels as compared to the situation in which $\alpha$ is zero.

In Fig. 1 we have plotted the streamfunction and PT profiles of the edge waves for the U and N case as well for the intermediate value $\alpha = 0.5$. We have used the excessively large value of $\beta = 5$ to properly show the differences with the $\beta = 0$ situation. Notice that the surface streamfunction attains different values for the different cases. In terms of the wind field of the edge waves, this implies that the same PT perturbation at the surface produces a different geostrophically balanced wind field for different basic states. It can be seen in Fig. 1 that for the N case, the surface winds of the edge wave are smaller than for the R or U cases. It is clear that therefore the zonal propagation of the N-case edge wave will be slower as compared to the R or U cases.

b. Interior PV perturbations: PVBs

The second class of initial perturbations is characterized by zero PT at the surface and a delta function spike of PV at a stipulated level $h > 0$:

$$
q(x, z, 0) = Q \delta(z - h) e^{ikx} \rightarrow q(x, z, t)
$$

$$
q(x - c_{pv} t, z, 0), \text{ with } Q \text{ the amplitude}^2 \text{ of the PV and } c_{pv} = U(z = h) \text{ because of PV conservation Eq. (1). In agreement with Eq. (10) the initial streamfunction } \psi(x, z, 0) \text{ can be written entirely in terms of } \psi_{pv}(x, z, 0). \text{ Mathematically it is clear, that the elliptic Green function } \phi_{pv} \text{ cannot alone as a mode for } t \neq 0, \text{ because } \psi_{pv} \text{ does not satisfy both the zero surface PT condition and the thermodynamic equation. From the physical point of view this is also to be expected: any PV anomaly has a nonzero wind field at the surface. Given the nonzero meridional PT gradient at the surface, this PV wind field generates a PT anomaly and therefore leads to a nonzero } \psi_0 \text{ at later times. In other words, } \psi_0 \text{ and } \psi_{pv} \text{ are coupled.}
$$

The vertical structure of $\psi_0$ is identical to the edge wave in Eq. (8). The effect of the PV wind field appears as a time-dependent amplitude $A(t)$ in $\psi_0$. We write

$$
q(x, z, 0) = Q \delta(z - h) e^{ikx} \rightarrow q(x, z, t)
$$

$$
= \phi_{pv}(z)e^{ikx} + A(t)\phi_{ew}(z)e^{ikx}.
$$

\[2\] Here $Q = 1$ corresponds with a PV anomaly of amplitude $0.33 F_4 \text{ s}^{-1}$ (see Table 1).
where \( A(t) \) is given by
\[
A(t) = a_0 + \left[ \frac{\hat{U}}{\partial z} \frac{\partial \psi_p(x)}{\partial z}, \right]_{z=0}^{\exp[i(k(c_w - c_p) t)]} c_w - c_p, \tag{12}
\]
with \( \theta_{cw} = \partial \phi / \partial z \) and the constant \( a_0 \) is chosen\(^3\) to satisfy the initial condition \( \phi(0, 0, t) = 0 \). The resulting streamfunction describes the time evolution of one PVB with \( q = Q(0 - h) \exp[i(kx)] \):
\[
\psi(x, 0, t) = \phi_p(x) e^{ikx} - \left[ \frac{\hat{U}}{\partial z} \phi_p(x) \right]_{z=0}^{\exp[i(kx - c_w t)]} \phi_{cw}(0).
\tag{13}
\]
where the first and second term on the rhs of Eq. (13) define \( \phi_p(x, 0, t) \) and \( \phi_{cw}(x, 0, t) \), respectively. Notice that the PVBs show completely regular behavior for \( c_p \to c_w \) and that the resonant and near-resonant CMs are included naturally in this approach. This is different from the modal decomposition technique, in which the near-resonant CMs become singular.

We have calculated \( \phi_p(x) \) in appendix A [see Eq. (A4)]. In Fig. 2 we show \( \phi_p(x) \) of initial PVBs at various heights (normalized to have unit KE) for the U case, the N case, and the case with \( \alpha = 0.5 \). Again \( \beta = 5 \) has been taken to exaggerate the differences with the \( \beta = 0 \) case. In all cases, the streamfunction attains its maximum at the position of the PV anomaly and decreases away from the position of the PV. However, the height dependence of the individual cases is different. What is generally seen is that the larger the value of \( \alpha \), the less rapid the decay of the streamfunction away from the position of the PV anomaly. This is of course a direct result of the fact that the buoyancy frequency decreases slowly with height if \( \alpha \neq 0 \) (especially the N case) leading to less stratification. This effect is strongest for PVBs with PV at high levels.

Given the two ways to initialize the system (with boundary PT and interior PVBs) we will next explain how to them to calculate SVs.

### 4. Construction of singular vectors

#### a. Nonmodal basis

Using PVBs together with edge waves, initial perturbations with any distribution of interior PV and surface PT can be created. This implies that we have constructed a nonmodal basis that can be used for SV calculations. Each nonmodal basis function (PVB) consists of two modal components, a CM and an edge wave, superposed in such a way that the initial PT vanishes at the surface.\(^4\) An extensive discussion of the use of this nonmodal basis has been presented in DVO and the advantages have been summarized in the introduction.

#### b. Optimization norm and growth factors

Singular vectors are constructed to produce optimal finite-time growth in some specified (vector) norm. This norm may be specified differently for initial and final time. In the present paper we will use total kinetic energy (KE) as the initial norm and SKE (the surface contribution to the KE) as the norm at optimization time. The physical motivation for using these norms is to investigate the possible importance of the SV as a precursor disturbance for rapid cyclogenesis at the surface. Another possibility would be to use SKE at initial time as well. However, as shown in DVO, this is not a good choice if more than one PVB is used.\(^5\)

Based on the two norms, the optimization measure \( \Gamma(0, t) \) is defined as
\[
\Gamma(0, t) = \frac{\text{SKE}(t)}{\text{KE}(0)} = \frac{\int_{0}^{H} \psi(z, 0) \phi(z, 0) dz}{\int_{0}^{H} \psi(z, 0) \phi(z, 0) dz}, \tag{14}
\]

\(^3\) The choice \( a_0 = 0 \) leads to the pure CM, a single modal structure propagating with the speed of the basic-state flow at height \( h \) (Thorncroft and Hoskins 1990). A linear resonance occurs if \( c_p = c_w \) and the streamfunction of the CM becomes singular for near-resonant situations (Chang 1992). The PVB (with a different choice of \( a_0 \)) does not suffer from the singular behavior.

\(^4\) For details see Fig. 2 in DVO.

\(^5\) Mathematically, SKE is a seminorm. This implies that the SKE may become zero for nontrivial perturbations (perfect shielding), which would lead to an infinite growth factor if SKE is used as the initial norm.
where $H$ is the integration height and the bar means integrating over one horizontal wavelength in the $x$ and $y$ directions and multiplying by $(k^2 + f^2)/2$. In the literature $\Gamma(0, t)$ is referred to as the growth factor or the finite-time growth rate if the initial and final norms are equal. Note that with the present choice of norms, $\Gamma$ (in a dimensional form) would carry the units of an inverse length, whereas $\Gamma$ is unitless if the norms are equal.

A nondimensional optimization time $t_{opt} = 5.16$ (48 h in real time, see Table 1) has been chosen as well as a fixed zonal wavenumber $k = 2$. In the present study we have not included a meridional structure of the perturbations. An integration height $H = 2$ (20 km) is chosen because perturbations (with zonal wavenumber $k = 2$) at these heights have a negligible contribution to the surface development (DVO).

Given the analytical expression of the PVB in Eq. (13), a completely analytical determination of the SV is possible (Fischer 1998). The SV optimizes the projection coefficient for each PVB. If $M$ PVBs are included, the optimization problem is $(M \times M)$ dimensional. More details are presented in appendix B.

It should be noted that a recent paper by Dirren and Davies (2004) investigates a very similar setup. For the $f$-plane Eady model with upper rigid lid, they introduce a PVB in a similar way and (numerically) study the sensitivity of the height of the PV to the finite-time PT amplitude at the surface. Upon removal of the upper rigid lid, this would be identical to computing the amplitude $\theta(z = 0, t)$ from Eq. (13) for a given range of heights $h$ and a given value of the PV (i.e., $Q = 1$). In the present paper, however, the PV is determined through the initial constraint of unit KE.

### 5. Kinetic energy growth in singular vectors

#### a. Diagnosing the singular vector evolution

When the initial structure (in terms of interior PV and boundary PT) of the SV has been determined by the optimization procedure, it is important to understand which are the key mechanisms that cause the SKE growth. These key mechanisms are studied from the decomposition of $|\psi_{ad}|^2$:

$$|\psi_{ad}|^2 = |\psi|^2 + |\psi_{pu}|^2 + |\psi_{pu}^*|\psi_{pu} + |\psi_{pu}^*|^2$$

where $|\psi|^2 = \langle \psi(t)\psi(t) \rangle$, all functions are evaluated at $z = 0$, and the bar again indicates that integration over one wavelength in the zonal and meridional direction has been performed [and multiplied by a factor $(k^2 + f^2)/2$]. Given the analytical solution of the PVB [Eq. (13)], this decomposition is straightforward.

#### b. Growth mechanisms

The contributions on the rhs of Eq. (15) show that three mechanisms cause SKE growth in this model. SKE may increase due to 1) a growing edge wave $q_\theta$ (resonance), 2) PV unshielding, and 3) PV–PT unshielding. For future reference we investigate in which way each of these mechanisms is influenced by the specific choice of the basic state.

1) **Growing edge wave—resonance**

From the modal theory on the CMs in the Eady model without the upper rigid lid it is known that there is one level at which the PV is in exact resonance with the surface edge wave (Chang 1992). This is the level at which the phase speed of the PV and the surface edge wave coalesce. This resonant PV anomaly causes $q_\theta$ to grow linearly in time.

However, not only PV residing exactly at the resonant level will generate a large PT wave. Near-resonant PV anomalies generate a large PT wave $q_\theta$ as well. In the present paper resonance is therefore defined as the (temporary) growth of the edge wave $q_\theta$ resulting from advection of the basic-state PT field by the winds of the PV perturbations. The time scale for the resonant growth is determined from the difference in phase speed between the interior PV and the boundary PT. If this difference in phase speed is large, the growth will quickly stop. The existence of this intrinsic time scale implies that the long optimization time SVs (with $t_{opt} \to \infty$) are dominated by PV near the steering level of the surface edge wave.

The instantaneous growth of the edge wave (i.e., the growth rate) is influenced by three factors [see also Eq. (13)]. First, the growth of the edge wave is a result of the advection of basic-state PT by the winds due to the PV. Obviously, stronger surface winds due to the PV produce a larger growth rate of the edge wave. We will call this the **PV effect** of the edge wave growth. The surface winds of a PV anomaly with a fixed amplitude of PV decrease with increasing height of the PV anomaly. This is the reason that a SV optimizing SKE will favor PV in the lower parts of the troposphere.

Second, for fixed surface winds due to the interior PV, the growth rate of the edge wave will be larger if the meridional gradient in the basic-state PT at the surface is larger (larger vertical wind shear of the basic state at the surface). We will call this the **$\overline{\Sigma}$ effect** of the edge wave growth.

Finally, the wind field structure of the PT anomaly is important. Figure 1 shows that the wind field of a PT anomaly with a given amplitude of PT is different for the U and N cases. For a fixed basic-state shear at the
surface and fixed surface wind speeds due to the PV, the growth rate of the edge wave (in terms of kinetic energy growth) is therefore smaller if the winds associated with a PT anomaly of unit amplitude \((\theta = 1)\) are smaller. Because this decrease of the surface winds of the PT anomaly with \(\theta = 1\) occurs for an increasing value of \(\alpha\) (decrease of the buoyancy frequency with height) this will be called the \(N\) effect of the edge wave growth.

2) PV \textit{UNSHIELDING}

The second way to generate kinetic energy growth at the surface is by the mechanism known as PV unshielding. The individual PV anomalies residing at the different levels cannot amplify because of the absence of any basic-state PV gradient in the interior. However, their wind fields may initially partially shield each other. Due to the existing shear of the basic state, the PV anomalies unshield and the wind field amplifies. The kinetic energy growth rate due to PV unshielding is influenced as well by the basic-state properties. There is a \(\Lambda\) effect and an \(N\) effect of PV unshielding. The \(\Lambda\) effect of PV unshielding is simply the effect that PV unshielding occurs more rapidly if the shear is larger. For the basic states considered in the present paper, the U cases have a shear that is increasing with height. The PV unshielding will therefore be faster in the upper layers. The \(N\) effect of PV unshielding is related to the streamfunction structure of the PV anomaly. If the streamfunction decays slowly away from the height of the PV, PV unshielding of individual PV anomalies occurs more effectively (for fixed shear) than would be the case if the streamfunction decays more rapidly. Moreover, for a fixed height of the PV and nonzero \(\beta\), the surface-to-TKE ratio of this PV anomaly increases for increasing \(\alpha\). This ratio becomes larger the higher the PV is located. It is clear that a larger ratio produces a much larger contribution from the PV to the surface wind field.

3) PV–PT \textit{UNSHIELDING}

Finally, the third mechanism that causes growth of kinetic energy is termed PV–PT unshielding. As its name suggests, this is the unshielding that occurs when the surface winds of the PV and PT anomalies, which are initially out of phase, become aligned due to the difference in propagation speed. Clearly PV–PT unshielding is influenced by the difference in phase speed between the PT and the PV anomaly. This will be called the \(\Lambda\) effect of PV–PT unshielding. In general the kinetic energy growth rate resulting from PV–PT unshielding is maximized if the difference in phase speed is maximized, that is for a PV anomaly far away from the resonant level, at which the phase speed of the PV and the edge wave coincide.

6. One PV building block

In the first experiment we have used only one PVB. The SV is determined by choosing the height of the PVB such that \(\Gamma(0, t_{\text{opt}})\) is maximized for the given time interval. Two definitions will be used. The optimal growth height is defined as the height of the PV at which \(\Gamma(0, t_{\text{opt}})\), Eq. (14), reaches its optimum. The optimal \(\Gamma\) is the value that \(\Gamma(0, t)\) attains at \(t = t_{\text{opt}} = 5.16\) (48 h) if the PV is located at its optimal growth height. Figure 3 displays \(\Gamma(0, t_{\text{opt}})\) as a function of the height of the PVB. The thick lines represent the R case.

\(a.\) \textit{The R case}

For the R case \((\beta = 0)\) the optimal growth height for the given optimization time is obtained in a region below the resonance level (which for \(\beta = 0\) is located at \(h = \mu^{-1}\)). For smaller optimization times the optimal
growth height is found near the lower surface while the resonant level is approached from below if the optimization time is increased. This is similar to what has been found by Dirren and Davies (2004) who used a norm on PV rather than on KE at initial time. The reason that the optimal position of the PV anomaly for small optimization times is close to the lower boundary is the PV effect of the edge wave growth. Perturbations with high surface winds already present at initial time are favored because they produce larger growth of the edge wave and of the kinetic energy.

In Fig. 4 the relative contributions of the different processes to the SV time evolution are shown; PV unshielding does not occur and \( |\phi_{\mu}(t)|^2 \) is time independent. Initially, \( \phi_{\mu} \) is zero, and the SKE of the SV is entirely due to the PV. As time increases, the kinetic energy of the SV increases and the relative contribution from PV becomes smaller. The growth of the edge wave is most important for the SV development. PV–PT unshielding contributes negatively. In DVO (their Fig. 5) it is shown that this implies that the PV resides below the resonant level.

b. The U case

For the U case, the resonance level lowers because of the increased shear (see Bishop and Heifetz 2000, especially their appendix F). The buoyancy frequency profile remains unmodified. From the PV effect on the edge wave growth (see also Fig. 2) one therefore anticipates that the optimal \( \Gamma \) will increase for nonzero \( \beta \). This is indeed observed in Fig. 3. Furthermore, the time evolution of the relative contributions to the SKE, displayed in Fig. 4, shows that the optimal growth height for the U case is much closer to the resonance level (as compared to the R case), as the contribution from PV–PT unshielding is almost zero. For larger values of \( \beta \) (not shown) it has been found that the optimal growth height is even above the resonance level, which is a direct result of the \( \Lambda \) effect of PV–PT unshielding (i.e., the shear that increases with height). In this case of large \( \beta \), the contribution from PV–PT unshielding can even exceed the contribution from the growing edge wave. In general we have an interplay between the kinetic energy growth due to a growing edge wave becoming stronger for PV at low altitudes, on the one hand, and PV–PT unshielding becoming stronger for PV at higher altitudes, on the other hand (and only adding up with the SKE if the PV resides above the resonance level).

c. The N case

For the N case the growth of the edge wave reduces slightly because of the different structure of the edge mode (the N effect of the edge wave growth). On the other hand, because of its reduced propagation speed, the height of the resonant level is reduced, which enhances the growth (the PV effect of the edge wave growth). The net result of the opposing effects for the N case is a slightly reduced optimal \( \Gamma \) (Fig. 3). It depends on the exact choice of \( \alpha \) and \( \beta \) whether the optimal growth height resides above or below the resonant level. For the N case with \( \beta = 0.5 \) the optimal growth height is slightly below the resonant level and PV–PT unshielding contributes negatively (see the thick lines in Fig. 4).
7. Two PV building blocks

For the SV constructed with one PVB, the only growth mechanisms have been the growing edge wave (resonance) and PV–PT unshielding. The contribution from the PV was time independent. To make the system more realistic, the effect of PV unshielding is included in this section. The simplest way to do this, is to add one additional PVB and vary the positions of both PV anomalies in order to optimize $\Gamma(t_{\text{opt}})$. In DVO it has been shown that the optimal $\Gamma$ obtained using a KE norm at optimization time increases significantly when PV unshielding of only two PV anomalies is included. The two PV anomalies are located at different positions above the resonant level and are initially almost completely out of phase. It has been shown in DVO that resonance remains to be the dominant growth mechanism at the surface, a result that is also valid for the problem in which a continuous PV distribution is used.

In Fig. 5 the height dependence of $\Gamma$ is shown for various values of $\beta$. In this figure, one PV anomaly is positioned at its optimal position and the height of the other is varied to examine the height dependence of $\Gamma$. In Fig. 5 the optimal position of the first PVB (at the position of the line) coalesces with the absolute minimum of $\Gamma$ in Fig. 5. The reason for this minimum is that if both PVBs are located at the same height, PV unshielding does not occur and $\Gamma$ attains the same value as $\Gamma$ of one single PVB at that level (see Fig. 3).

An observation holding for all cases (also noted in DVO) is the increase of the optimal $\Gamma$ for the two-PVB problem (more than an order of magnitude) as compared with the one-PVB problem (Fig. 3). The additional growth is caused by PV unshielding in the following way. If positioned sufficiently out of phase, the initial PV anomalies will shield each other’s wind field. This shielding occurs not only at the surface but, more importantly, also in a large part of the interior domain. The optimization procedure therefore increases the amplitudes of both initial PV anomalies to satisfy the initial norm. When time increases unshielding of the individual PV anomalies generates a surface wind field, which is larger than in the one-PVB case. This larger wind field causes the edge wave to grow more rapidly.

To illustrate the above reasoning, we have displayed the time evolution of the two-PVB SV streamfunction (and its components $\psi_{\text{tr}}$ and $\psi_{\text{pe}}$) for the R case in Fig. 6. The time evolution is similar to the evolution shown in DVO (their Fig. 7), but the initial structure is located somewhat lower in the atmosphere as compared to DVO. Initially, the SV is completely described by $\psi_{\text{tr}}$. Unshielding of the two PVBs occurs and the total streamfunction starts to propagate downward. After the surface has been reached, a PT wave is rapidly excited (the resonance effect) and after two days the perturbation streamfunction at the surface is dominated by the contribution from $\psi_{\text{pe}}$. The time evolution of the relative contributions to the SKE of the SV for the three cases is shown in Fig. 7.

a. The R case

In DVO the optimal positions of both PV anomalies were found to lie above the resonance level. In the present paper, the different choice of norm at optimization time, positions one PV anomaly below the resonance level, just as in the one-PVB case. As a result of this lower position, PV–PT unshielding contributes negatively to the growth of the SV, which is shown in Fig. 7. At optimization time the main contribution to the SKE is due to the growing of the edge wave (reso-
nance). The contribution from PV unshielding is relatively small.

b. The U case

For the U cases displayed in Fig. 5, the optimal $\Gamma$ increases for increasing values of $\beta$, similar to the one-PVB case. The optimal $\Gamma$ increases because of two effects. The first is again the PV effect of the edge wave growth, namely, the lowering of the resonant level for nonzero $\beta$, which gives a larger growth rate of the edge wave. The second effect is that PV–PT unshielding contributes less negatively to the growth, which is again due to the shear increasing with height favoring PV anomalies at higher altitudes (the $\Delta$ effect of PV–PT unshielding). In Fig. 7 it is shown that at optimization time, the largest contribution to the SKE is due to resonance. The contribution from PV unshielding is small and the contribution from PV–PT unshielding is negative throughout the time evolution. As in the previous section increasing $\beta$ would favor PV–PT unshielding and show that for $\beta > 1.5$, PV–PT contributes positively to the kinetic energy growth at the surface at $t_{\text{opt}}$.

c. The N case

For the N case, Fig. 3 showed a decrease of the optimal $\Gamma$ for increasing $\beta$ in the one-PVB problem. This decrease does not occur for the two-PVB problem and Fig. 5 shows that the optimal $\Gamma$ increases for increasing

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*Fig. 6. (top) Time evolution of the two-PVB SV streamfunction $\psi(x, z, t)$ (the R case) and its components (middle) $p$ and (bottom) $\psi_u$ toward $t_{\text{opt}} = 5.16$. Contour interval 0.2, negative values are dashed.*
values of $\beta$. The position of the maximum in Fig. 5 occurs roughly at the same (or even a little higher) level for different values of $\beta$. However, the height of the resonant level is reduced for nonzero $\beta$. We conclude that PV unshielding becomes more effective for nonzero $\alpha$. This is the N effect of PV unshielding explained in section 5 (see also Fig. 2). As compared to the U case the PV streamfunction structures of the N case decay less rapidly with height. This results in better shielding capacities of the PV anomalies at different heights. On the other hand, the N effect of the edge wave (lower wind speeds for a given amplitude of PT at the surface) reduces the SV growth due to growth of the edge wave. For the two-PVB case studied in this section, the larger growth due to enhanced shielding properties of the initial PV anomalies (the N effect of PV unshielding) is more important than the reducing effect of the different edge wave structure (the N effect of the edge wave growth), and the optimal $\Gamma$ increases as compared to the R case. From Fig. 7 it is observed that resonance has produced the largest contribution. The contribution from PV–PT unshielding is negative and the contribution from PV unshielding is also small. Again, for increasing $\beta$, PV–PT unshielding contributes positively to the growth because the unshielding mechanisms become more efficient at higher altitudes (the $\Lambda$ effect of PV–PT unshielding).

One final note concerns the evolution toward a vertical PV tower at optimization time. It can be shown that PV unshielding continues after the optimization time has been reached. This implies that a vertical tower has not yet been reached at optimization time.

8. The continuous problem

In the previous section we have shown that the optimal $\Gamma$ increased an order of magnitude (cf. the one-PVB case) by including PV unshielding of only two PV anomalies. Moreover in all cases the optimal $\Gamma$ increases for nonzero $\beta$.

To approach the continuous situation, this section finally addresses the problem in which interior PV is allowed at all (60) levels. We have chosen to let $\beta$ vary between 0 and 2.5. Although the latter value is rather large for the atmosphere at midlatitudes, it serves to explain the qualitative differences between the U and N cases. The growth of the kinetic energy of the SV and the contributions of the different mechanisms have been displayed in Fig. 8. In all panels in Fig. 8 the thick lines represent the R case ($\beta = 0$), the thin lines show jumps of $\beta = 0.5$.

a. The R case

The thick lines in Fig. 8 display the time evolution of the different contributions to the SKE for the R case. Although the norm at optimization time is different from the one used in DVO, similar results are obtained. As compared to the two-PVB problem, $\Gamma$ increases significantly (not shown). We do not show the initial period in which the amplitudes are very small. It can be shown that initially (but not really visible in the figure), cyclolysis occurs at the surface because the generated PT wave is almost completely out of phase with the PV wind field. After this initial period of surface cyclolysis, the SV rapidly attains a large amplitude at the surface (and in the interior, not shown). Similar to the one- and two-PVB cases, resonance is the most important process at the surface but the contribution of PV–PT unshielding to the SKE of the SV is negative during the complete time evolution toward $t_{opt}$. Just after the optimization time has been reached the contribution from the PV is maximized, implying that an almost vertically aligned tower of PV has been formed. The SV contin-
ues to amplify after $t = t_{\text{opt}}$ because the advection of basic-state PT has not stopped.

b. The U case

For the U case, the effect of increasing $\beta$ is to enhance the growth significantly (see Fig. 8, top-left panel). This is similar to what is observed in the one- and two-PVB situations discussed before. The largest contribution to the SKE at optimization time is from the resonance. The contribution from the PV is maximized just prior to $t = t_{\text{opt}}$ and increase for increasing values of $\beta$. The contribution from PV–PT unshielding

Fig. 8. Time evolution of the absolute contributions of the different growth mechanisms to the SKE of the SV for the continuous problem (60 PVBs) for various values of $\beta$ [$\beta = \{0(\text{thick}), 0.5, 1.0, 1.5, 2.0, 2.5\}$, numbered 1–6]. The vertical lines denote the position of the optimization time. Please note that the different panels have different scales.
depends on the value of $\beta$. Contributing negatively in the R case, the effect of nonzero $\beta$ is to make PV–PT unshielding contribute positively to the growth of the SKE. For $\beta = 0.5$, the contribution is almost zero, indicating that the net PV wind field propagates with almost the same speed as the PT wave. There are a couple of reasons that the U case has resulted in more growth than in the R case. First, the increased shear of the basic state leads to stronger winds from PV (the $\Lambda$ effect of PV unshielding). Second, the increase in shear with height favors PV above the resonance level, and the contribution from PV–PT unshielding becomes positive (the $\Lambda$ effect of PV–PT unshielding). With a gradual increase of $\beta$ this leads to a maximum of the SKE just after $t_{\text{opt}}$. After $t_{\text{opt}}$ the edge wave still amplifies and resonance completely dominates the longtime value of the SKE.

c. The N case

Similar to the time evolution of the U case, the N case shows enhanced growth of SKE as compared to the R case (see Fig. 8, right panels). On the other hand, the additional growth is less than in the U case. Furthermore, the growing edge wave provides the main contribution to the SKE at optimization time and dominates the longtime kinetic energy of the SV (as in the U and R cases). The contribution from the PV is maximized just before $t_{\text{opt}}$. PV–PT unshielding contributes positive at $t_{\text{opt}}$ only if $\beta > 1$.

Apart from these similarities with the U case there are at least two differences between the U and N cases. First, it can be seen that the contribution from the growing edge wave at optimization time is almost independent of $\beta$. In section 5 we showed that the different structure of the edge wave reduces its kinetic energy growth rate (the $N$ effect of the edge wave growth). However, we have also discussed that PV unshielding becomes more and more efficient for increasing $\beta$ in the N cases (the N effect of PV unshielding, see also the bottom-right panel in Fig. 8). The results of this more efficient PV unshielding are stronger surface winds due to the PV and the edge wave growth rate is increased (the PV effect of the edge wave growth). Apparently the result of these counteracting effects acts to produce a contribution to the kinetic energy growth, which is almost independent of $\beta$. A comparison between the U and N cases furthermore shows the following at-first-sight remarkable fact. Although we have argued before that PV unshielding becomes more and more effective in the N case due to the N effect of PV unshielding, the total contribution from the PV is smaller in the N case than in the U case. The $\Lambda$ effect of PV unshielding (see section 5) causes this apparent inconsistency. In the U case the shear increases with height, whereas in the N case this shear is constant. Two PV perturbations at a given height difference unshield more rapidly in the U case than in the N case. On the other hand, the N effect of PV unshielding (more effective unshielding) possibly generates kinetic energy more effectively in the N case. For which of the two cases the contribution from the PV to the growth is largest depends on the specific choice of $\alpha$ and $\beta$. In the cases discussed here, the kinetic energy growth is larger in the U case than in the N case.

d. Matching with observations

After these general investigations, an additional calculation is performed using an estimate for $\alpha$ based on observational data of $N^2$ profiles. Charney and Drazin (1961) have calculated summer and winter profiles of $N^2(z)$ for the 30°–60°N belt. These profiles show a sharp decrease with height in the troposphere, which could be simulated with a suitable choice of $\alpha$. For the winter profile they get the dimensional values $N^2 = 3.4 \times 10^{-4} \text{s}^{-2}$ at the surface and $N^2 = 1.5 \times 10^{-4} \text{s}^{-2}$ at the level of the tropopause. Together with an average value of the surface shear in the basic-state zonal velocity of $3 \text{m s}^{-1} \text{km}^{-1}$, the value for $\alpha$ using the relation for $N^2$ in Eq. (5) would be $\alpha \approx 0.7$. Using the relation for $\overline{U}$ in Eq. (5) this produces zonal wind speeds of $34.5 \text{m s}^{-1}$ at the tropopause level.

To compare the results with the results of the previous sections, we have performed a number of calculations with $(\alpha = 0.7, \beta = 0.53)$ in which the surface buoyancy frequency increases from $N^2 = 1$ to $N^2 = 3.4$. The results are shown in Fig. 9. Increasing the surface value buoyancy frequency reduces the growth of the SV considerably (top-left panel in Fig. 9) and all processes are affected. As can be seen from Eq. (9) the phase speed of the edge wave reduces significantly when increasing the surface buoyancy frequency. This phase speed decreases because the surface winds of an edge wave of a given PT decrease for increasing $N_0$.

Similarly, the surface winds of interior PV perturbations decrease for increasing value of $N_0$. Therefore, the growth rate of the edge wave will become less.

As a result of the reduced propagation speed of the edge wave, the maximum in the contribution from the resonance shifts to larger time values (top-right panel). The PV still attains its maximum contribution around $t_{\text{opt}}$. PV–PT unshielding contributes negatively during the complete time evolution toward $t_{\text{opt}}$ for all chosen values of $N_0$.
9. Summary and concluding remarks

In the preceding sections we have investigated in which way a nonzero value of $\beta$ influences the structure and time evolution of SVs for the Eady model without the upper rigid lid. The standard Eady basic state (with linear shear and constant buoyancy frequency) has been modified in the presence of $\beta$ such that the mean PV gradient remains zero (Lindzen 1994). The adjustment process is controlled by a parameter $\alpha$, which modifies either the basic-state velocity $U(z)$ ($\alpha = 0$), or the buoyancy frequency profile $N^2(z)$ ($\alpha = 1$), or both ($0 < \alpha < 1$). Based on earlier work on the $f$-plane Eady model (DVO), the PV perspective is used to construct PVBs (PV building blocks), which in turn are used as an analytical, nonmodal basis to determine the SV. In the present paper a KE norm has been used at initial time. To concentrate on rapid surface cyclogenesis, the SKE norm has been chosen at the optimization time. The general conclusion of the present paper is that a nonzero value of $\beta$ leads in almost all cases to a larger optimal $\Gamma$ than would be obtained in absence of $\beta$. Whether or not this increase is significant, depends largely on the exact choice of the basic state. However, the SV growth is strongly reduced for an observationally matched mean profile of the buoyancy frequency. This reduction is mainly a consequence of the large value of the buoyancy frequency at the surface. Some final remarks are added to put the obtained results into perspective.

1) The growth of the SV in the semi-infinite model depends significantly on the exact value of the buoyancy frequency at the surface. In idealized SV studies in the literature, the buoyancy frequency usually is given the reference value $N^2 = 10^{-4} \text{s}^{-2}$. Based on the present study, it is to be expected that the choice of buoyancy frequency also plays a role in situations in which baroclinic instability occurs, for instance, in the case of the Eady model with the upper rigid lid (Morgan 2001; Morgan and Chen 2002).

2) The conventional way to calculate SVs numerically is based on a modal decomposition. Although the purely resonant mode is excluded in this approach,

![Figure 9](image-url)
it is easy to go from the modal perspective to the PV perspective. The strategy (which is nothing more than a simple coordinate transformation) is the following. Given the projection coefficients of the initial SV in the modal perspective, compute the PV at the different interior levels. This gives the initial PV distribution (and hence the projection coefficients of the PVBs). To compute the projection coefficients of the boundary edge waves in the PV perspective, add up all the boundary PT contributions from the CMs and the edge waves in the modal components. For given wavelength \((k, l)\), this results in an upper and lower edge wave of different amplitude. The amplitude of these edge waves are the projection coefficients of the edge waves in the PV perspective.

3) In the present study it is found that the SV growth increases for nonzero \(\beta\). In the Eady problem in which the upper rigid lid is present as discussed in Lindzen (1994), the stability properties of the discrete normal modes are affected by \(\beta\) as well. If the basic-state PV gradient is zero (by adjusting the basic-state velocity and buoyancy frequency) a long-wave cutoff appears. In this sense, \(\beta\) acts to stabilize the ultralong modes. However, it can be shown that the growth rate of the most unstable mode increases\(^6\) for nonzero \(\beta\). This increase can be understood by applying the results of our analysis. The basic-state shear and buoyancy frequency at the upper rigid lid influence the propagation of the upper-level edge wave. If the phase speed of the lower- and upper-level edge wave are sufficiently similar, phase locking occurs and instability sets in. In the U case the basic-state meridional PT gradient (and hence the shear) increases with height. For given winds from the lower edge wave, the growth rate of the upper-level edge wave will be larger (one may call this the \(\Theta\) effect of the upper-level edge wave growth, see section 5). At the same time phase locking occurs more easily because the phase speed of the upper-level edge mode is more reduced than in the original R case with \(\beta = 0\). In the N case, the main cause for the larger growth rate is the buoyancy frequency that decays slowly with height. As a result, the coupling between the modes becomes stronger. The phase speed of the upper-level edge mode is reduced more effectively, which facilitates phase locking with the lower edge wave. This can be compared with the PV effect of the edge wave growth. The result is a larger growth rate. This should be contrasted with the N effect of the edge wave growth in the absence of the rigid lid. In the latter case the height dependence of the buoyancy frequency reduced the phase speed of the edge wave as a result of which the edge wave with \(\theta = 1\) possesses much less kinetic energy. In the case with rigid lid, this effect is compensated by the facilitation of phase locking.

4) The inclusion of the upper rigid lid provides the system with a new (exponential) growth mechanism, which will inevitably dominate the longwave SV growth in the long run. This implies that it is not unlikely that the initial longwave SVs (especially for long optimization times) will be different\(^7\) in the model in which the upper rigid lid is retained [see, e.g., the differences between shortwave and longwave SVs considered in the literature for a KE norm; Mukougawa and Ikeda (1994); Morgan (2001); Morgan and Chen (2002)]. If, as is done in the present paper, the focus is on the generation of SKE in a relatively short time, we expect that PV unshielding and resonance will remain the important mechanisms for SV growth at the surface.

5) As already addressed shortly in section 2b, Lindzen (1994) considered a slightly different modification of the basic-state zonal wind. His approach was to retain a fixed mean tropospheric wind shear, rather than a fixed surface shear and surface zonal wind, as we did for the semi-infinite model. The introduction of nonzero \(\beta\) in Lindzen’s \(\alpha \neq 1\) cases, leads to a decreased surface wind shear. In the present paper, the \(\Theta\) effect of the edge wave growth (section 5b), was independent of \(\beta\), but this clearly changes if the wind shear at the surface is modified. This point once more illustrates that, despite its mathematical simplicity, the finite-time instability of strictly zonal flows remains a delicate issue, which depends sensitively on the choice of the optimization norms and the basic state.

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\(^6\) With a rigid lid at 10 km, the instantaneous growth rate \(\sigma = kc\), of the most unstable mode (with \(l = 0\)) gives \(\sigma = 0.31\) (the R case), \(\sigma = 0.38\) (the U case, \(\beta = 0.53\)), and \(\sigma = 0.34\) (the N case, \(\beta = 0.53\)), again if \(m_0 = 1\) has been chosen for the surface wind shear.

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\(^7\) This is illustrated by the fact that an interior PVB in the model with the upper rigid lid induces both a lower and an upper edge wave. Since the mean PV gradients at the surface and the tropopause are of opposite sign, these PVB-generated edge waves are \(\pi\) out of phase initially, which is known to be suboptimal for the exponential growth (Heifetz et al. 2004; Dirren and Davies 2004).
Solving the PV Equation

Suppose one is given the inhomogeneous problem

\[ \phi'' + f(z)\phi' + g(z)\phi = q(z), \]  

(A1)

where the prime indicates differentiation with respect to the variable. Let \( \phi_1(z) \) and \( \phi_2(z) \) be two independent solutions of Eq. (A1) with \( q(z) = 0 \). Solutions of Eq. (A1) with nonzero \( q(z) \) are given by (Kamke 1967)

\[
\phi(z) = C_1\phi_1(z) + C_2\phi_2(z) + \int_0^z \frac{\phi_1(\xi)\phi_2(z) - \phi_2(\xi)\phi_1(z)}{W(\xi)} \, q(\xi) \, d\xi, \tag{A2}
\]

where \( W(\xi) = \phi_1(\xi)\phi_2'(z) - \phi_2(\xi)\phi_1'(z) \) is the Wronskian. The constants \( C_1 \) and \( C_2 \) are determined by the boundary conditions.

For the problem we address in this paper, regularized solutions to the homogeneous PV equation [Eq. (2) with \( q(z) = 0 \)] are

\[
\phi_1(z) = \frac{K_d[\gamma(z)]}{K_d[\gamma_0]}, \quad \phi_2(z) = \frac{I_d[\gamma(z)]}{I_d[\gamma_0]}, \tag{A3}
\]

where \( \gamma(z) = \gamma_0 \sqrt{1 + abz} \) and \( \gamma_0 = 2\mu N_0/(ab) \) with \( b = \beta N_0^2/m_0 \). Here \( I_d[\gamma(z)] \) and \( K_d[\gamma(z)] \) are modified Bessel functions of the first and second kind of order \( n \), respectively (Abramovitz and Stegun 1970). Using Eq. (A2) the streamfunction associated with a spike of PV at \( z = h \) [Eq. (2) with \( q(z) = Q\delta(z - h) \), satisfying \( \partial\phi_p/\partial z(z = 0) = 0 \) and \( \phi_p(z \to \infty) = 0 \) becomes

\[
\phi_p(z) = \frac{Q N_0^2}{W(h)} \left\{ H(z - h)[\phi_1(h)\phi_2(z) - \phi_2(h)\phi_1(z)] - \phi_1(h)[\phi_1(z) + \phi_2(z)] \right\}, \tag{A4}
\]

where \( \phi_1 \) and \( \phi_2 \) are given by Eq. (A3), \( H(z - h) \) is the Heaviside step function\(^8\) and where

\[
W(h) = 2\mu^2 N_0^2 \frac{K_d[\gamma(h)]I_d[\gamma(h)] + K_d[\gamma(h)]I_d[\gamma(h)]}{K_d[\gamma_0]I_d[\gamma_0]}, \tag{A5}
\]

where \( \gamma(h) = \gamma(z = h) \). For \( \alpha \to 0 \) one obtains (Thornicroft and Hoskins 1990; Chang 1992)

\[
\phi_p(z) = \frac{Q N_0}{\mu} \left[ H(z - h) \sinh[\mu N_0(z - h)] - e^{-\mu N_0h} \cosh[\mu N_0z] \right]. \tag{A6}
\]

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Computing the SV Analytically

The analytical expression of the PVB, Eq. (13) in the main text, makes it possible to compute the SV completely analytically. We follow Fischer (1998), who computed the SV for the discrete normal modes of the \( f \)-plane Eady model with the upper rigid lid. In the present paper an arbitrary number of PVBs is used and therefore the contribution from the continuum modes (CMs) is automatically included.

Let \( M \) be the number of levels at which PVBs are positioned. The streamfunction of the SV is written as

\[
\psi_{SV}(x, z, t) = \sum_{j=1}^{M} a_j \psi_j(x, z, t), \tag{B1}
\]

where \( a_j \) is the (complex) projection coefficient of SV on the \( j \)th PVB. Arranging the projection coefficients \( a_j \) in a vector \( \mathbf{a} \), the SKE and KE become

\[
\text{SKE}(t) = \mathbf{a}^H \cdot \mathbf{\Psi}_{\text{SKE}}(t) \cdot \mathbf{a}, \quad \text{KE}(t) = \mathbf{a}^H \cdot \mathbf{\Psi}_{\text{KE}}(t) \cdot \mathbf{a}, \tag{B2}
\]

where \( \mathbf{A}^H \) denotes the conjugate transpose of object \( \mathbf{A} \).

The matrix elements are given by

\[
[\mathbf{\Psi}_{\text{SKE}}(t)]_{ij} = \int_0^H \tilde{\psi}_j^*(z, t)\psi_i(z, t) \, dz, \tag{B3}
\]

where \( z_0 = 0 \) is the earth surface. The computation of the integrals in the last expression is tedious but straightforward, involving multiple products of the Bessel functions; at this point, one may wish to approximate the integrals numerically. The SV optimizing \( \Gamma(0, t) \) defined in Eq. (14) is formed by the eigenvector \( \mathbf{a} \) corresponding to the largest eigenvalue of the eigenvalue problem:

\[
[\mathbf{\Psi}_{\text{SKE}}(t) - \lambda \mathbf{\Psi}_{\text{KE}}(0)] \mathbf{a} = 0, \tag{B4}
\]

where \( \mathbf{\Psi}_{\text{KE}}(0) = \mathbf{\Psi}_{\text{KE}}(t = 0) \). In a similar way the SV optimizing total kinetic energy at optimization time is computed by replacing \( \mathbf{\Psi}_{\text{SKE}}(t) \) with \( \mathbf{\Psi}_{\text{KE}}(t) \) in Eq. (B4). The method is easily generalized to different norms.

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