Tropical Cyclone Intensification from Asymmetric Convection: Energetics and Efficiency

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ABSTRACT

Prior studies of the linear response to asymmetric heating of a balanced vortex showed that the resulting intensity change could be very closely approximated by computing the purely symmetric response to the azimuthally averaged heating. The symmetric response to the purely asymmetric part of the heating was found to have a very small and most often negative impact on the intensity of the vortex. This result stands in contrast to many previous studies that used asymmetric vorticity perturbations, which suggested that purely asymmetric forcing could lead to vortex intensification.

The issue is revisited with an improved model and some new methods of analysis. The model equations have been changed to be more consistent with the anelastic approximation, but valid for a radially varying reference state. Expressions for kinetic and available potential energies are presented for both asymmetric and symmetric motions, and these are used to quantify the flow of energy from localized, asymmetric heat sources to kinetic energy of the wind field of the symmetric vortex.

Previous conclusions were based on simulations that used instantaneous temperature perturbations to represent rapid heat release in cumulus updrafts. Purely asymmetric heat sources that evolve over time and move with the local mean wind are shown to also cause vortex weakening. Weakening of the symmetric vortex is due to extraction of energy by the evolving asymmetries that undergo significant transient growth due to downgradient transport of momentum across the radial and vertical shears of the symmetric wind field. While much of this energy is returned during the axisymmetrization of the resulting potential vorticity anomalies, there is typically a net loss of energy for the symmetric vortex. Some variations on the rotation rate and duration of the heat sources can lead to intensification rather than weakening, as does a deeper (more barotropic) vertical structure of the symmetric vortex. However, it is reaffirmed that these asymmetrically forced changes are small compared to the response to the azimuthally averaged heating of an isolated heat source.

Following the work of Hack and Schubert, the efficiency of the intensification process, defined as the ratio of injected heat energy to the kinetic energy change of the symmetric vortex, is computed for vortices of different sizes and strengths. In the limit of small perturbations, the efficiency does not depend on the temporal distribution of the heating. The efficiency is shown to increase with the intensity of the vortex and with the Coriolis parameter, with substantial efficiency increases for weak vortices. Potential applications of these results for predicting tropical cyclone formation and rapid development are discussed.

1. Introduction

The source of energy for tropical cyclones is the release of latent heat due to condensation in moist convection. For developing tropical storms, such convection is quite disorganized relative to the cyclonic wind field, and highly sporadic in nature. Nonetheless, a portion of the heat released in these convective updrafts is transformed into energy of two forms: the kinetic energy (KE) of the quasi-symmetric wind field and the available potential energy (APE) associated with the warm core, which is necessary to hold the cyclonic wind field in hydrostatic and gradient wind balance. The goal of this study is to understand, and to quantify, this transformation of highly disorganized heat energy into the highly symmetric KE and APE of the developing storm.

A sense of these heating distributions can be seen from Tropical Rainfall Measuring Mission (TRMM) satellite-derived precipitation rates obtained from the
Naval Research Laboratory, as shown in Fig. 1. These figures show TRMM estimated rain rates from passes over Hurricanes Katrina (2005), Ophelia (2005), and Rita (2005) in their earlier stages of development. At the times shown, Katrina was a 30-kt depression (Fig. 1a), Ophelia was a 45-kt tropical storm (Fig. 1b), and Rita was a 60-kt tropical storm (Fig. 1c). In each case, the precipitation is very asymmetrically distributed around the respective cyclone centers. Furthermore, the areas of strong precipitation are localized in clusters ranging from 10–100 km in scale. We presume that these precipitation fields are a reasonable indication of the horizontal distribution of the latent heat release that drives the development of each storm. Numerous other examples of asymmetric convection patterns in developing tropical cyclones have been seen or inferred in both observations (Simpson et al. 1998; Lonfat et al. 2004; Reasor et al. 2005) and numerical models (Frank and Ritchie 2001; Davis and Bosart 2002; Hendricks et al. 2004; Montgomery et al. 2006; Nolan 2007).

The process by which asymmetrically distributed heat sources lead to intensification of a symmetric, bal-
anced vortex was previously studied by Nolan and Montgomery (2002, hereafter NM02) and Nolan and Grasso (2003, hereafter NG03). In these papers, the rapid heat release associated with deep convection was approximated with instantaneous temperature perturbations. In cylindrical coordinates, any distribution of instantaneous heating can be decomposed into a symmetric part that is equal to the azimuthally averaged heating and a purely asymmetric part (with zero mean) that can be further decomposed into a Fourier series around the axis of the vortex. A key conclusion of NG03 was that the vortex intensification caused by an instantaneous temperature perturbation could be very closely approximated by its projection onto the purely symmetric motions, that is, the symmetric response to the azimuthally averaged heating. This is because the impact of the purely asymmetric part of any such heating was one to two orders of magnitude smaller than the symmetric part, and almost always negative, that is, leading to a weakening of the mean vortex.

This last result is surprising, given that numerous previous studies of asymmetric forcing on tropical cyclone vortices found that purely asymmetric perturbations often lead to an intensification of the symmetric circulation, in terms of either azimuthally averaged wind speed or kinetic energy (Carr and Williams 1989; Montgomery and Kallenbach 1997; Nolan and Farrell 1999b; Möller and Montgomery 1999, 2000; Enagonio and Montgomery 2001; Shapiro 2000). The essential difference between these studies and NG03 is that asymmetric vorticity perturbations (or sources) were used in the earlier studies, rather than temperature perturbations or heat sources. For two-dimensional, incompressible flows, nondivergent vorticity perturbations were added to the vortex (e.g., Carr and Williams 1989; Montgomery and Kallenbach 1997; Nolan and Farrell 1999b), and for shallow-water or three-dimensional systems the perturbations were defined by quasi-balanced potential vorticity perturbations (e.g., Enagonio and Montgomery 2001; Möller and Montgomery 2000). Vorticity perturbations were used because they were considered to be the end product of a fast adjustment to strong localized heating from convection. The use of two-dimensional or balance models necessitates bypassing this process.

The importance of capturing the adjustment process is suggested by a minor observation in NG03 (p. 2726, footnote 4). The symmetric response to asymmetric forcing (in the linear limit) was computed by using divergences of eddy heat and momentum fluxes as forcing to a symmetric model. When the symmetric model was forced with eddy flux divergences saved every 10 min, asymmetric forcing was found to lead to vortex intensification in some cases; but when data every 3 min were used, the results were substantially different and weakening was always observed. The results with 3-min data were almost identical to simulations using a three-dimensional, fully compressible model with a 30-s time step. Thus, accurate representation of the rapidly evolving hydrostatic adjustment process and associated gravity wave radiation seems to be critical in determining the response of a symmetric, balanced vortex to asymmetric heating.

The conclusions that asymmetric forcing causes weakening, and that the intensification is closely approximated by the symmetric response to the azimuthally averaged heating have been supported by recent analyses of high resolution, full-physics simulations of tropical cyclones. Two recent studies of Hurricane Opal using forecasts from the Geophysical Fluid Dynamics Laboratory Hurricane Prediction System found that asymmetric intensification mechanisms had a small impact compared to symmetric mechanisms during Opal’s rapid intensification (Möller and Shapiro 2002; Persing et al. 2002). In an idealized framework, Möller and Shapiro (2005) forced an intensifying hurricane over short times with asymmetric temperature forcing quite similar in structure to the perturbations used in NM02 and NG03. For many cases, this resulted in a very brief strengthening of the cyclone followed by a longer weakening. The authors showed that the results were highly dependent on the interactions of the new, forced asymmetries with preexisting asymmetries in the core of the vortex, which occasionally led to seemingly contradictory results (such as different results for different azimuthal wavenumbers). Furthermore, the interpretation was clouded by the fact the cyclone was rapidly intensifying, such that it was difficult to say whether the effects of asymmetric forcing were due to a dynamic response of the vortex or simply the fact that the asymmetric heating was perturbing a highly nonlinear system.

Montgomery et al. (2006) analyzed the effects of latent heat release and vorticity generation by convective towers in a cloud-resolving simulation of tropical cyclogenesis starting from a weak, midlevel vortex. The azimuthally averaged heating and the eddy momentum fluxes from the simulation were used to force the quasi-balanced response of a balanced vortex using a modern version of the Eliassen (1951) balanced vortex model [similar to those used by Shapiro and Willoughby (1982) and Hack and Schubert (1986)]. Montgomery et al. found that the effect of the azimuthally averaged heating in driving the symmetric circulation was an or-
der of magnitude larger than the other forcing terms, such as eddy heat and momentum fluxes. This suggests that the conclusions of NG03 may apply to substantially weaker vortices than the tropical storm vortex used in that study.

Nonetheless, given the weight of the prior work on intensification by asymmetric forcing, and the mixed results of Möller and Shapiro (2005), it seems worthwhile to validate the results of NG03 with more realistic heat forcing that is injected over time and may move with the mean flow, and to consider the results for different basic-state vortices. Furthermore, we will look more closely at the energetics of the interactions between the evolving asymmetries and the symmetric vortex to better understand the process of symmetric vortex intensification from asymmetric convection.

In section 2, we introduce improved equations for perturbations to balanced vortices, along with expressions for kinetic energy and available potential energy, and their exchanges. In section 3, we present the vortexes used for this study. In section 4, we study the response of balanced vortices to asymmetric heat sources that evolve in time and may also be moving with the mean flow. Section 5 reintroduces the concept of the efficiency of a tropical cyclone based on the fraction of heat energy that is converted into KE of the symmetric wind field. Section 6 discusses how the results might change as nonlinear effects become significant. A summary and conclusions are presented in section 7.

2. Model equations and parameters

a. Background and presentation

NM02 modified the linearized anelastic equations to account for the significant variations in density and temperature associated with the strong warm core that holds a tropical cyclone in gradient and hydrostatic balance. This was done ad hoc by simply extending parts of the anelastic equations for incompressibility and temperature to include radial variations, for example,

\[ \frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{1}{p_{rel}(z)} \frac{\partial}{\partial z} [wp_{rel}(z)] \rightarrow \frac{1}{\rho(r,z)} \frac{\partial}{\partial r} [ru(r,z)] + \frac{1}{\rho(r,z)} \frac{\partial}{\partial z} [wp(r,z)], \]

while the buoyancy term in the vertical momentum equation was modified in a similar fashion

\[ g \frac{\theta'}{\theta_{ref}(z)} \rightarrow g \frac{\theta'}{\theta(r,z)}, \]

where “ref” subscripts refer to density and temperature profiles of a reference state that varies only in the vertical, while the overbars refer to vertically and radially varying density and temperature fields that hold the basic-state baroclinic vortex in hydrostatic and gradient wind balance. NM02 referred to this set as the asymmetric hurricane equations (hereafter, AH equations).

In retrospect, we have come to appreciate two problems with the AH equations: First, the AH equations were in error as they did not place the reference density inside the pressure gradient terms, as is required for the anelastic equations, and, second, a formal validation of anelastic equations with a radially varying reference state had never been performed. The first error is easily fixed, and the second issue has been addressed by Hodys and Nolan (2007), where a multiple scaling technique was used to derive anelastic equations valid for small perturbations to a balanced, baroclinic vortex. The most significant change is that while the anelastic equations typically use the expression on the left-hand side of (2.2) (e.g., Lipps and Hemler 1982; Bannon 1996; Yamazaki and Peltier 2001), the newly derived equations use the expression on the right. These “vortex anelastic equations” were shown to be slightly superior to the original anelastic equations when compared to identical simulations using a fully compressible model.

In this paper, we consider balanced vortices without any secondary circulations that might be driven by friction or persistent convection in the eyewall. All perturbations have arbitrary variation in the radial and vertical directions, and in time, but vary harmonically in the azimuthal direction; that is,

\[ u'(r, \lambda, z, t) = \text{Re} \left[ \sum_{n=0}^{\infty} u_n(r, z)e^{i\lambda \theta} \right]. \]

(2.3)

where, from here on, variables with subscripts refer to complex functions in the \((r, z)\) plane that contain the amplitude and phase information of the disturbances. For each azimuthal wavenumber \(n\), the equations of motion are

\[ \frac{\partial u_n}{\partial t} + in\Omega u_n - (2\Omega + f)u_n = -\frac{1}{\rho} \frac{\partial}{\partial r} \left( \frac{p_n}{\rho} \right) + F_u, \]

(2.4)

and

\[ \frac{\partial \theta_n}{\partial t} + in\Omega \theta_n = w_n \frac{\partial \theta}{\partial z} + u_n \left( \frac{\partial \theta}{\partial r} + \Omega + f \right) = -\frac{1}{\rho} \frac{\partial}{\partial r} p_n + F_{\theta}, \]

(2.5)
where the overbars over the perturbations represent azimuthal averages. The perturbation available potential energy for each wavenumber is

$$A_n = \int_0^R \int_0^Z \frac{1}{2} \frac{\overline{p}^2}{\overline{\theta}^2} z^2 \theta^2_0 2\pi r \, dr \, dz,$$

(2.11)

where \( \overline{N}^2 = (g/\overline{\theta})(\partial \overline{\theta}/\partial z) \) is the square of the basic-state Brunt–Väisälä frequency. Energy is exchanged between the perturbation kinetic and available energies, and the basic-state vortex, according to the following relations,

$$\frac{dK_n}{dt} = \int_0^R \int_0^Z \left[ -\left( \frac{\partial V}{\partial r} - \frac{V}{r} \right) \theta_n \dot{v}_n - \frac{\partial V}{\partial z} \frac{\overline{\theta} \theta_n}{\overline{\theta}} \dot{w}_n \right] 2\pi r \, dr \, dz
+ \frac{g}{\overline{\theta}} \frac{\overline{\theta} \theta_n}{\overline{\theta}} \dot{w}_n \theta_n + \frac{\overline{\theta} \theta_n}{\overline{\theta}} \theta_n \dot{w}_n \theta_n \right] 2\pi r \, dr \, dz,$$

(2.12)

$$\frac{dA_n}{dt} = \int_0^R \int_0^Z \left[ \frac{\overline{p}^2}{\overline{\theta}^2} \frac{\overline{\theta} \theta_n}{\overline{\theta}} \left( -\frac{\partial \overline{\theta} \theta_n}{\partial r} \dot{u}_n \theta_n - \frac{\partial \overline{\theta} \theta_n}{\partial z} \dot{w}_n \theta_n \right)
+ \frac{\overline{\theta} \theta_n}{\overline{\theta}} \theta_n \dot{w}_n \theta_n \right] 2\pi r \, dr \, dz,$$

(2.13)

where the \( F \)'s are frictional forces on each variable, including the damping regions.

The \( K_n \) exchanges energy with the basic-state vortex via the first two terms in the integral, which represent momentum fluxes across the vortex shear. It is also exchanged with \( A_n \) via the vertical heat transport term (third term in the integral), and lost due to friction (last three terms). Also, \( A_n \) exchanges energy with the basic-state vortex via the radial transport of heat across the radial gradient of \( \overline{\theta} \), and with \( K_n \) via the vertical heat transport term; it is lost due to friction (third term) and gained or lost through the injection of potential temperature (fourth term). The equivalent heat injection associated with a tendency \( \dot{\theta} \) is given by

$$\dot{Q} = \frac{c_p \overline{p} \overline{\theta} \dot{\theta}}{\overline{\theta}},$$

(2.14)

where \( c_p \) is the heat capacity of dry air at constant pressure.

Since the azimuthal mean of a symmetric perturbation is not zero, the symmetric perturbation energies cannot be decoupled from the basic state as in (2.10) and (2.11). Rather, we define the symmetric kinetic energy perturbation as the difference between that of the basic state and the total symmetric flow (the sum of the basic state and the perturbations). Taking the difference of the two gives
where \( \theta \) refers to the basic-state wind field. We take a similar approach for symmetric available potential energy, defining it as the difference between the APE of the basic state and the APE of the sum of the basic state and the symmetric perturbations:

\[
A_0 = \int_0^R \int_0^Z \frac{1}{2} \bar{\theta}_r \bar{\theta}_r N_{ref}^2 (\theta_0^2 + 2\theta_r \theta_0) 2\pi dr dz;
\]

(2.16)

where \( \theta_{ref} \) and \( N_{ref}^2 \) refer to the temperature and stability of the background (or reference) environment, and \( \theta_p = \bar{\theta} - \theta_{ref} \) is the perturbation potential temperature that holds the vortex in balance (the warm core). The reference values are used since the perturbation APE is the difference between the warm cores of two symmetric vortices whose available energies are defined in reference to the larger environment. The alternative use of local values of \( \theta \) and \( N^2 \) in (2.16) makes only small changes in the numerically computed values for \( A_0 \), but using the reference values produces the most accurate balance with changes in \( K_0 \). As will be shown below, this balance is extremely close in our calculations, supporting (2.16) as a useful expression for symmetric perturbation APE.

c. Domain and grid structure

The domain for all calculations ranges from \( z = 0 \) to \( Z = 20 \) km and \( r = 0 \) to \( R = 300 \) km; this extends the domain outward from 200 km as in NM02 and NG03. The same grid stretching procedure is used (NM02, appendix B), with the radial stretching parameters modified so that the inner core is better covered by a grid with approximately equal grid spacing of 2.5 km between \( r = 0 \) and \( r = 80 \) km; the grid spacing expands to 9.7 km near the outer boundary. In contrast to previous work, no grid stretching is used in the vertical direction, but 50% more points are used in the vertical. This is to account for the fact that much of the dynamics occur at and above the altitudes of the heat release, which is maximized between 5 and 8 km. The grid is shown in Fig. 2.

3. Basic-state vortices and their environment

The linearized initial value problems presented in NM02 and NG03 used an idealized vortex similar to a weak tropical storm. This “tropical storm vortex” was constructed from a Gaussian vorticity profile. While useful and popular owing to their analytic simplicity, Gaussian vortices have been shown by Mallen et al. (2005) to be unrealistic in their representation of the “outer core” wind fields of tropical cyclones since the Gaussian vorticity goes quickly to zero in the radial distance between 1 and 2 times the radius of maximum winds (RMW). In this region and beyond, the velocity profile decays like \( 1/r \), while observed profiles tend to decay much more slowly, like \( 1/r^2 \), where typically \( \frac{1}{5} < a < \frac{1}{3} \). Two mathematical models that represent these slower decay rates reasonably well are the modified Rankine vortex, in which the solid-body rotation in the inner core is matched to a predefined \( 1/r^a \) profile, and the Rankine-with-skin vortex, in which the core is surrounded by a region of lower vorticity.

For comparison purposes to previous results, it is helpful for some calculations to use the same basic state as in NM02 and NG03. Therefore, three vortices are used for this study: 1) the same tropical storm vortex (hereafter, the TS vortex); 2) a Rankine-with-skin vortex, with the same maximum wind speed \( (v_{max}) \) at the same RMW as the tropical storm vortex (hereafter, the RWS vortex), and 3) a weaker Rankine-with-skin vortex with its maximum winds elevated to \( z = 4 \) km, decaying to even smaller values at the surface (hereafter, the ERWS vortex). This represents a midlevel vortex that has not yet undergone the “transition” to a warm-core tropical cyclone and is similar to initial conditions in idealized genesis simulations, like those of Montgomery et al. (2006) and Nolan (2007).

The vorticity of the TS vortex is defined by

\[
\zeta(r) = \zeta_{max} \exp \left[ -\left( \frac{r}{b} \right)^2 \right],
\]

(3.1)

with \( \zeta_{max} = 1.5 \times 10^{-3} \) s\(^{-1}\) and \( b = 45 \) km. This generates a wind profile with \( v_{max} = 21.5 \) m s\(^{-1}\) and RMW = 50.4 km (stated incorrectly as 49.0 km in NM02). The vorticity of the RWS profiles are defined by

\[
\zeta(r) = \begin{cases} 
\zeta_1, & 0 \leq r \leq r_1 - d_1 \\
(\zeta_1 - \zeta_2)(r_1 - r + d_1)/2d_1, & r_1 - d_1 \leq r \leq r_1 + d_1 \\
\zeta_2, & r_1 + d_1 \leq r \leq r_2 - d_2 \\
(\zeta_2 - \zeta_3)(r_2 - r + d_2)/2d_2, & r_2 - d_2 \leq r \leq r_2 + d_2 \\
0, & r_2 + d_2 \leq r < \infty,
\end{cases}
\]

(3.2)
where \( S(x) = 1 - 3x^2 + 2x^3 \) is the cubic Hermite polynomial, which satisfies \( S(0) = 1, S(1) = 0, \) and \( S'(0) = S'(1) = 0. \) With \( \xi_1 = 1.06 \times 10^{-3} \text{ s}^{-1}, \xi_2 = 1.06 \times 10^{-4} \text{ s}^{-1}, r_1 = 46 \text{ km}, r_2 = 150 \text{ km}, d_1 = 20 \text{ km}, \) and \( d_2 = 75 \text{ km}, \) this formula generates a vortex with \( u_{\text{max}} \) and \( RMW \) identical to the TS vortex. Vorticity and velocity profiles for the TS and RWS vortices are shown in Fig. 3. For the ERWS vortex, the vorticity and velocity profiles are exactly \( \frac{1}{2} \) of the RWS vortex.

The velocity profiles of the TS, RWS, and ERWS vortices are extended into the vertical with functions of the form

\[
\bar{v}(r, z) = V(r) \exp \left( -\frac{|z - z_0|^\alpha}{\alpha L_z^\alpha} \right),
\]

where \( L_z \) indicates the depth of the barotropic part of the vortex, \( \alpha \) is the decay rate away from the barotropic zone, and \( z_0 \) is the altitude of the maximum wind speed. For the TS and RWS vortices, \( \alpha = 2.0, L_z = 5823 \text{ m}, \) and \( z_0 = 0. \) For the ERWS vortex, \( \alpha = 2.0, L_z = 3400 \text{ m}, \) and \( z_0 = 4000 \text{ m}. \)

The pressure and temperature fields that hold each vortex in hydrostatic and gradient wind balance are computed through an iterative procedure described in Nolan et al. (2001, section 4b). The environmental temperature profile is the Jordan (1958) mean hurricane season sounding. The Coriolis parameter is set to \( f = 5.0 \times 10^{-5} \text{ s}^{-1} \) for all calculations. The wind and perturbation potential temperature fields of the three vortices are shown in Fig. 4.

4. Response to moving and evolving asymmetric heat sources

4a. Spatial and temporal structure of heat forcing functions

Due to steady improvements in the power of both observational and computing technology, the scales of the convective features observed and simulated in tropical cyclones have decreased down to the size of individual cumulus updrafts. Indeed, we now understand that for developing storms almost all of the vertical motion, and associated latent heat release, is contained in updrafts that can be as small as 1 km across—
where \( r_b \) and \( z_b \) are the center of the heating in the \( r-z \) plane, \( \sigma_r \) and \( \sigma_z \) are the radial and vertical widths of the heating, and \( F(t) \) is a time distribution to be described shortly. The exponential variation in the vertical has been changed from the second to the fourth power, such that it is more sharply confined in the vertical direction. For all calculations, \( z_b = 6 \) km, \( \sigma_z = 3 \) km, and \( \sigma_r = 10 \) km. The \( r-z \) structure of the heat source for \( r_b = 40 \) km is shown in Fig. 5a. This structure is inspired by latent heating in tropical convection as observed by satellite platforms (Olson et al. 1999, 2002, 2006; Yang et al. 2002) and simulated in numerical models (Hendricks et al. 2004; Montgomery et al. 2006).

Except where noted, the time evolution of the heating is

\[
F(t) = \exp \left[ - \left( \frac{t - 1.5 \times t_{\text{forc}}}{pt_{\text{forc}}} \right)^4 \right] \exp(-\text{int}t),
\]

where setting the coefficient \( p = 0.55163 \) normalizes \( F(t) \) such that the area under its curve is equal to \( t_{\text{forc}} \). Thus, for a given value of \( t_{\text{forc}} \), the heating is maximized at the time \( t = 1.5 \times t_{\text{forc}} \), and the total temperature change injected is equal to \( \theta_{\text{max}} \times t_{\text{forc}} \). The complex exponential term rotates the physical location of the heating according to the angular velocity \( \omega = \nu_{\text{rot}}/r_p \), where \( \nu_{\text{rot}} \) is the rotation speed. The amplitude of \( F(t) \) is shown in Fig. 5b for \( t_{\text{forc}} = 30 \) min, 1 h, and 2 h.

b. Energetics of a control case

Before exploring the effects of varying the duration and rotation rates of the asymmetric heat sources, we present the results of a control case. This example uses the TS vortex as a basic state, while the heating has the same size and altitude parameters as above, along with \( \theta_{\text{max}} = 10 \) K h\(^{-1}\), \( n = 3 \), \( r_p = 40 \) km, \( t_{\text{forc}} = 1 \) h, and \( \nu_{\text{rot}} = 15.1 \) m s\(^{-1}\), which is 75\% of the local maximum wind speed at the surface at \( r = r_p \).

Qualitatively, these heatings generate asymmetric motions and a symmetric vortex response identical to those discussed in detail in NM02 and NG03: there is a rapid response to the asymmetric heating with asymmetric gravity wave radiation and adjustment to hydrostatic balance. During this process, asymmetric vorticity perturbations are generated, which are then sheared into trailing spirals by the radially and vertically varying wind. The asymmetries also propagate outward and upward as predicted by vortex–Rossby wave theory (Montgomery and Kallenbach 1997; Möller and Montgomery 2000). The evolving asymmetries interact with the symmetric vortex via eddy heat and momentum fluxes, and the symmetric vortex responds by adjusting to a new balanced state with modified wind and temperature fields.
Fig. 4. Radius–height sections of the basic-state vortices used in this study: (a), (c), (e) azimuthal velocities of the TS, RWS, and ERWS vortices (in m s\(^{-1}\)); (b), (d), (f) potential temperature perturbations (in K), which hold these vortices in balance. Note shading values are smaller for the ERWS vortex. Contour intervals are indicated on each plot.
NG03 evaluated these changes in two ways: in terms of the change in the surface pressure at the center of the vortex and in terms of the change in the wind speed at the lowest model level (which we hesitate to call the surface since the frictional boundary layer is neglected in all three vortices). We may also now evaluate these changes in terms of the change in the KE and APE of the entire symmetric wind field, that is, \( K_0 \) and \( A_0 \) in (2.15)–(2.16).

Figure 6a shows the asymmetric energy of perturbations generated by this heating. When the heating begins to intensify, the asymmetric energy increases very quickly. In these figures, energy values are normalized by the total asymmetric APE injected during the simulation, that is, the time-integrated quantity of the fourth term in (2.13). As the asymmetric energy begins to rise, the symmetric energy begins to decline (Fig. 6b), suggesting an immediate interaction of the heat-forced asymmetries with the symmetric vortex. Note that the asymmetric energy reaches a value above 1.0, suggesting that the energy lost by the symmetric vortex is acquired by the asymmetries.

This is proved in Fig. 7, which shows the rates of energy change for \( K_n \) and \( A_n \), their sum, and the energy exchanges with the basic state in (2.12) and (2.13). Energy exchange rates in this and subsequent figures are normalized by the maximum rate of APE injection, that is, the volume integral of the fourth term in (2.13) at the time of maximum heating. The actual rate of change of total energy \( (K_n + A_n) \) almost perfectly matches the rates predicted by the exchange equations (2.12) and (2.13), for both the sum and the two parts. This gives considerable confidence in the equations of motion and the energy analyses.

As shown in Fig. 7d, the asymmetries extract a large amount of energy from the vortex during the heat release. Most of this energy comes from the vertical shear term (dashed line), but some also comes from the radial shear term (solid line). After the period of significant heating, the asymmetries return energy to the vortex during axisymmetrization, as can be seen from the negative values of the radial shear term (solid line) and the radial temperature advection term (dash-dot line), which are both negative after \( t = 3.5 \) h. However, the area under the total exchange curve is positive, indicating the asymmetries have extracted energy from the vortex. Normalized by the total injected APE, the energy extracted by the asymmetries has a value of 0.24, while the change of total energy in the symmetric vortex is \(-0.16\). The difference may be due to the numerical and explicit diffusion in the model, which along with the damping regions near the domain boundaries, act on the perturbations, not on the total flow. Thus, the diffusion acts to return the vortex to its initial state, damping out some of the changes caused by the simulated adjustment processes.

Also shown in Figs. 6c,d are the surface pressure change with time and the change in the low-level wind field at \( t = 12 \) h. As seen in NG03, the heating has caused a contraction of the low-level vortex. While this contraction is associated with a fall in the surface pressure, the wind speed has decreased at the RMW (located almost exactly at \( r = 50 \) km), and the symmetric KE has decreased. NG03 found that asymmetric heatings in almost all radial (and vertical) locations resulted in a decrease of the wind speed at the RMW.

c. The effect of injecting heat over time

We evaluate whether the main conclusions of NG03—that purely asymmetric forcing always leads to
weakening—are valid for more realistic convective heating. For the TS vortex, simulations similar to the control, for wavenumber $n = 3$ and heating centered at $r_b = 40$ km and at 80 km, were performed for rotating heat sources with $u_{rot} = 15.1$ m s$^{-1}$ and $t_{torc} = 10$ min, 30 min, 1 h, and 2 h.

The results of these simulations are shown in Fig. 8, in terms of the symmetric energy changes, the time evolution of the central pressure, and the final change to the lowest level wind field. Looking first at the figures on the left for $r_b = 40$ km, the symmetric energy decreases for all forcing times. The pressure increases slightly for the two shorter times, and decreases slightly for the longer times. For all forcing times, the vortex is intensified inside the RMW and weakened at the RMW, suggesting a contraction of the low-level wind field. The results for $r_b = 80$ km are on the right-hand side of the figure, and universally indicate vortex weakening. It is interesting to note that asymmetric heating well outside of the RMW still results in a decrease in the low-level winds inside and at the RMW that is comparable or even greater than the decreases caused by heating inside the RMW.

d. Heat sources that move with the flow

Along the same lines, we perform simulations with the same parameters in the previous section, but we fix $t_{torc} = 1$ h, and vary the speed at which the convective heating moves around the vortex. The rotation speed is described by the percentage of the maximum basic-state azimuthal wind at radius $r_b$. For example, at $r_b = 40$ km, the TS vortex has a maximum wind speed of 20.5 m s$^{-1}$, so in the figures that follow, 75% means $u_{rot} = 15.1$ m s$^{-1}$.
Figure 9 shows the same data as in Fig. 8 but for varying rotation speeds of the convective heating. Results for \( r_b = 80 \) km again show consistent weakening of the symmetric vortex. For \( r_b = 40 \) km, the results for slower moving heat sources (0% and 25%) show symmetric energy increase; however, these are associated with surface pressure increases. The surface wind changes all show inner-core increase surrounded by decrease near the RMW.

Thus, there are, in fact, some asymmetric heating patterns that can lead to increases in the energy of the symmetric vortex. However, no heating patterns led to significant increases in the low-level wind speed at the RMW. Examining Fig. 8e and Fig. 9e, one might think that by moving \( r_b \) outward 20 to 30 km this would be achieved. However, NG03 found that, as the heating is moved outward, the change in the low-level azimuthal wind adjusts so that the change at the RMW is almost always negative or, at best, a very small positive value. This was found for moving and evolving heat sources as well (not shown).

e. Effects of other basic states and heating parameters

As shown in Fig. 10, further simulations with the TS vortex for varying wavenumber \( n \) and for the RWS and ERWS vortices (with the control parameters) do not change the qualitative results: purely asymmetric heating generally leads to vortex weakening, usually in terms of the symmetric energy, and always in terms of the low-level wind. Variations of other parameters such as the sizes, shapes, and radial locations of the heating give similar conclusions (not shown).
Fig. 8. Summaries of changes to the symmetric vortex for purely asymmetric heating that is injected over different time intervals for (left) $r_b = 40$ km, and (right) $r_b = 80$ km: (a), (b) change in KE of the symmetric vortex; (c), (d) change in central surface pressure; (e), (f) change in the lowest-level wind profile at $t = 12$ h. Energies are normalized by the total injected APE.
Fig. 9. As in Fig. 8 but for purely asymmetric heat sources moving at different speeds with the flow.
Fig. 10. As in Fig. 8 but with $r_s = 40$ km for all cases, with (left) varying azimuthal wavenumber and (right) varying vortex type.
f. Asymmetric energy exchanges

NG03 attempted to explain weakening of the vortex by studying the evolution of potential vorticity (PV) perturbations generated by the heating. In the linearized system, the perturbation PV is

\[ q_n = \frac{1}{p} (\omega_n \cdot \nabla \mathbf{u} + \mathbf{\omega} \cdot \nabla \theta_n) \]

\[ = \frac{1}{p} \left( \xi_n \frac{\partial \mathbf{u}}{\partial r} + \eta_n \frac{\partial \mathbf{u}}{\partial \lambda} + \zeta_n \frac{\partial \mathbf{u}}{\partial z} + \overline{\xi} \frac{\partial \theta_n}{\partial r} + \overline{\eta} \frac{\partial \theta_n}{\partial \lambda} + \overline{\zeta} \frac{\partial \theta_n}{\partial z} \right) \]  

where \( \xi, \eta, \) and \( \zeta \) are the radial, azimuthal, and vertical components of the three-dimensional vorticity vector \( \omega \). In a barotropic vortex, a heating-induced temperature perturbation would generate coherent, opposite-signed PV anomalies directly above and below the heat source, each associated with the last term in (4.3). Owing to the radial vorticity \( \overline{\xi} \) associated with the vertical wind shear, a temperature perturbation in a baroclinic vortex produces a PV perturbation that is the sum of four PV perturbations: two associated with the last term of (4.3) and two associated with the fourth term of (4.3). These latter two anomalies lie on either side of the temperature perturbation in the radial direction. As it turns out, these radial PV anomalies are about twice as strong as the vertical anomalies, such that the resulting PV is dominated by the radial anomalies, as shown in Fig. 11 (see also NG03, their Fig. 18). This initial condition (in terms of vorticity or potential vorticity) is quite different than those used in the earlier studies using two-dimensional flow or quasi-balanced dynamics. Being closer to the center of the vortex, the inner ring of PV perturbations rotates faster than the outer ring, such that the inner asymmetries eventually come into phase with the outer asymmetries of same sign. During this phase of the evolution, the vorticity has an upshear tilt that evolves into a more coherent structure, such that the asymmetries gain energy from the basic-state vortex through downgradient eddy momentum fluxes (Orr 1907; Farrell 1982; Nolan and Farrell 1999a). NG03 inferred that this early extraction of energy was the reason that asymmetric temperature perturbations cause weakening.

There are some cases above in which the final value of \( K_0 \) is in fact positive, suggesting some vortex strengthening. Why do the asymmetries extract energy in some cases, but deposit energy in others? Asymmetric energies and exchanges, for a simulation with the same parameters as the control case, except for \( u_{rot} = 0 \), are shown in Fig. 12. Symmetric energy for this case is shown in Fig. 9a (thin solid line), and pressure is shown in Fig. 9c (thin solid line). The area under the total exchange curve (thick line in Fig. 12a) is negative, indicating that there has been a net transfer of energy from the asymmetries to the symmetric vortex. The normalized energy loss by the asymmetries is 0.23, while the normalized increase in total symmetric energy is 0.20 (not shown). For comparison, the energy exchanges for the control case are shown again in Fig. 12b, which show an energy gain by the asymmetries of 0.24, with an energy loss by the symmetric motions of 0.16 (Fig. 6b).
Why does the change in rotation speed lead to a different result? To understand this, we first examine the eddy momentum fluxes and their divergences generated by the asymmetric motions in these two cases ($\Omega_{\text{rot}} = 75\%$ and $0\%$ of the local flow speed). Figure 13 shows eddy momentum flux vectors and their associated tendencies on the symmetric azimuthal wind, $u_1$ at $t = 1.5$ h and $t = 3.5$ h for the two simulations. At $t = 1.5$ h, they are similar and show substantial vertical transport of momentum. Note, however, that there is a significantly larger radially outward component of the momentum transport at low levels for the case with $\Omega_{\text{rot}} = 75\%$ (left-hand side of figure). This is consistent with the larger radial transport term in the energy exchanges of Fig. 12b. For $\Omega_{\text{rot}} = 75\%$, this extraction of energy from radial momentum transport continues for another hour, while for $\Omega_{\text{rot}} = 0\%$, it has decreased to zero and changed to negative values by $t = 2$ h. At $t = 3.5$ h, the fluxes are quite different. The case with $\Omega_{\text{rot}} = 75\%$ has significant radial transports of momentum, inward at low levels and outward at upper levels. For $\Omega_{\text{rot}} = 0\%$, the fluxes are smaller and are only at upper levels.

Why are the eddy fluxes different at lower levels? The answer seems to be related to the vertical structures of the perturbations generated by the heating. For the heating that is moving with $75\%$ of the surface wind speed, the heat source stays in phase with the $w_n$ and $\theta_n$ fields that it generates, creating more vertically coherent PV anomalies that reach closer to the surface. For the stationary heating, the $w_n$ and $\theta_n$ fields generated are advected away from the heat source and are also substantially sheared in the vertical direction due to the vertical wind shear. These differences are depicted in the PV fields for the two cases as a function of height and azimuth at $r = 33$ km (the center of the inner perturbation), at $t = 2$ h and $t = 4$ h, as shown in Fig. 14. The forcing with $\Omega_{\text{rot}} = 75\%$ (figures on left) generates stronger PV anomalies, which penetrate to lower altitudes than those generated by forcing with $\Omega_{\text{rot}} = 0\%$. This leads to stronger interactions with the low-level part of the symmetric vortex, thus leading to more energy extraction at earlier times.

Our conclusion that the baroclinicity of the symmetric vortex is the reason why asymmetric heating leads to vortex weakening suggests that the results will be different for barotropic, or more barotropic vortices. To explore this possibility, we repeated the control case with $\Omega_{\text{rot}} = 75\%$ but with three different versions of the TS vortex, modified as follows: 1) a “deeper” vortex, with $L_z$ and $\alpha$ in (3.3) both increased by 50%, labeled the Deeper-TS vortex; 2) a “deeper and stronger” vortex, with $L_z$ and $\alpha$ increased by 50% and the azimuthal wind speed doubled so that the wind profile is representative of a mature hurricane, labeled Deeper-TSx2; and 3) a “barotropic” TS vortex with no wind decrease with height, labeled Baro-TS. The vertical profiles of the wind field for the TS vortex, the Deeper-TS vortex, and the Deeper-TSx2 vortex are shown in Fig. 15 (Baro-TS not shown).

The symmetric responses to the asymmetric heating of the control case for each of these more barotropic vortices are shown in Fig. 16. For the Baro-TS vortex, the symmetric energy change is quite different, chang-
ing from negative to positive. For the Deeper-TS vortex, the symmetric energy change is also positive, though not so much as for the Baro-TS vortex. The symmetric energy change is positive for the Deeper-TSx2 vortex as well, but not any more so than for the Deeper-TS vortex. The central pressure change and the surface wind change, however, are essentially the same for all three of the more barotropic vortices, with wind decrease still occurring at the RMW and a pressure fall indicative of vortex contraction.

The preceding analyses have shown that, while purely asymmetric heating in baroclinic vortices always leads to a decrease in the mean surface winds, it can sometimes lead to an increase (rather than a decrease) in the kinetic energy of the mean vortex. Whether net extraction or deposition of energy occurs depends on subtle differences in the evolution of the forcing and the vertical structure of the symmetric vortex. However, NG03 found that, regardless of their sign, the effects of the asymmetries were always far smaller than the response to the azimuthally averaged heating. This will now be demonstrated.

g. The net impact of a moving, isolated heat source in three dimensions

To consider the total impact of a moving and evolving cluster of convection, we define a convective heating distribution in three-dimensional, Cartesian coordinates.

Fig. 13. Eddy momentum flux tendencies on the symmetric azimuthal wind for (a),(c) the control simulation and (b),(d) the stationary heating simulation: (a), (b) at $t = 1.5$ h; (c), (d), at $t = 3.5$ h. Gray scales and contour intervals are identical in the two plots, but the flux vectors are not scaled and are shown only for illustration of the directions and magnitude changes of the fluxes.
where now $x_b$ and $y_b$ indicate the center of the heat source, which is then prescribed to move about the center of the vortex,

$$x_b = r_b \cos \omega t, \quad y_b = r_b \sin \omega t. \tag{4.6}$$

The projection of this heating is taken onto each azimuthal wavenumber for $n = 0$ to 8, and this heating is used to drive asymmetric and symmetric perturbations. Furthermore, in addition to the wavenumber-0 contribution, the sums of the divergences of the eddy momentum and temperature fluxes over all eight asymmetric wavenumbers are used to force the symmetric motions.

Using the more realistic wind profile of the RWS vortex, we consider a moving, rotating heat source, with $r_b = 40$ km and $\omega = 75\%$ of the local angular velocity, $\dot{\theta}_{\text{max}} = 40$ K h$^{-1}$ with $t_{\text{forc}} = 1$ h, the equivalent of 40 K kg$^{-1}$ m$^2$ s$^{-1}$.)

In practice, rather than calculating the components of the Fourier series at each time step, the projections onto each wavenumber are computed at $t = 0$ and then these forcing functions are spatially rotated in time.

FIG. 14. Azimuth–height profiles of perturbation Ertel’s PV at $r = 33$ km for the control case (left) and the stationary heating case (right): (a), (b) $t = 2$ h; (c), (d), $t = 4$ h. For easier comparison, the grayscale range and contour intervals are the same ($5.0 \times 10^{-7}$ K kg$^{-1}$ m$^2$ s$^{-1}$) for all four plots, and the zero contours have been suppressed.
injected over 1 h. Figure 17 shows $K_0$ and $A_0$ as a function of time for this simulation, for a simulation in which only the symmetric projection of the heating forces the vortex, and for one in which only the summed asymmetric eddy fluxes force the vortex. Also shown are the symmetric perturbation wind field $u_0$ at $t = 12$ h for each case. The energy plots for the first two cases are nearly identical, highlighting once again the extent to which the response to symmetric forcing dominates over asymmetric forcing. The changes in the wind field, however, are not identical, and the difference is equal to the change due to the asymmetric parts ($n = 1$ to 8) of the forcing (Fig. 17f). These changes are not trivial compared to the symmetric response (as was seen in NG03 for much weaker forcing), but consist of approximately equal areas of opposite sign—thus causing very little change in $K_0$ as compared to the response to symmetric heating.

5. Efficiency of the intensification process

a. Prior work

The problem of understanding the conversion of heat energy released in tropical cyclones into balanced wind and temperature fields is a special case of the general geostrophic adjustment problem so widely studied in meteorology and oceanography. Schubert et al. (1980) studied the transient evolution and final states resulting from perturbations to the geopotential fields of balanced, barotropic vortices. Schubert and Hack (1982) extended these ideas to study the transverse circula-

Fig. 15. Vertical profiles of the azimuthal wind at the RMW for the TS vortex and two modified versions, which are more barotropic in the lower troposphere.

Fig. 16. As in Fig. 8 but for the TS vortex, a TS vortex with a deeper low-level barotropic region (Deeper-TS), a similar vortex but that is twice as strong (Deeper-TSx2), and a purely barotropic TS vortex (Baro-TS): (a) change in symmetric KE with time; (b) change in surface-central pressure with time; (c) change in the lowest-level wind profile at $t = 12$ h.
Fig. 17. (left) Symmetric energies (in J) and (right) azimuthal wind changes (in m s$^{-1}$) for the response of the RWS vortex to an isolated, moving heat source: (a), (b) for all wavenumbers ($n = 0$ to 8); (c), (d) for symmetric heating only; (e), (f) for the asymmetric heatings ($n = 1$ to 8) only.
tions and implied intensification of barotropic vortices due to radially and vertically varying heat sources. They defined the efficiency as the fraction of temperature change maintained in the vortex (i.e., not lost to gravity waves via adjustment) as compared to the actual heat injected. Hack and Schubert (1986) computed the transverse circulations caused by heat sources in more realistic, baroclinic vortices and defined the “dynamic efficiency” as the ratio of the rate of change in kinetic energy of the wind field to the rate of heat injection.

The common theme among these three studies was confirmation of the intuitive fact that the intensification or response of the vortex was strongly dependent on the strength and structure of the vortex and the location of the heat source. The dynamic efficiency was typically a maximum of a few percent in the upper part of the center of the vortex, at the same altitude as the strongest baroclinicity of the surrounding wind field. Another commonality of these studies is that they used the same assumptions as the classic “balanced vortex” model of Eliassen (1951). Such models are purely diagnostic, predicting the steady transverse circulation that exactly compensates for the heating so as to keep the symmetric vortex in hydrostatic and gradient wind balance. Evolution of the symmetric vortex is inferred from the tendencies due to the advection of the basic state wind and temperature by the transverse circulation.

Given the sporadic and disorganized nature of the convection, it is natural to wonder whether these results carry over to rapidly evolving heat sources and the unsteady, nonhydrostatic motions they generate. As we shall show in the following sections, efficiencies such as those computed by Schubert and Hack do not depend on the time distribution but only on the total heating, at least as long the adjustment process can be reasonably well represented by linear dynamics.

b. More on symmetric energetics

NG03 only considered the response to instantaneous symmetric heatings, that is, symmetric temperature bubbles. How do the results depend on the time distribution of the heating? Fig. 18 shows symmetric energy changes as a function of time for two simulations: one with a 1-K temperature perturbation (bubble) initialized at \( r_b = 40 \) km and the other for a simulation with a 1-K temperature change injected with \( t_{\text{torc}} = 1 \) h. In both cases, the curves are normalized by the total heat energy \( Q \) injected into the symmetric vortex \( Q \) is equal for these two cases). The final adjustment for the two simulations are nearly identical, and this is supported by Figs. 18c and 18d, which show the change in the symmetric wind field at \( t = 8 \) h for each case.

To understand why the final response of the symmetric vortex does not depend on the time distribution of the heating, we recall that any linear model can be represented as a linear dynamical system,

\[
\frac{dx_0}{dt} = T_0 x_0 + S_0, \tag{5.1}
\]

where \( x_0 \) is the state vector, \( T_0 \) is the time evolution operator, \( S_0 \) represents any source terms, and the zero subscripts remind us that we are considering only the symmetric dynamics. For \( S_0 = 0 \), (5.1) has the solution

\[
x_0(t) = x_0(0) \exp(T_0 t). \tag{5.2}
\]

We have seen in NG03 and above that for any finite-time heating, the vortex adjusts to a new, balanced state, which would be steady if not for the diffusion in the model. Neglecting this diffusion, the adjustment to a new steady state can be represented mathematically by the statement

\[
limit_{t \to \infty} \exp(T_0 t) = C, \tag{5.3}
\]

where \( C \) is a constant matrix. For a time-evolving heat source where \( S_0(t) = 0 \) for all \( t > t^* \),

\[
x_0(t) = \int_0^t \exp(T_0(t - t')) S_0(t') \, dt'
\approx C \left[ \int_0^{t^*} S_0(t') \, dt' \right] \quad \text{for } t \gg t^*. \tag{5.4}
\]

The exponential term in the first integral of (5.4) is the Green’s function for temporal evolution from time \( t' \) to \( t \). Equation (5.4) simply states that the final adjustment is the sum of the adjustment to each infinitesimal part of the heating.

Such is not the case for the symmetric response to asymmetric dynamics. The effect of the asymmetries on symmetric motion is a second-order quantity (e.g., \( \bar{u}' \bar{v}' \)), so its amplitude is nonlinearly dependent on the disturbance amplitude, such that a larger symmetric response occurs for faster, stronger asymmetric heating.

Returning to the symmetric response, Fig. 18b shows \( K_0 \) and \( A_0 \) for three simulations in the RWS vortex with \( r_b = 40, 60, \) and \( 80 \) km. As predicted by Schubert and Hack (1982), the increase in \( K_0 \) is far larger for heating closer to the center of the vortex. Interestingly, the fraction of the energy retained as KE (wind energy), instead of APE (the warm core), is also larger for convective heating at smaller radius.
c. Examples of efficiency

We define the kinetic energy efficiency (KEE) of the vortex as the ratio of energy retained as wind KE to the injected heat energy, that is, $K_0/Q$. The efficiencies shown in Fig. 18 are very small, less than 2%, even for convective heating inside the RMW. However, comparing the efficiencies for heating at different radii (1.9%, 1.4%, 1.1% for $r_b/\lambda_1005 = 40, 60, \text{ and } 80$ km) demonstrates the substantially larger impact that convection has when it is closer to the center of a developing cyclone: the heating at $r_b = 40$ km is nearly twice as efficient as heating at $r_b = 80$ km.

Taking this analysis to its logical next step (as in Hack and Schubert 1986), we compute the KEE for heat injected into each individual grid point in the domain. Rather than performing separate simulations for each grid point, this is achieved through computation of the matrix exponential\(^2\) in (5.2), for $t = 24$ h, and then use of (5.2). We also compute the available energy efficiency (AEE) and the total energy efficiency (TEE). All three are shown for the TS vortex in Fig. 19. KEE is far larger than AEE, and is maximized in the center of the upper part of the warm core, near the maximum baroclinicity of the symmetric wind field, as was shown before by Hack and Schubert. The appearance of negative values in some of these figures indicate that the adjustment to heat injected at that location results in a

\(^2\) This and all calculations were performed with Matlab 7.0 and recent upgrades.
change to a new balanced state with less kinetic or available energy than the initial state.

KEE profiles for the RWS and ERWS vortices are shown in Fig. 20. The profiles for the TS vortex and the RWS vortex are very similar, with the higher efficiency region slightly broader for the RWS vortex due to its broader wind field. Recall that along with being elevated, the ERWS vortex is half as strong as the RWS vortex; not surprisingly then, the maximum efficiency (1.44%) is about half of that for the RWS vortex (2.79%).

d. Dependence on vortex parameters and environment

Except perhaps for the ERWS vortex, which represents a disturbance that has not yet organized into a

Fig. 19. Efficiency diagrams for the TS vortex: (a) kinetic energy efficiency, (b) available potential energy efficiency, (c) total efficiency.

Fig. 20. Kinetic energy efficiency diagrams for (a) the RWS vortex and (b) the ERWS vortex.
tropical cyclone, the high efficiency values at the cores of the TS and RWS vortices are probably irrelevant. Deep convection in such systems is almost always displaced from the centers of circulation, where the higher surface winds lead to larger fluxes of moisture from the ocean, and the turbulent swirling boundary layer maximizes vertical motion just inside the RMW (Eliassen 1971; Eliassen and Lystad 1977; Nolan 2005). To get a more meaningful estimate of the KEE for these vortices, we define the relevant KEE for a vortex as the volume-weighted average of KEE over the region \(0.5 \times \text{RMW} < r < 1.5 \times \text{RMW}, 2 \text{ km} < z < 12 \text{ km}\). These last two parameters are arbitrary choices intended to bracket the region where latent heat release is possible, but the results that follow are not sensitive to their values.

Hack and Schubert (1986) showed that, as their quasi-balanced vortex intensified, the maximum efficiency greatly increased, suggesting that tropical cyclone development was a self-amplifying process whereby stronger vortices were better and better able to use the latent heat release they generated. However, they did not present the effects of changes in vortex intensity, size, shape, etc. Some calculations along these lines of thought are shown in Fig. 21a. The three curves show the relevant KEE as a function of \(v_{\text{max}}\) for each vortex (note that here the data for ERWS are for elevated vortices with the same \(v_{\text{max}}\) as the other two vortices). The curve stops for the ERWS vortex because, due to its larger upper-level baroclinicity, its warm core becomes convectively unstable for \(v_{\text{max}} > 27 \text{ m s}^{-1}\). The sensitivity to \(v_{\text{max}}\) is very large for all three vortices. For example, the relevant KEE is about 0.4% for all three vortices with \(v_{\text{max}} = 10 \text{ m s}^{-1}\), but increases to about 1.1% for \(v_{\text{max}} = 20 \text{ m s}^{-1}\). While 1.1% is not a large efficiency, it is nearly 3 times larger than for a vortex with half the wind speed.

Figure 21b shows the same calculation for the three vortices while varying the Coriolis parameter from an equivalent latitude of 0° to 40°N. While the fractional increase in efficiency is not large for the TS and RWS vortices, it is quite large for the weaker ERWS vortex, nearly doubling over the range. Between 15°N (the deep Tropics) and 30°N (the subtropics), the relevant efficiency increases by a factor of 1.42. This perhaps explains the ability of disturbances in the subtropics and midlatitudes to organize into tropical cyclones despite having shallower and weaker convection.

Could the effects of varying vortex strength and varying latitude be combined into a single parameter that correlates with the efficiency of the vortex? An excellent candidate for such a parameter would be the inertial stability

![Relevant KEE versus Max Wind Speed](image1)

![Relevant KEE versus Latitude](image2)

![Relevant KEE versus Mean Inertial Stability Parameter](image3)

**Fig. 21.** Relevant kinetic energy efficiencies for the three vortices as a function of (a) maximum surface wind speed, (b) latitude, (c) inertial stability. For a fair comparison, the ERWS vortex in (a) is adjusted to have the same \(v_{\text{max}}\) as the other two vortices. The same efficiency data used in (a) and (b) are plotted in (c).
which is the product of the modified Coriolis parameter and the absolute vertical vorticity (Hack and Schubert 1986; Shapiro and Montgomery 1993; Kepert 2001). Figure 21c shows the same data from Figs. 21a,b, but with the relevant KEE values plotted instead against the mean value of $I^2$ volume-averaged over the range $0 \leq r \leq 3 \times$ RMW, $0 \leq z \leq 12$ km. Much like the averaging area used for the definition of relevant KEE, this region is arbitrary, but it is chosen to account for the fact that the circulations generated by heating at any location are very nonlocal and “feel” the effects of inertial stability over a larger region (Shapiro and Willoughby 1982; Hack and Schubert 1986; NG03). Remarkably, most of the data seems to collapse along a line of constant proportionality between KEE and $I^2$. However, data from variations in $f$ vary differently than those from variations in intensity. This seems to be so because $I^2$ is dominated by terms associated with the mean wind: variations in the mean wind speed make large changes in the amplitude of $I^2$, while variations in $f$ change the distribution of $I^2$. Further studies of these relationships are ongoing (Stern and Nolan 2006).

6. The effects of nonlinearity

In trying to use our results to understand real tropical cyclones, two questions immediately come to mind: How well does the linear approximation hold for the response to convective heating in weak tropical cyclones and do the results change as nonlinearity becomes important?

The maximum values of the asymmetric $u$, $v$, and $w$ velocities during the simulation discussed in section 4g are 1.8, 1.5, and 0.4 m s$^{-1}$, respectively, suggesting that this simulation approaches (but does not exceed) the limit of where we can expect linear calculations to be accurate, as does the fact that the amplitudes of the symmetric changes caused by the nonlinear eddy fluxes of the asymmetries are about one-fifth of those caused by the symmetric heating (Figs. 17d,f). The heating rate used in that example had a maximum of 40 K h$^{-1}$ over a period of about one hour, spread out over a Gaussian bubble with $\sigma_r = 10$ km. Maximum heating rates of this magnitude, and in fact even much higher, have been diagnosed in high-resolution simulations of tropical cyclones (Hendricks et al. 2004; Montgomery et al. 2006; Nolan 2007). However, these very high heating rates (as much as 100 K h$^{-1}$) are confined to the small cores of the intense updrafts on the scales of 1–5 km, which rarely last as long as an hour. Accounting for its larger scale and duration, the heating injected by the isolated heat source in section 4g is at least comparable to the heating injected in one or more of the most intense updrafts. Nonetheless, the energy changes of the symmetric vortex were still dominated by the response to the symmetric heating, suggesting that nonlinear effects were not yet significant.

A thorough evaluation of the effects of increasing asymmetric heating rates on the intensification of the vortex could certainly be worthwhile, but it would require a fully nonlinear model and is beyond the scope of this paper. A first-order estimate of the effects of weak nonlinearities can be presented, however. There are two essential effects of nonlinearity: the interactions of each wavenumber with each other and the changes to the dynamics of the vortex as it is changed by the symmetric and asymmetric motions. This latter effect can be approximated by updating the basic-state fields and the linear dynamical operators to incorporate the changes predicted by the symmetric model as time progresses. This was performed for a simulation with the same parameters as for the isolated, moving heat source used in section 4g, in which the basic-state vortex was updated every 1 h during a 12-h simulation for three cases: with forcing from the $n = 2$ perturbations (which of all the asymmetries caused the largest response), with forcing from the $n = 0$ heating, and with the two together. Furthermore, the maximum value of the heating rate was doubled to $\theta_{max} = 80$ K h$^{-1}$.

The results are summarized in Fig. 22 in terms of the lowest-level wind change at $t = 12$ h. For the linear simulations, the asymmetric forcing moves the maximum increase inward and increases it by about 20%, but still decreases the response at the RMW. The effect of including the weakly nonlinear updates in each case is to decrease the low-level wind changes and to amplify the negative effects near the RMW. The decrease of the response is primarily due to the increased “stiffening” of the vortex as it is strengthened by the symmetric part of the heating. This is, of course, a crude approximation to nonlinear dynamics, but this example shows that, as the fully nonlinear regime is approached, the results remain qualitatively, even somewhat quantitatively, similar to those shown above.

7. Conclusions

The process of intensification of a balanced, baroclinic, tropical cyclone–like vortex by convection displaced from the vortex center has been revisited with an improved linear model. Along with improvements to the equations, this model allowed for the specification
of moving and evolving heat sources. These heat sources were modeled after the heating distributions of deep convection inferred from satellite and radar observations and numerical model simulations.

The ultimate conclusion of NG03 that the purely asymmetric part of any such heating generally leads to weakening of the symmetric vortex was confirmed for a variety of simulations with variations in the rotation speed and duration of the heat sources. The conclusion that the symmetric response to the asymmetric motions was much smaller than the symmetric response to the azimuthally averaged heating was also confirmed, even for heating rates of substantial amplitudes (e.g., 40 K h$^{-1}$ over 1 h). Weakly nonlinear effects were shown to decrease the amplitude of the final response, but not to change any of the aforementioned conclusions.

The linear, nonhydrostatic model was used to evaluate the efficiency of the symmetric intensification process, as first introduced by Schubert and Hack (1982). For the range of heating rates where the linear dynamics are reasonably accurate, it was proved that the final change in the vortex strength does not depend on the temporal distribution of the heating. Efficiency diagrams for the three different vortices were created, showing the net fraction of heat energy introduced at any location in the vortex that is converted into kinetic energy of the symmetric wind field. Naturally, these efficiency rates are quite small with maximum values of a few percent along the center axis in the upper part of the warm core.

A more relevant efficiency was defined in the area above and around the RMW and was computed for varying strengths of the vortices and for varying latitudes. Not surprisingly, these relevant efficiencies increase with vortex strength and with latitude. The more compelling results were the large changes in efficiency for the weaker ERWS vortex for changes in latitude and strength. If relocated from 15° to 30°N, this vortex acquires a 40% increase in efficiency, while, if changing its maximum wind speed from 10 to 20 m s$^{-1}$, its efficiency nearly triples.

The results of this paper have potentially useful applications for the science of forecasting intensity change for developing tropical cyclones. With ever-increasing advances in our ability to observe moist physical processes in tropical cyclones, it could become possible to accurately diagnose the convective heating surrounding developing tropical cyclones on a more frequent basis (Olson et al. 1999, 2002, 2006; Rodgers et al. 2000; Yang et al. 2002). While there are already established meth-

![Change in Lowest Level Wind at t = 12 h](image-url)
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