A Generalized Momentum Framework for Looking at Baroclinic Circulations

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ABSTRACT

This paper introduces the concept of potential momentum, which is a nonlocal measure of the thermal structure that has momentum units. Physically, it may be interpreted as the zonal momentum that the flow would realize through an adiabatic redistribution of mass that made the isentropic thickness uniform poleward of a reference latitude. At the surface, a poleward temperature gradient is equivalent to an easterly reservoir of potential momentum. Potential momentum gives a global picture of the thermal field that also takes into account the meridional structure.

When the mean flow is redefined in terms of the total momentum (the standard zonal momentum plus this newly defined potential momentum), the mean-flow response to the eddy forcing can be formulated locally. This allows one to relate in equilibrium the eddy absorption and the restoration of the mean flow. Using the concept of potential momentum, vertical propagation can be related to thermal forcing by means of an equation analogous to the relation between meridional propagation and surface friction. Based on these ideas, it is argued that the time-mean eddy propagation is largely constrained by the strength and structure of the forcing.

The new formalism is applied to a forced–dissipative two-layer model to study the dependence of meridional and vertical propagation and of the global circulation on changes in the diabatic and frictional forcing time scales. It is found that baroclinic adjustment, in the form of a fixed potential vorticity (PV) gradient, is a good approximation because the intensity of the circulation only depends weakly on the forcing time scale. Another remarkable result is that, owing to the robustness of this PV gradient, thermal homogenization is always enhanced with stronger friction.

1. Introduction

The classical energy cycle proposed by Lorenz (1960) provides an invaluable tool for diagnosing and understanding the maintenance of the atmospheric circulation. In Lorenz’s formulation, the energy of both eddies and mean flow is conceptually split into kinetic and potential components. Thus, the energy exchange between eddies and the mean also occurs in both forms: baroclinic conversions are associated with an exchange of potential energy and barotropic conversions with a kinetic energy exchange. However, it has been known since Charney and Drazin (1961) that the existence of nonzero baroclinic or barotropic conversions does not automatically imply an interaction between eddies and the mean flow, as these conversions may be such as to cancel each other. Under nonacceleration conditions, the kinetic and potential energy exchanges between eddies and mean flow are exactly equal and opposite, and there is no net input or dissipation of energy into the system. The situation has been amply discussed by Plumb (1983), who also shows that the interpretation of the energy conversions depends on the averaging scheme used, and is to a large extent ambiguous.

On the other hand, Andrews and McIntyre (1978) show that the distinction between eddies and mean is most meaningfully expressed in a Lagrangian framework. Unfortunately, it is nearly impossible to formulate such a framework in practice, due to the chaotic nature of Lagrangian trajectories. A convenient approximation that only requires Eulerian averaging is...
provided by the Transformed Eulerian Mean (TEM) formalism (Edmon et al. 1980). This formalism avoids the cancellations inherent to the traditional Eulerian averaging so that in the quasigeostrophic limit, the net interaction between eddies and the mean flow is encapsulated in the eddy PV flux [or Eliassen–Palm (EP) divergence] \( \vec{v} \vec{q} \). This is manifest in the following conservation equation for the eddies (e.g., Andrews et al. 1987):

\[
\frac{\partial \vec{A}}{\partial t} = -\nabla \cdot \vec{F} + D_{\text{eddy}},
\]

(1)

where \( \vec{A} \approx (1/2) q^2 \vec{f} \) is the eddy pseudomomentum, a quadratic measure of eddy amplitude. \( D_{\text{eddy}} \) represents the eddy forcing/damping by all nonconservative processes, and \( \nabla \cdot \vec{F} = \vec{v} \vec{q} \) is the EP divergence, and represents eddy growth or decay at the expense of the mean flow. This wave–mean flow interaction term is also closely related to wave propagation, as \( \vec{F} \approx c_g \vec{A} \) in the Wentzel–Kramers–Brillouin (WKB) limit, where \( c_g \) is the group speed of the eddies. The relation between wave propagation and eddy–mean flow interaction has led to a modern view of the general circulation in which the distinct stages of baroclinic and growth and barotropic decay of the classical Simmons and Hoskins (1978) paradigm are regarded in a unified framework (e.g., Thorncroft et al. 1993). A recent application of this paradigm is provided by the work of Seager et al. (2003), who explain the shift of the extratropical jet during the ENSO cycle in terms of changes in the refraction index.

In this context, it seems more natural to diagnose the general circulation using wave activity (or pseudomomentum) rather than energy as in the classical treatment. To a large extent, this is achieved by the EP-flux diagram introduced by Edmon et al. (1980). This diagram characterizes the surface as the divergence of a wave flux, part of which is absorbed aloft and part of which propagates meridionally to be dissipated at lower latitudes. Based on these ideas, one could characterize/quantify the circulation more physically in terms of its net eddy and mean flow components, and their sources and sinks, without the artificial distinction between barotropic and baroclinic processes of the traditional framework. However, there are some difficulties when implementing a budget of this kind. First, wave activity is not a sign-definite quantity, unlike energy, so one has to be careful when estimating the eddy component of the circulation globally. Second, the conservation relation Eq. (1) is subject to large errors due to the linearization of pseudomomentum and to the implicit eddy dissipation as enstrophy cascades to the smallest scales, which makes it impractical for budgeting the circulation. Finally, it is not clear how to define the mean flow counterpart of wave activity. In other words, there is not a mean flow quantity that increases in the same proportion as \( \vec{A} \) decreases in the presence of a wave–mean flow interaction \( \vec{v} \vec{q} \). The difficulty arises because, although \( \vec{v} \vec{q} \) is the only eddy forcing of the mean flow, the response of the mean flow to this forcing is nonlocal. The goal of this paper is to characterize the impact of the eddy forcing on the mean flow locally, by essentially providing an equation for the mean that is complementary to relation (1) for the eddies.

The paper is divided into two main parts. Section 2 contains the main theoretical development. We derive a local conservation relation for the mean flow by introducing the concept of potential momentum. A physical interpretation of this concept is provided, and the extratropical circulation is reinterpreted in terms of the new framework. Section 3 applies these diagnostics to the two-layer model and compares the relative importance of mechanical and thermal forcing for the maintenance of the equilibrium state. It is argued that these questions are best addressed in the potential momentum framework.

2. Potential momentum

a. Mathematical framework

As explained above, our aim is to derive a conservation equation for the mean flow that is complementary to Eq. (1). The starting point is the QG TEM momentum equation:

\[
\frac{\partial \vec{U}}{\partial t} = f_0 \vec{v}^* + \nabla \cdot \vec{F} - \alpha_M^{-1} \vec{U},
\]

(2)

where \( \vec{v}^* \) is the residual meridional velocity and \( \alpha_M^{-1} \) a frictional time scale.

The main difficulty is the appearance of the zonal momentum forcing by the residual circulation. Although \( \vec{v} \vec{q} \) represents the only forcing of the mean flow by the eddies, the mean flow responds in a nonlocal manner to this forcing. In particular, the \( f_0 \vec{v}^* \) term may be thought to redistribute the forcing to enforce geostrophic balance. It is simple to eliminate this term through vertical integration, which leads to a budget equation for the vertically integrated momentum. The same can be done with Eq. (1), leading to a pair of equations describing the exchange between the eddy and mean flow integrated momentum. However, this is
not necessarily very useful: since neither $\overline{U}$ nor $\overline{A}$ are sign-definite quantities, the integration involves severe cancellation (Zurita-Gotor and Lindzen 2004b). Indeed, the global integral of $\overline{A}$ is trivially zero when surface delta-function contributions are included (Held 1985).

An alternative way to eliminate $\overline{\Theta}^*\theta$ is through the use of the thermodynamic equation, which is given in its (QG) TEM version by

$$\frac{\partial \overline{\Theta}}{\partial t} + \overline{w}^*\Theta_z = -\alpha_T(\overline{\theta} - \overline{\theta}_R),$$

(3)

where $\overline{w}^* = \overline{w}_0 + \delta_z(\overline{w}/\overline{\Theta}_z)$ is the vertical component of the residual circulation, $\overline{w}_0$ is the (ageostrophic) vertical velocity, $\overline{\Theta}_z$ is the reference stratification of QG theory, which is only a function of height, and $\overline{\theta}$ represents the deviations from that reference state, which may include in general a zonally symmetric part $\overline{\theta}_R$. Finally, $\overline{\theta}_R$ is a radiative equilibrium profile toward which the flow is relaxed with a constant time scale $\alpha_T^{-1}$. Note that we use for simplicity a Boussinesq framework, but the results are easily generalized.

We will be concerned in this paper with the case of a localized baroclinic zone within a meridionally unbounded domain. In particular, we assume that some reference latitude $\gamma_0 = 0$ exists such that $\overline{\Theta}^*\theta = 0$. Multiplying Eq. (3) by $-f_0/\overline{\Theta}_z$, integrating meridionally between $\gamma_0$ and $\gamma$ and differentiating with respect to $\gamma$, Eq. (3) can be written (taking into account continuity):

$$\frac{\partial \overline{M}}{\partial t} + f_0 \overline{w}^* = -\alpha_T(\overline{M} - \overline{M}_R),$$

(4)

where we define the potential momentum $M$ as

$$M = \int_{\gamma_0}^{\gamma} \int_{\theta_0}^{\theta} \left( \frac{f_0}{\overline{\Theta}_z} \overline{\Theta}_z \right) dy',$$

and $\overline{M}_R$ is the potential momentum in radiative equilibrium, obtained by application of the above operator to $\overline{\theta}_R$.

Note that $M$ has momentum units and that Eq. (4) formally looks like a momentum equation. Comparing Eqs. (2) and (4) we can see that, in this framework, the role of the mean circulation $\overline{w}^*$ is simply to transform potential into physical momentum. Adding both equations together this term disappears and we get

$$\frac{\partial}{\partial t} (\overline{M} + \overline{U}) = \nabla \cdot \overline{F} - \alpha_T(\overline{M} - \overline{M}_R) - \alpha_M \overline{U}.$$  

(6)

Thus, when the generalized momentum $\overline{M} + \overline{U}$ is considered, the eddy PV flux/EP convergence appears as the only forcing of the mean flow. Moreover, this generalized momentum is forced locally, while the remotely induced circulation $\overline{w}^*$ only determines the partition between $\overline{M}$ and $\overline{U}$. This partition must be such that thermal wind is satisfied; that is,

$$\frac{\partial^2 \overline{M}}{\partial y^2} = \frac{\partial}{\partial z} \left( \frac{f_0^2}{N^2} \frac{\partial \overline{U}}{\partial z} \right).$$

(7)

Equation (6) provides the mean flow counterpart to the conservation of eddy pseudomomentum [Eq. (1)], allowing one to quantify the local impact on the mean flow of an EP divergence $\nabla \cdot \overline{F}$. Both equations may be written in compact form:

$$-\frac{\partial}{\partial t} (\overline{M} + \overline{U}) = \nabla \cdot \overline{F} + D_{\text{mean}},$$

(8)

$$\frac{\partial}{\partial t} \overline{A} = -\nabla \cdot \overline{F} + D_{\text{eddy}},$$

(9)

where $D_{\text{mean}} = -\alpha_M \overline{U} - \alpha_M(\overline{M} - \overline{M}_R)$, $D_{\text{eddy}} = -\alpha_M \overline{w}^* + (\alpha_M \overline{w}^* - \alpha_M \overline{w}_0 \overline{q}) - \alpha_M (\overline{w}^* \overline{q} + D_0)/\overline{\Theta}_z$, and $m^*$ is the eddy component of $\overline{M}$ [cf. Eq. (5)]. Apart from the large-scale mechanical and thermal damping, we also included in $D_{\text{eddy}}$ the term $D_0$ to account for the small-scale dissipation. This term is expected to be significant owing to the role of an enstrophy norm in $\overline{A}$.

This pair of equations, (8)–(9), reflects a three-way balance between the forcing of the mean flow $D_{\text{mean}}$, the eddy dissipation $D_{\text{eddy}}$, and the wave–mean interaction $\nabla \cdot \overline{F} = \overline{v} \overline{q}$. They thus provide the basis for a “wave activity” budget of the circulation, a concept that will be exploited below. Note that in terms of wave activity there is no distinction between barotropic and baroclinic conversions. Another important difference is that, while the balances implied by Eqs. (8)–(9) apply locally everywhere, the Lorenz energetic formulation is global. Since $D_{\text{mean}}$, $D_{\text{eddy}}$, and $\nabla \cdot \overline{F}$ integrate globally to zero, care must be taken when formulating global budgets. Section 2d describes how this can be done with the present formulation.

The reader may have noticed that a more conventional way to eliminate $\overline{w}^*$ in Eq. (2) involves the $y$ differentiation of that equation instead of the $y$ integration of Eq. (3) performed here. This alternative manipulation produces the well-known zonal-mean quasi-geostrophic potential vorticity (QGPV) equation, of which Eq. (6) is the $y$ integral. The mean PV is also forced locally in that framework, in which the nonlocality only arises through the PV inversion. Although both expressions are, of course, intimately related, we believe that it can be advantageous in some cases to write the mean flow balance in the form given by Eq. (6). This formulation makes explicit the role of the eddy PV flux as a conversion term between eddies and mean
flow, which is needed for constructing closed budgets of the circulation. One could also construct these budgets by integrating the QGPV equation meridionally and defining the mean flow in terms of $\int \overline{\mathbf{q}} \ dy$, with no consideration of potential momentum. However, we believe that there is additional value in introducing this concept and in understanding how potential momentum is generated and destroyed. As shown in the next sections, potential momentum provides a natural definition of the baroclinic eddy source, including the effects of the meridional structure. Moreover, the partition between $\overline{M}$ and $\overline{U}$ allows one to relate the vertical/meridional wave propagation to the diabatic/mechanical restoration, and hence to understand the time-mean wave propagation in terms of the structure of the forcing.

b. Isentropic view and boundary contributions

The definition (5) implies that the potential momentum is simply the zonal momentum profile that produces the same PV distribution as the baroclinic term in the full 3D basic state:

$$\frac{\partial \overline{M}}{\partial y} = - \frac{\partial}{\partial z} \left( \frac{f_y}{\Theta_z} \overline{q}_z \right).$$ (10)

If the stratification were equal to its reference value $\Theta_z$ everywhere, then $\overline{q}_z(y, z) = 0$ [recall that $\bar{q}$ is defined as the difference from the reference state $\Theta(z)$], and the potential momentum would be zero. Thus, $\overline{M}$ is associated with deviations of the isentropic thickness from its reference value. An easterly (westerly) meridional shear in $\overline{M}$ is associated with a positive (negative) $\theta_z$, implying isentropic thicknesses smaller (larger) than in the reference profile.

Moreover, Eqs. (2) and (4) imply that $\overline{M}$ can only be destroyed in the adiabatic limit through a mass circulation $\overline{\mathbf{q}}^*$ that also generates $\overline{U}$. Thus, the potential momentum can be interpreted as the zonal momentum that the flow would realize through an adiabatic redistribution of mass that made the isentropic thickness uniform poleward of the reference profile. Note that this adiabatic redistribution would also conserve the zonal-mean PV gradient:

$$\overline{q}_y = \frac{\partial}{\partial y} \overline{U} + \frac{\partial}{\partial z} \left( \frac{f_y}{\Theta_z} \overline{q}_z \right) = \frac{\partial}{\partial y} (\overline{U} + \overline{M}).$$ (11)

An advantage of the potential momentum formulation is that it presents the wave–mean flow interaction equations, (8)–(9), in a local framework. However, the inherent nonlocality of the problem has only been hidden in the definition of the mean flow: $\overline{M}(y)$ is not a local function of temperature but depends on the full thermal structure equatorward of that latitude [viz. Eq. (5)]. Thus, while the shear of $\overline{M}$ is unambiguously defined, the actual value of $\overline{M}$ depends on the boundary conditions, or the choice of reference latitude $y_0$. It is assumed here that a localized baroclinic zone $y_0 < y < y_L$ exists, outside of which the mean flow remains unmodified. As long as $y_0$ is chosen in the unperturbed region, the definition of $\overline{M}$ is insensitive because for $y < y_0$ the stratification equals the reference stratification and $\overline{q}_y = 0$.

By construction, $\overline{M}(y_0) = 0$. Moreover, if the reference stratification $\Theta_0(z)$ is properly defined, $\overline{q}_y$ has no horizontal mean and Eq. (5) integrates to zero, implying that $\overline{M}(y_L) = 0$ as well. Taking this into account, it is easy to see (refer to Fig. 1) that easterly (westerly) values of $\overline{M}$ are associated with isentropes that open up (close down) with latitude. The former is the case in the extratropical troposphere in which the isentropic slope increases with height at fixed latitude (Stone and

\[ \begin{array}{c}
A \\
\begin{array}{c}
\Theta_z \\
y_L \\
\theta_z > 0 \quad \theta_z = 0 \quad \theta_z < 0 \\
M_y < 0 \quad M_y = 0 \quad M_y > 0
\end{array}
\end{array} \]

\[ B \]

\[ \begin{array}{c}
M < 0 \\
y_0 \\
y_L
\end{array} \]

Fig. 1. Sketch illustrating the relation between potential momentum and isentropic thickness. (a) On the left side the flow is more stably stratified than the reference $\Theta(z)$, implying $\overline{q}_y > 0$ and $\overline{M}_y < 0$, while the reverse is true on the right. This distribution gives easterly potential momentum. (b) In the presence of a surface temperature gradient the isentropic layer disappears, and there is a jump in thickness at the intersection latitude. This is equivalent to a delta-function of easterly potential momentum at that latitude.
Nemet 1996). This easterly potential momentum jet gives a negative contribution to the interior PV gradient by the baroclinic term [Eq. (11)] that tends to balance the positive contributions resulting from beta, the horizontal curvature, and Boussinesq effects (Zurita and Lindzen 2001).

It is useful to regard the lower boundary as an isothermal surface (Bretherton 1966), which requires the inclusion of a delta-function interior PV gradient right above it. In the potential momentum framework, this delta-function PV gradient results from the contribution to the total PV gradient and may be obtained as a generalization of Eq. (5):

$$M_S = -\frac{f_0}{\Theta_x} \int_0^y n(y')\delta(z) dy',$$  \hspace{1cm} (12)

where, consistent with the QG framework, the mean surface temperature has been subtracted from $\Theta$. Thus, Eq. (12) also integrates to zero and $\overline{M}_S$ vanishes at both endpoints. When surface temperature decreases with latitude, as observed, Eq. (12) also produces an easterly potential momentum and a negative delta-function contribution to $\overline{M}_y$ [Eq. (11)]. Physically, this can also be interpreted in terms of the isentropes opening up with latitude, provided that we regard the disappearance of the isentrope $\theta_i$ equatorward of the latitude $y_i$ as a jump in thickness between 0 and its interior value at that latitude (see Fig. 1b).

The characterization of the surface temperature gradient as a potential momentum reservoir provides a link with conceptual models of the circulation based on isentropic averages in which the eddy heat flux acts as a momentum source for the column via the form drag on isentropes that intersect the surface (Schneider 2005). Similarly, in the TEM framework the surface is characterized as a region of EP divergence and, thus, as a momentum source. Equation (12) allows one to quantify this source. In the initial value problem, $\overline{M}_S(t_0)$ gives an estimate of the maximum source, which would be obtained by fully depleting the surface potential momentum; that is, $\Delta\overline{M}_S = -\overline{M}_S(t_0)$. In the forced–dissipative equilibrium, Eq. (4) shows that the time-mean source is a function of the potential momentum depletion at the surface: $\Delta\overline{M}_S = \overline{M}_S - \overline{M}_{SR}$.

It is important to note that this source is a nonlocal function of $\overline{\Theta}$, more closely related to the standard deviation of surface temperature than to its actual gradient. As an example, it is instructive to consider the inviscid life cycle simulations of Simmons and Hoskins (1978). In these simulations, the surface temperature gradient is eliminated over the original baroclinic zone, but large temperature gradients appear on the sides. Figure 2a shows an idealized representation of their initial and final states with a dashed and solid line, respectively. For this example the net temperature jump across the baroclinic zone is unchanged and the maximum temperature gradient actually increases. Yet Fig.
2b shows that the surface potential momentum is nearly depleted, as the standard deviation of surface temperature is reduced over the baroclinic zone. This depletion in $\bar{M}_S$ results from the net poleward transport during the life cycle [cf. Eq. (4)]:

$$\Delta \bar{M}_S = - \int_0^t \int f \bar{a} \delta(t) \, dt' .$$  \hspace{1cm} (13)

The previous example suggests that potential momentum might offer a fuller view of the wave activity source than more traditional, local measures of the baroclinicity, like the maximum temperature gradient. On the negative side, the nonlocal character of $\bar{M}_S$ also introduces some ambiguity. To see this, consider an idealized surface temperature distribution consisting of a baroclinic zone with width $2 \times a$, and constant temperature gradient $\bar{\theta}_h < 0$ (dashed line in Fig. 2a). Taking as reference endpoints $y_0 = -b, \ y_L = b$, with $b > a$, the surface potential momentum at the channel center can be shown to be sensitive to the choice of $b$:

$$\bar{M}_{S_{y=0}} = - \int_0^a \left( \frac{a}{2} - b \right) a \bar{\theta}_h \delta(z) .$$

Mathematically, the sensitivity arises because the temperature at large positive and negative $y$ is different from its mean value of zero. This sensitivity is not altogether undesirable since, physically, the magnitude of the wave source should depend on the size of the meridional domain over which the thermal field is rearranged, which is essentially what $b$ represents. As discussed above, $y_0 = -b$ should always be chosen within the region of unmodified mean flow to ensure that $\bar{\theta}(y_0) = 0$. It is easy to show that, as long as this is true, the change in potential momentum $\Delta \bar{M}_S = \bar{M}_S - \bar{M}_S(t_0)$ is independent of the choice of endpoints, even though each of these terms independently is a function of $b$. In other words, changing $b$ does not affect the actual wave activity source, only the theoretical maximum $\bar{M}_S(t_0)$. As $b$ increases, the source represents a smaller fraction of this theoretical maximum because the fluid rearrangement involves a narrower latitude range than anticipated.

Consider the vertical integral of $\bar{M}$. Using Eqs. (5), (12), and including the surface delta contribution in the integral we obtain

$$\int_0^a \bar{M} \, dz = - \int_0^z \frac{f_a}{\bar{\theta}_h} \int_0^y \bar{\theta}_a \, dy = \int_0^z \bar{M}_R \, dz .$$ \hspace{1cm} (14)

In a semi-infinite domain, $\bar{\theta}_a$ would be the temperature at infinity, which cannot be changed by the dynamics (assuming that the eddies extend over a finite mixing depth). Alternatively, in the presence of an upper rigid lid $\bar{\theta}_a = 0$, as the temperature of the top boundary is constant (the mean can again be subtracted) when the boundary temperature gradient is included as an interior delta-function PV gradient. In both cases, Eq. (14) implies that the vertically integrated $\bar{M}$ cannot be changed and agrees, in particular, with its radiative equilibrium value. This is the essence of the mixing depth constraint discussed by Zurita-Gotor and Lindzen (2004a). When the horizontal curvature of the zonal wind is negligible, it directly translates through Eq. (11) into a constraint on the vertically integrated PV gradient. Physically, the reason for this constraint is that the sum of the isentropic thicknesses of all layers must add up to a constant because we assumed that the mean potential temperature is fixed at two reference heights. As a result, the eddies and diabatic processes can only redistribute the mass between the different isentropic layers.

We can interpret the generation and destruction of $\bar{M}$ [Eq. (4)] using the isentropic framework. A poleward circulation (Fig. 3A) transfers mass from low to high latitudes along the isentropes, generating easterly potential momentum in the interior. This is accompanied by an equal generation of westerly $\bar{U}$ so that $\bar{M} + \bar{U}$ and $\bar{\theta}_a$ are conserved (the total momentum only changes if there is also a wave–mean flow interaction, i.e., a nonzero $\nabla \cdot \bar{F}$). The delta-function return flow of this circulation generates westerly potential momentum at the surface by reducing the surface temperature gradient/depleting the surface potential momentum reservoir. This $\bar{M}$ tendency is balanced in equilibrium by diabatic restoration. The mechanism is illustrated in Fig. 3B for a stably stratified fluid ($\theta_2 > \theta_1$): low-latitude heating transfers mass from $\theta_2$ to $\theta_1$, while the reverse is true for high-latitude cooling.

For a basic state like Charney’s, the interior potential momentum is zero because the isentropic thickness is independent of latitude and all the potential momentum is locked in the surface reservoir. For Eady’s basic state the interior potential momentum is also zero, and there is equal and opposite easterly (westerly) potential momentum over the lower (upper) boundary. As the flow equilibrates, the residual circulation redistributes vertically the easterly potential momentum initially locked at the surface, generating easterly poten-
Hence, we can relate locally in equilibrium the net eddy absorption $\nabla \cdot F$ to the restoration of the mean flow, both mechanical and thermal:
\[
\alpha_M \overline{U} + \alpha_T (\overline{M} - \overline{M}_R) = \overline{\nu' q'} = \nabla \cdot F. \quad (16)
\]

Moreover, there is also a one-to-one correspondence between the mechanical forcing $\alpha_M \overline{U}$ and meridional EP convergence $\partial_z F_z = -\partial_z (\overline{u' v'})$, on one hand, and between the thermal forcing $\alpha_T (\overline{M} - \overline{M}_R)$ and vertical EP convergence $\partial_z F_z = f_0 / \Theta_z \partial_z (\overline{\theta' \overline{\theta'}})$ on the other. The former relation is well known and can be obtained by integrating Eq. (15) vertically:
\[
\frac{\partial}{\partial t} \int \overline{U} \, dz + \alpha_M \overline{H U_S} = \int \overline{\nu' q'} \, dz = - \int \frac{\partial}{\partial y} \overline{u' v'} \, dz = \gamma', \quad (17)
\]
where $U_S$ is the surface wind, $H$ the depth of the boundary layer, and we took Eq. (14) into account. In equilibrium, Eq. (17) relates the (vertically integrated) meridional absorption $\gamma'$ to the net frictional restoring of the mean flow along the column.

An analogous relation for $\overline{M}$ can be obtained through the meridional integration\(^3\) of Eq. (15):
\[
\frac{\partial}{\partial t} \int \overline{M} \, dy + \alpha_T \int (\overline{M} - \overline{M}_R) \, dy = \int \overline{\nu' q'} \, dy = \int \frac{\partial}{\partial z} \left( \frac{f_0}{\Theta_z} \overline{\theta' \overline{\theta'}} \right) \, dy = \zeta. \quad (18)
\]

The similarity with the previous equation is another manifestation of the isomorphism between barotropic and baroclinic dynamics or, equivalently, between meridional and vertical wave propagation (e.g., Lindzen 1988). Equation (18) relates the (meridionally integrated) vertical absorption $\zeta$ to the diabatic forcing of potential momentum. In equilibrium:
\[
\gamma' = \int \alpha_M \overline{U} \, dz = \alpha_M \overline{H U_S}, \quad \zeta = \int \alpha_T (\overline{M} - \overline{M}_R) \, dy. \quad (19)
\]

Because the meridional convergence of $\overline{u' v'}$ (the vertical convergence of $\overline{\theta' \overline{\theta'}}$) is typically one-signed in the vertical (in the horizontal), there is little cancellation in

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\(^3\)To obtain this relation, one needs to assume that the surface wind $U_S$ integrates meridionally to zero. Then, the meridionally integrated $\overline{U}$ must be independent of time at each level due to our assumption of a localized baroclinic zone (since temperature is unchanged at large positive and negative $y$, the meridionally averaged vertical shear is also independent of time). Even when initially nonzero, the meridionally integrated $\int \overline{U} \, dy$ is quickly damped to zero with the frictional time scale $\alpha_M$.\(^3\)
these terms when integrating Eqs. (17) and (18); \( \gamma \) and \( Z \) thus provide a robust measure of the net meridional and vertical components of the Eliassen–Palm divergence. Note also that \( \gamma' - \partial_t (\bar{v}' \bar{\theta}') \) integrates meridionally to zero, whereas \( Z - \partial_z (\bar{v} \bar{\theta}) \) does so vertically. In most situations \( Z \) only changes sign once in \( z \), whereas \( \gamma \) may change sign twice in \( y \). We can thus interpret the net meridional/vertical divergence (i.e., the integral of these terms over the regions where they are positive) as a characteristic maximum meridional/vertical EP flux:

\[
Y = \int \gamma(Y) \, dy \sim \bar{u} \bar{v} \max;
\]
\[
Z = \int Z \gamma(Z) \, dz \sim \frac{f_0}{\Theta_z} \bar{v} \bar{\theta} \max,
\]

where \( \gamma \) is the Heavyside function. We shall refer in the following to \( Y \) and \( Z \) as the meridional/vertical propagation, based on the WKB relation between the EP flux and wave propagation (Edmon et al. 1980). Equation (19) then suggests that meridional and vertical propagation should be enhanced with faster mechanical and thermal forcing, respectively.

d. Global circulation budget

We next turn our attention to the problem posed in the introduction of constructing a wave activity budget of the circulation. Locally, this budget is very simple:

\[
- D_{\text{mean}} = \nabla \cdot \mathbf{F} = D_{\text{eddy}},
\]

which supports a momentum cycle consisting of just three steps: 1) the forcing of the mean momentum \( \mathbf{M} + \mathbf{U} \) by heating and friction, 2) the conversion of this mean momentum into wave activity through the wave–mean flow interaction term \( \nabla \cdot \mathbf{F} = \bar{v}' \bar{\theta}' \), and 3) the nonconservative dissipation of the eddies. This argument implies that, in a time-mean sense, \( D_{\text{mean}} \) is the ultimate source for the eddies, even though the magnitude of this source is itself largely eddy-determined due to the restoring nature of the forcing.

However, a global integral of \( \bar{A}, \mathbf{M} + \mathbf{U} \), or the above conversion terms is not very meaningful because neither of these terms is sign-definite. In fact, Bretherton (1966) showed that \( \bar{v} \bar{q}' \) always integrates globally to zero, and Eq. (21) implies that so must \( D_{\text{mean}} \) and \( D_{\text{eddy}} \) in equilibrium. Hence, all terms in Eq. (21) trivially vanish when integrating globally. To avoid this problem, we define the global circulation \( C \) as the integral of these terms over the regions where they are positive (or, equivalently, minus their integral over the regions where they are negative, or half the integral of their absolute value):

\[
C = \frac{1}{2} \int |D_{\text{mean}}| \, dy \, dz.
\]

This is not just a mathematical convenience but is also physically meaningful, as one can associate the regions with positive/negative \( \nabla \cdot \mathbf{F} \) with the sources and sinks of wave activity. In other words, \( C \) can be regarded as the net wave source/sink. Assuming that the time-mean PV fluxes are downgradient, the wave source (sink) is associated with regions of negative (positive) PV gradient. Equation (21) then implies that \( D_{\text{mean}} \) is negative over the former and positive over the latter, while the reverse is true for \( D_{\text{eddy}} \). In practice, this separation is established between the surface and the interior troposphere (Held 1999), so that \( D_{\text{mean}} \) is negative (positive) at the surface (in the interior). This is a generalization of the barotropic result that the mean flow acceleration must be westerly over regions with negative PV gradient (Zurita-Gotor and Lindzen 2004a). In the 3D case this result still holds, but the westerly mean flow acceleration at the surface results from the depletion of the surface reservoir of easterly potential momentum, rather than from an actual \( \mathbf{U} \) acceleration.

At the surface \( \mathbf{q}_y < 0 \), so that \( \bar{v} \bar{q}' > 0 \), \( D_{\text{mean}} < 0 \) and \( D_{\text{eddy}} > 0 \). The easterly forcing of the mean flow is associated with the diabatic restoration of the easterly potential momentum, while the westerly eddy forcing is associated with the diabatic damping of the thermal anomalies. In the interior, there is everywhere a westerly thermal forcing of the mean, and there is also over the boundary layer a westerly (easterly) mechanical forcing that adds to (subtracts from) \( D_{\text{mean}} \) wherever the surface winds are easterly (westerly). For the eddies, small-scale dissipation is also required to absorb the enstrophy cascade. The different processes may have varying impact over different regions, but as long as \( \mathbf{q}_y > 0 \) and \( \bar{v} \bar{q}' < 0 \) they should all act to make \( D_{\text{mean}} > 0 \) and \( D_{\text{eddy}} < 0 \) in the interior.

d. Global circulation budget

3. Two-layer diagnostics

a. Model description and formulation

The framework put forward in the preceding section emphasizes the relation between wave propagation/wave–mean flow interaction and the forcing of the mean flow for an equilibrated system. In this section we illustrate these concepts for the simple two-layer QG model. We also investigate, using the same model, the relation between wave propagation and the structure/strength of the forcing by changing the diabatic and mechanical forcing time scales. As discussed above, the former is associated with vertical propagation, and the
latter with meridional propagation. Some of the questions that we ask are, is meridional propagation enhanced over vertical propagation when the mechanical forcing is enhanced (and vice versa); does the enhancement in either time scale always lead to a stronger net circulation; does surface friction enhance or weaken vertical propagation?

The model equations that we solve are standard:

\[
\frac{\partial q_n}{\partial t} = -J(\psi_n, q_n) - \alpha_T(-1)^n \frac{\psi_1 - \psi_2 - \psi_R}{\lambda_R^2} - \delta_n \alpha_M \nabla^2 \psi_n - \nu \nabla^4 \psi_n, \tag{23}
\]

where \( q_n = \nabla^2 \psi_n + (-1)^n(\psi_1 - \psi_2)/\lambda_R^2 + \beta y \) stands for the potential vorticity in the upper \((n = 1)\) and lower \((n = 2)\) layers, and \( \psi_n \) is the corresponding streamfunction. Here \( \lambda_R \) is the Rossby radius based on the layer depth \( H \), and \( \alpha_T^2 \) and \( \alpha_M^2 \) are the diabatic and frictional forcing time scales (friction only applies to the lower layer). We also include fourth-order hyperdiffusion to get rid of the small scales. Temperature is proportional to \( \psi_n \) and the flow is forced to a radiative equilibrium profile \( \Theta_R \) defined by the following thermal wind:

\[
- \frac{\partial \psi_R}{\partial y} = U_{\max} \exp(-y^2/\sigma^2). \tag{24}
\]

Most of the runs described below are characterized by the following parameters: \( U_{\max} = 40 \) m s\(^{-1}\), \( \sigma = 2500 \) km, \( \beta = 1.6 \times 10^{-11} \) m\(^{-1}\) s\(^{-1}\), and \( \lambda_R = 700 \) km. This produces a significantly supercritical radiative equilibrium \( U_{\max}/\beta \lambda_R^2 \approx 5.1 \). The diffusion coefficient is \( \nu = 5 \times 10^5 \) m\(^2\) s\(^{-1}\) and the control values of \( \alpha_T^2 \) and \( \alpha_M^2 \) are 20 and 3 days, respectively. In the analysis described below these time scales are varied for the zonal mean component of Eqs. (23), but kept constant for the eddy component. Thus, only the mean flow forcing but not the eddy damping is changed, which reduces the ambiguity and allows one to explore the sensitivity to the forcing time scales over a wider parameter range.

The domain is meridionally bounded by rigid walls, where standard boundary conditions are formulated \((\psi_k = \psi_m = 0)\). The channel length is \( 32 \times 10^3 \) km and the domain width \( 48 \times 10^3 \) km. The choice of such a large domain ensures an unmodified mean flow near the walls: in particular, the temperature jump across the domain remains unchanged, with the exception noted below. This makes the potential momentum diagnostics insensitive to the location of these meridional boundaries (see section 2b). We use a spectral transform method with 60 waves zonally and 120 meridionally: with a standard \( 2/3 \) truncation, this gives an effective resolution of 400 km in \( x \) and 300 km in \( y \).

The potential momentum is defined for this model integrating Eq. (5) over both layers:

\[
\mathcal{M}_n = -(-1)^n \int_0^\pi \frac{\psi_1 - \psi_2}{\lambda_R^2} \, dy. \tag{25}
\]

Thus, potential momentum is simply minus/plus the meridional integral of the midlevel temperature. The different sign in each layer results from setting \( \theta = 0 \) at the level above or below (the upper or lower rigid lid), consistent with the discussion following Eq. (12). A profile in which temperature decreases with \( y \) implies an easterly potential momentum jet in the lower layer and an equal westerly potential momentum jet in the upper layer. As a result, \( \mathcal{M}_n \) sums up to zero over both layers at all latitudes, consistent with the discussion following Eq. (14). For brevity, we will identify hereafter \( \mathcal{M} \) (without subscript) with the lower-layer potential momentum \( \mathcal{M}_2 \), with the understanding that the upper layer \( \mathcal{M} \) is equal and opposite. The same is done for the radiative equilibrium profile \( \mathcal{M}_R \).

On the other hand, the wave–mean flow interaction term \( \nabla \cdot \mathbf{F} \) and mean-flow forcing \( D_{\text{mean}} \) [Eq. (8)] are in discretized form:

\[
\nabla \cdot \mathbf{F}_n = \psi_n \delta/u_n = -\frac{\partial}{\partial y} \left( u_n \psi_n \right) + (-1)^n \frac{f_0}{\mathcal{R}_R^2} \nabla^2 \psi, \tag{26}
\]

\[
D_{\text{mean},n} = -\alpha_T(-1)^n(\mathcal{M} - \mathcal{M}_R) - \delta_n \alpha_M \mathcal{U}_2. \tag{27}
\]

In the lower layer, \( \mathcal{M}_R < \mathcal{M} < 0 \) in general, as the easterly potential momentum is depleted from its radiative equilibrium value. The diabatic restoration then produces a negative (easterly) contribution to \( D_{\text{mean}} \). Additionally, there is a frictional contribution to \( D_{\text{mean}} \). Since in general \( \mathcal{U}_2 > 0 \) over the baroclinic zone, this frictional term is typically easterly there as well and reinforces the diabatic forcing. In the upper layer, the only contribution to \( D_{\text{mean}} \) is the westerly thermal forcing. In the following, we will also drop the subscript in Eq. (27) and refer to the lower-layer forcing of the mean flow as

\[
D_{\text{mean}} = -\alpha_T(\mathcal{M} - \mathcal{M}_R) - \alpha_M \mathcal{U}_2. \tag{28}
\]

As discussed in section 2d, we quantify the circulation \( c \) by integrating \( D_{\text{mean}} \) over the region where it is one-signed [Eq. (22)]: \( D_{\text{mean}} \) is westerly over the upper layer and dominantly easterly over the lower layer. Assuming that \( D_{\text{mean}} \) does not change sign over the lower layer (Fig. 8 shows that this is a good approximation), one can then estimate the circulation by integrating the above equation over the lower layer. Note that the frictional term integrates meridionally to zero with no net contribution to \( c \). Its only effect is to concentrate the
wave activity source meridionally over the region of surface westerlies.

b. Potential momentum diagnostics

Before examining the forced–dissipative case, it is illustrative to consider the equilibration of the inviscid problem, when $a_M = a_T = 0$ both for the eddies and for the zonal mean (hyperdiffusion is still included). This may be regarded as the two-layer quasigeostrophic equivalent of the inviscid life cycle of Simmons and Hoskins (1978). Figure 4 shows a time series from this run, for the zonal mean temperature (top), lower-level wind (middle) and potential momentum change from the initial radiative equilibrium state (bottom). As in the classical life cycle, the flow initially equilibrates through the barotropic governor mechanism of James (1987). Between model days 60 and 70 a strong barotropic westerly jet develops over the original baroclinic zone with easterlies appearing on the sides. The temperature gradient is eliminated at the center of the

Fig. 4. For the inviscid life cycle, time series of (top) midlevel temperature, contour interval (c.i.) = 5 K, (middle) lower-layer wind, c.i. = 10 m s$^{-1}$, and (bottom) potential momentum change $M - M_R$, c.i. = 100 m s$^{-1}$, as a function of latitude. Also shown on the right are the initial (dashed) and final (solid, averaged over last 20 days) profiles. Latitudes are nondimensionalized with the Rossby radius $\lambda_R$, and only part of the domain is shown.
channel, but enhanced temperature gradients appear laterally, over the region of low-level easterlies. As time progresses, the central region with homogenized temperature broadens, the regions with easterlies move outward, and so do the collocated baroclinic zones, which also sharpen. Around day 70, the eddy momentum fluxes reverse, suggesting a saturation of the meridional absorption and reflection of the wave activity back into the central region. Anomalous negative heat flux follows (not shown) with low-level easterlies appearing at the center of the channel between days 75 and 80. However, this situation is quickly reversed and low-level westerlies reappear at the center of the channel after day 80. At the same time, weaker baroclinic development is observed over the outer baroclinic zones, most notably over the southern flank. As a result of this secondary, off-center life cycle the temperature gradient is restored again at the center of the channel. However, the low-level westerlies persist and strengthen over that region because the bulk of the wave propagation is still outward. Eventually, a steady state is reached (see right panels of Fig. 4) with a central temperature gradient that is almost as strong as in the initial state, but is also accompanied by a strong barotropic jet. This is what stabilizes the flow. [Figure 6 (thick, solid) the low-level PV flux is larger and more source in the lower layer: in the presence of friction propagation also implies a stronger and more localized eddy panel of Fig. 6 shows, this enhanced meridional propagation also implies a stronger and more localized eddy fluxes are enhanced with friction. As the lower-right main effect of surface friction is to broaden the region with depleted potential momentum, which is not surprising given the connection between this depletion and the net Lagrangian mass transport [cf. Eq. (13)]. The previous example demonstrates the importance of barotropic processes for the inviscid life cycle. We have also examined the evolution of the same flow when surface friction with time scale 1 day is included (with no thermal forcing). The results support the notion that the flow is more efficient in reducing the midchannel baroclinicity when surface friction limits the creation of meridional shear and prevents the stabilization by the barotropic governor. The equilibrium PV gradient (shown with thick, solid line in Fig. 6) is very similar to the previous case (dashed) and only marginally negative. But without the positive PV gradient contribution by the horizontal jet curvature, this occurs through a large reduction in the midchannel temperature gradient, in contrast with the recovery of this gradient noted in the inviscid case. However, the initial development is very similar to the previous run (see Fig. 5), and this state of affairs is only reached after a complex evolution, which will not be described here for brevity.

In terms of potential momentum the initial evolution also resembles the inviscid simulation, up to day 90 or so. At that time the potential momentum depletion stabilizes in the inviscid run, but further depletion is observed in the presence of friction. The lower panel of Fig. 5 shows that this is also accompanied by the meridional expansion of the depleted region. This is consistent with the temperature field shown on the top panel. The depletion beyond day 90 is associated with the slow warming and cooling of the outer regions (note the profile on the right), which results in further reduction of the standard deviation of temperature. This process proceeds until the temperature at the meridional walls is also modified (not shown), beyond which point the concept of potential momentum becomes ambiguous.

The comparison between both runs suggests that the main effect of surface friction is to broaden the region with modified flow. This is consistent with the arguments of section 2c, which imply that the meridional EP fluxes are enhanced with friction. As the lower-right panel of Fig. 6 shows, this enhanced meridional propagation also implies a stronger and more localized eddy source in the lower layer: in the presence of friction (thick, solid) the low-level PV flux is larger and more concentrated than in the inviscid case (dashed). Finally,

4 Note that the domain shown in the figure is not the full computational domain.
the potential momentum depletion is also enhanced with friction, which is at odds with the speculation (Robinson 2000; Zurita-Gotor and Lindzen 2004b) that surface friction might restore the low-level baroclinicity. We will return to this point in section 3c.

We next shift our attention to the forced–dissipative problem. All results described below are taken from runs integrated for 3000 days starting from radiative equilibrium. The statistics were calculated after discarding the first 500 days of spinup. For reference, Fig. 7 describes the mean state for the control run ($\alpha_f = 20$ days and $\alpha_M = 3$ days, both for the eddies and the mean). The generalized momentum balance [cf. Eq. (21)] for the same run is shown in Fig. 8. The upper panels show the partition of $\nabla \cdot \mathbf{F}$ in both layers. While in the lower layer the eddy momentum flux is negligible, in the upper layer the meridional absorption $\bar{\xi}_y F_y$ is smaller than the vertical absorption $\bar{\xi}_z F_z$ but still significant. This results in a broader PV flux in the upper than in the lower layer (e.g., Held 2000). The medium panels show that the mean flow forcing $D_{\text{mean}}$ balances the EP-flux convergence $\nabla \cdot \mathbf{F}$ locally to a very good approximation. In the upper layer, the only mean flow forcing is the diabatic restoration of westerly po-

\begin{figure}[h]
  \centering
  \includegraphics[width=\textwidth]{example.png}
  \caption{As in Fig. 4 but for the run with surface friction $\alpha_f = 1$ day, and using a different contour interval, c.i. = 5 m s$^{-1}$, for the middle panel.}
\end{figure}
potential momentum. An equal easterly potential momentum forcing is observed in the lower layer, where there is also a frictional contribution to $D_{\text{mean}}$, easterly over the central region of low-level westerlies and westerly over the lateral regions of low-level easterlies. Its net effect is therefore to concentrate the easterly $D_{\text{mean}}$ at the center of the channel, in agreement with the narrower EP source seen in the lower layer. Consistent with the dominance of $\partial_z F_z$ over $\partial_y F_y$ in the upper panel, the diabatic restoration dominates over the frictional forcing (see section 2c). Note that because $\nabla \cdot F$ and $D_{\text{mean}}$ are very nearly one-signed over each layer, their meridional integral is a good estimate of the net circulation $C$ [cf. Eq. (22)]. Finally, the lower panel shows the partition of $D_{\text{eddy}}$. Since in this case there is an unresolved contribution to $D_S$ by numerical diffusion, we simply estimated $D_S$ as a residual assuming the balance in Eq. (21) to hold. The resulting term, shown dash–dotted, includes both the resolved (from $\nu$) and unresolved (numerical) dissipation. Here $D_S$ was found to be the dominant term in $D_{\text{eddy}}$ in both layers, which is not surprising owing to the tendency of enstrophy to

---

**Fig. 6.** (left) Time series of meridional temperature gradient in (top) K $(1000 \text{ km})^{-1}$ and (bottom) low-level PV gradient, normalized by beta at the center of the channel. (top right) Meridional equilibrium profile (last 20 days average) of normalized PV gradient. (bottom right) Time-integrated PV flux during the life cycle. In all panels the dashed line corresponds to the inviscid simulation (Fig. 4) and the thick, solid line corresponds to the simulation with surface friction $\alpha_M = 1$ day (Fig. 5).

---

**Fig. 7.** For the control run ($\alpha_y^{-1} = 20 \text{ days}, \alpha_M^{-1} = 3 \text{ days}$), time-mean (left) zonal wind and (right) PV gradient, normalized by $\beta$. The broken lines in both plots show the radiative equilibrium distributions.
cascade to small scales. With enhanced resolution the resolved diffusion takes up a larger fraction of the required $D_S$, but is still smaller than needed (not shown). Besides the resolution limitation, this could also be due to errors in the linearization of pseudomomentum, the contribution of nonlinear advection to the eddy enstrophy balance and the unsteadiness of the mean PV field. Due to these large numerical errors, it would be impractical to construct a wave activity budget of the circulation based on $\vec{A}$. The concept of potential momentum allows us to construct such a balance based on the mean flow.

c. Sensitivity to diabatic and frictional forcing

To better understand what controls the net meridional and vertical propagation in the equilibrated system, and how this propagation is related to the potential momentum depletion and global circulation, we

![Graphs showing components of the time mean: (top) EP flux convergence $\nabla \cdot F$ (solid = vertical, dashed = meridional, thick solid = total), (middle) mean-flow forcing $-D_{\text{mean}}$ (solid = diabatic, dashed = frictional, thick solid = total), and (bottom) eddy dissipation $D_{\text{eddy}}$ (solid = thermal, dashed = mechanical, dash–dotted = small scale, thick solid = total).]
have performed a series of runs varying the diabatic and frictional time scales from their control run values. In contrast with most previous studies (e.g., Stone and Branscome 1992), the perturbed values of $\alpha_T$ and $\alpha_M$ are only used for the zonal mean component of the flow, whereas for the eddy component the control values are still used. A similar device has been used, for instance, by Robinson (1997). This allows one to consider extreme values of the forcing time scales ($\alpha_T^{-1}$ and $\alpha_M^{-1}$ are as small as 0.01 day in some runs) without directly affecting the eddies. It also eliminates the unavoidable ambiguity when the forcing and the damping are simultaneously changed, which may lead to a non-monotonic dependence on the forcing time scale (G. Chen 2005, personal communication).

As discussed in section 2c, one may estimate the horizontal and vertical propagation by integrating the source terms $\mathcal{Y}$ and $\mathcal{Z}$ [cf. Eqs. (17), (18)] over the regions where they are positive. In terms of the mean-flow forcing:

$$ Y = \alpha_M \int \bar{U}_2 \mathcal{Y}(\bar{U}_2) \, dy; \quad Z = \alpha_T \int (\bar{M} - \bar{M}_R) \, dy, $$

where it was taken into account that the meridionally integrated $\bar{M} - \bar{M}_R$ is positive over the lower layer and negative over the upper layer. As discussed in section 2d, this positive sign reflects the depletion of the lower-level easterly potential momentum from radiative equilibrium, as required to balance the downgradient PV fluxes. Also assuming that $\bar{M} - \bar{M}_R$ is one-signed over the lower layer (see Fig. 8), we can approximate the net circulation as

$$ C = - \int D_{\text{mean},2} \, dy = Z. $$

We first consider the effect of varying the diabatic time scale $\alpha_T^{-1}$ from 0.01 to 80 days. The results are summarized in Fig. 9. A priori, we expect the flow to move further away from radiative equilibrium as the forcing weakens, which would imply enhanced potential momentum depletion $\bar{M} - \bar{M}_R$. On the other hand, we also expect the net diabatic restoration (or vertical propagation) $\alpha_T (\bar{M} - \bar{M}_R)$ to decrease with weaker $\alpha_T$. Figure 9a shows that both are observed: as $\alpha_T$ decreases the potential momentum depletion increases, but the net vertical propagation $Z$ drops. This implies that $Z \sim \alpha_T^0$, with $0 \leq n \leq 1$. In the weakly forced limit $\alpha_T^{-1} \geq 10$ days the best fit is $Z \sim \alpha_T^{0.66}$ with an RMS error under 0.1%.

Although the midchannel baroclinicity decreases somewhat (not shown), the main reason why potential momentum is more depleted with larger $\alpha_T^{-1}$ is that the adjusted region broadens (Fig. 10a). On the other hand, the lower-layer wave activity source becomes more concentrated with weaker diabatic forcing (Fig. 10b), though this is somewhat less obvious. Both observations are consistent with what we found in the unforced problem when friction was added, suggesting that meridional propagation might also be important in this case. The net meridional propagation $Y$, displayed in Fig. 9b, is roughly an order of magnitude smaller than $Z$. As $\alpha_T^{-1}$ increases, $Y$ decreases, which might seem at first to contradict the above argument. However, this is only because $Z$ and the net circulation $C$ decrease: what is more relevant is that $Y$ represents a larger fraction of the circulation as the diabatic forcing weakens (Fig. 9c). This supports our conjecture that the ratio between meridional and vertical propagation depends on the structure of the forcing, as measured by $\alpha_T/\alpha_M$.

The local balance Eq. (21) relates the mean flow adjustment to the net wave–mean flow interaction, but some additional constraint is needed for closure. In a similar model, Stone and Bransome (1992) found a remarkable robustness of the midchannel supercriticality $\xi = (U_1 - U_2)/\beta \lambda_R^2$ against changes in $\alpha_T$ and $\alpha_M$. Based on this observation, they conjectured a weak baroclinic adjustment hypothesis: essentially, the flow adjusts to some fixed supercriticality $\xi \approx 2.4$. Interestingly, our control run is very close to that supercriticality, which may also be expressed as $\bar{q}_{v1} / \beta \approx -1.4$ when the PV gradient contribution by the meridional curvature of $\bar{U}_2$ is negligible. However, we have found that $\bar{q}_{v1} / \beta$ is much more robust against changes in the forcing than $\xi$ itself, particularly when $\alpha_M$ is reduced, and this curvature becomes significant. Although this modified version of Stone’s weak baroclinic adjustment fits well the control run and the differences in $\bar{q}_{v1}$ are moderate in the weakly forced limit $\alpha_T^{-1} \geq 10$ days, Fig. 9e shows that the lower-layer PV gradient can change significantly with stronger forcing. Stone and Bransome (1992) could not reach this limit because such large values of $\alpha_T$ stabilize the flow when they also apply to the eddies.

To the extent that Newtonian forcing can be considered realistic, the same diabatic time scale should apply
to the mean and the eddies. Thus, Stone and Branscome’s weak baroclinic adjustment hypothesis (and other constraints alike, e.g., Schneider 2005) might still be relevant, as long as the eddies are only unstable in the weakly forced limit. However, it is still illustrative for the purpose of understanding the circulation to examine further the limit of large $\mathcal{H}_{925}$. Additionally, $q_y$ also varies in the weakly forced limit—albeit slowly—and is indeed known to reach the value $q_y/\mathcal{H}_{925} \approx 0.8$ for $\alpha_r = \infty$ (see Fig. 6). On the other hand, a diffusive closure is found to work reasonably for all time scales considered, including the weakly forced limit discussed above. Figure 9d shows that

$$
\nabla_z \tilde{q}_z = -D(q_{y z}^2) \tilde{q}_{y z}
$$

(31)

fits the data well, with $D \sim |q_{y z}|^n$ and $n \approx 1.5$. This is a weaker sensitivity than predicted by the high-criticality scaling arguments of Held and Larichev (1996) and L-
Fig. 10. Meridional structure of (top) the potential momentum change and (bottom) the lower-layer source $D_{\text{mean}} = \nabla \cdot \mathbf{F}$ for the diabatic time scales indicated. The curves have been normalized to have equal area.

peyre and Held (2003) ($n = 3$), although there is some hint that the $\overline{\nu q} = f(\overline{q})$ relation is getting steeper at large $\alpha_f$, which should be closer to a local homogeneous limit. In any case, this value of $n$ still gives a quite steep (nearly third power) dependence of the lower-layer PV flux on the lower-layer PV gradient, which explains why it is so hard to change the lower-layer PV gradient. We can be more quantitative by combining this with the power law\(^7\) presented above $Z \sim \alpha_f^{0.66}$. Taking into account that $Z \sim C \sim \overline{\nu q}/q_{22}$, we get a very weak dependence $q_{22} \sim \alpha_f^{0.26}$, which implies that $q_{22}$ only changes by a 20% when $\alpha_f$ is doubled.

We next discuss the effect of changing the mechanical forcing time scale $\alpha_f^{-1}$. In this case, values between 0.01 and 25 days are considered.\(^8\) The results are summarized in Fig. 11 and are for the most part consistent with those discussed above. Most notably, we find again that, as the forcing time scale $\alpha_f^{-1}$ increases, the mean westerlies increase but the net meridional propagation $Y$ drops (i.e., meridional propagation is enhanced with stronger friction). This implies that $Y \sim \alpha_f^{0.4}$, with $0 \leq n \leq 1$. There is also a hint that this relation approaches a power law in the weak friction limit, although this has not been explored fully. In general, the net vertical propagation $Z$ and the circulation $C$ also decrease as friction weakens (Fig. 11b), albeit by a much smaller margin than when changing $\alpha_f$. Not surprisingly, the meridional to vertical propagation ratio $Y/Z$ increases as friction is enhanced (Fig. 11c), consistent with the changes in meridional structure described in Fig. 12.

The results in Fig. 11e show that the lower-layer PV gradient varies much less in this case than when changing $\alpha_f$. This implies that the weak baroclinic adjustment hypothesis—in the form of a fixed PV gradient, rather than a fixed criticality—works even better than before. The observed variability in $q_{22}$ is, in fact, too small to infer anything about diffusive closures (Fig. 11d). The main reason for this robust adjustment is that the net circulation $C \sim Z \sim \overline{\nu q}$ varies very little with friction (cf. to Fig. 9a), which translates via the diffusive closure into an even weaker $q_{22}$ dependence. Friction cannot affect the global circulation $C$ much because $Y/Z$ is always small, even for the largest $\alpha_f$ considered. This has important implications for the sensitivity of thermal homogenization on surface friction, which contradict the conjecture put forward by Zurita-Gotor and Lindzen (2004b) that friction might restore the lower troposphere temperature gradient. Because the PV gradient cannot change much, there must be a compensation between $\overline{M}$ and $\overline{U}$ [cf. Eq. (11)] as $\alpha_f$ changes (Fig. 11f). With weaker friction, the stronger westerly $\overline{U}$ must also be accompanied by a stronger easterly $\overline{M}$, which implies weaker potential momentum depletion from radiative equilibrium (similar to what we found for the unforced case in Fig. 6). Note that with varying thermal forcing there is also a tendency for changes in $\overline{M}$ and $\overline{U}$ to compensate (Fig. 9f), but with a different slope. In other words, the PV gradient changes are not as large as they would be if only $\overline{M}$ changed, but the $\overline{U}$ changes do not fully compensate the $\overline{M}$ changes either. The PV gradient can change in that case because the circulation changes.

4. Concluding remarks

In this paper we have introduced the concept of potential momentum and derived a conceptual framework for understanding the circulation of a forced–dissipative baroclinic system in terms of this generalized momentum. By eliminating the redistribution effect by the residual circulation, a local conservation equation for the mean flow has been derived in which the EP flux convergence is the only forcing term. This complements the TEM formulation by providing a local relation in equilibrium between the wave–mean flow interaction and the restoring of the mean flow.

A local framework is only possible because the concept of potential momentum is based on a nondimensional redefinition of the mean flow. This introduces some ambiguity through the boundary conditions, but also

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\(^7\) Although we lack a theory that predicts this law, we note that an exponent between 0 and 1 should be generally expected based on the arguments discussed above. Even with this upper bound a weak sensitivity of the lower-layer PV gradient on $\alpha_f$ is predicted.

\(^8\) The precise values considered are 0.01, 0.1, 1, 3, 5, 10, and 25 days.
has some advantages. Because of its global character, potential momentum provides a fuller description of the baroclinic source than more traditional local measures, such as the surface temperature gradient. In particular, potential momentum is also sensitive to changes in the width of the adjusted region. This has been illustrated with numerical simulations using a two-layer model.

We have also studied the sensitivity of the two-layer model circulation to changes in both the diabatic and frictional time scale. The main results of this analysis are: 1) vertical propagation is dominant over horizontal propagation so that the intensity of the circulation is more sensitive to changes in the diabatic time scale than it is to changes in friction; 2) the lower-layer PV gradient, which defines the Eliassen–Palm divergence via a diffusive closure, is very robust as a result of the weak dependence of the circulation on the forcing and the steepness of the empirical diffusivity; 3) this is the reason why weak baroclinic adjustment (Stone and Branscome 1992) is relevant for this model as long as the forcing does not change too much; and 4) the circulation is enhanced both with faster diabatic forcing and with faster friction for the zonal mean (albeit weakly in the latter case), implying enhanced thermal homogenization with stronger friction.

Fig. 11. As in Fig. 9 but when the frictional time scale is varied.
The potential momentum framework can also be used to relate the time-mean wave propagation to the local forcing of the mean flow. In particular, a simple relation exists between diabatic forcing and vertical propagation, analogous to the relation between surface friction and meridional propagation [Eq. (17)]. Based on these ideas, it is argued that the time-mean propagation should be controlled by the structure of the forcing, while the refractive index only adjusts to allow that. For instance, we found enhanced meridional propagation in the two-layer model with stronger friction. While this enhanced propagation is consistent with the changes in refractive index found in the time-mean state, we believe that interpreting the enhanced propagation in terms of these changes in the refractive index does not tell the whole story. What is more fundamental is that meridional propagation is enhanced because the stronger frictional forcing supports enhanced meridional EP fluxes.

To conclude, we note that the framework derived here has been restricted to the simple, QG limit. However, it is well known that the quasigeostrophic TEM formalism has an exact isentropic equivalent in terms of mass-weighted properties (Tung 1986). Although the physical concepts introduced here can be easily adapted to that framework, it still remains to be seen if the resulting wave–mean flow balances conserve the attractive simplicity of the QG case. The extension to the isentropic framework will be part of a subsequent study.

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