Internal Waves at an Interface between Two Layers of Differing Stability

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(Manuscript received 9 July 2008, in final form 4 December 2008)

ABSTRACT

Internal waves in a two-layer fluid are considered. The layers have different values of the buoyancy frequency, assumed to be constant in each layer. The density profile is chosen to be continuous across the interface and the flow is Boussinesq. The solution is an expansion in the wave amplitude, similar to a Stokes expansion for free surface waves. The results show that the nonlinear terms in the interfacial boundary conditions require higher harmonics and result in nonlinear wave steepening at the interface. The first few harmonics are scattered by the interface, whereas the higher harmonics are evanescent in the vertical. The second-order correction to the wave speed is negative, similar to previous results with a rigid upper boundary.

1. Introduction

The lapse rate of temperature in the earth’s atmosphere is climatologically constant in each layer, changing abruptly at the tropopause, stratopause, and mesopause, as is well known. A schematic of the temperature profile is shown in Fig. 1. The abrupt change in lapse rate is coincident with an abrupt change in the buoyancy frequency $N$, which is an upper bound for the frequency of internal waves. This rapid change in $N$ is known to have a dramatic effect on the propagation of internal waves, causing significant wave reflection, even in the small-amplitude linear approximation.

The altitude region near the tropopause has long been suspected of having higher levels of turbulence on average than other altitudes (Partl 1962). A recent statistical study by Wolff and Sharman (2008) supports this suspicion. Wolff and Sharman examine pilot reports of turbulent encounters (PIREPS) over the contiguous United States. They conclude that the 1-yr average of PIREPS for moderate or greater intensity turbulence has a maximum at the altitude corresponding to the average tropopause height.

Turbulence can be caused by wave overturning, and recent experiments by Anderson (2004) confirm that waves are forced to break at an altitude where $N$ has an abrupt change. These experiments were performed over an ice field in the Antarctic. During the day, the ice is colder than the air in the boundary layer, and as a result the temperature of the air increases approximately linearly up to an altitude of about 1 km. The temperature then decreases approximately linearly, resulting in a decrease in $N$ (Renfrew and Anderson 2006). This interface mimics the behavior of the temperature profile at the stratopause, where the buoyancy frequency decreases abruptly, rather than the tropopause, where it increases. The experiments found that traveling waves appear from an unknown source. Anderson (2004) documents these waves as they traverse the interface and overturn at exactly the altitude of the sudden change in buoyancy frequency. This experiment is a clear indication that nonlinear effects are important when the lapse rate and buoyancy frequency change suddenly.

Further observational evidence of aberrant wave behavior at the tropopause is reported by McHugh et al. (2008). The observations were made over Hawaii and indicate a duration of at least several days with large-amplitude wave activity. One important aspect of this observation is that the wave amplitude is large only in the vicinity of the tropopause. This feature again suggests that the tropopause experiences nonlinear dynamic effects due to the sudden change in buoyancy frequency.

Traveling gravity waves interacting with an idealized interface similar to the tropopause (and other such interfaces) is treated here with weakly nonlinear theory. The density profile is continuous at the interface but has
a finite change in the buoyancy frequency. The flow is modeled as inviscid and Boussinesq (Turner 1973), the horizontal mean flow is initially zero, and Coriolis effects are ignored. The Boussinesq approximation is valid for phenomena that are confined to a vertical extent that is less than a scale height. The region in the vicinity of the interface is sufficiently confined, and the contribution to the dynamics from a more general model would be of higher order.

Mountain waves in two-layer flow with continuous density have been considered previously (e.g., Eliassen and Palm 1961; Durran 1992). Eliassen and Palm (1961) treat the linear interfacial conditions. Durran (1992) includes nonlinear effects at the interface and finds a solution with a numerical Fourier transform. Durran’s results do not show unusual wave behavior at the interface; however, he does report that the position of the interface and the mountain height are important to wave behavior in general. The higher harmonics in Durran’s solution are a combination of forcing effects due to the mountain and nonlinear effects at the interface, and therefore it is difficult to isolate the interface dynamics.

Monochromatic waves incident to the $N$-discontinuous interface are treated below with an expansion in the wave amplitude, similar to a Stokes expansion for free surface waves (Stokes 1847). The waves are assumed to be periodic in the horizontal and propagate with permanent form. Internal waves of permanent form have been treated previously, most notably by Thorpe (1968) and Yih (1974). Yih (1974) showed that the background density profile must be adjusted to account for a nonlinear shift in the mean streamline. The correction to the background density results in a second-order correction to the wave speed, analogous to Stokesian waves (Stokes 1847). However, Yih (1974) showed that the correction for internal waves is negative, meaning that larger-amplitude waves travel more slowly, opposite to that for free-surface waves. Thorpe (1968) and Yih (1974) both considered a configuration in which the flow is bounded on top by a rigid lid, resulting in complete reflection of the internal waves. The value of correction to the wave speed depends strongly on the wave reflection. Partial wave reflection occurs in the lower layer of the configuration considered below, also resulting in an important correction to the wave speed. The present results show that this correction is also negative.

The waves are made steady by choosing a coordinate system that moves horizontally with the wave speed [also used by Stokes (1847), Long (1953), and Yih (1974)]. The governing equations are then reduced to Long’s equation (Long 1953). For the case considered here of Boussinesq flow with constant buoyancy frequency (in each layer) and no upstream shear, Long’s model becomes linear and is given by

$$\nabla^2 \delta + \kappa^2 \delta = 0,$$

where $\delta$ is the vertical displacement of streamlines from an upstream or background state and $\kappa$ is a constant.
At first glance, it would seem that the solution to the two-layer problem considered here is merely the sinusoidal solution

\[ \delta \propto \sin(mx) \sin(nz), \tag{2} \]

where \( m \) and \( n \) are constants unique to each layer. This solution does satisfy (1) exactly, but it only meets the boundary conditions at the mean interface. An accurate nonlinear solution must meet the boundary conditions at the actual interfacial position rather than the mean, which is not known beforehand. As a result, in a multilayer fluid (2) is only a linear solution, accurate for infinitesimal wave amplitudes only. Important nonlinear effects result when the displacement of the interface is included. Further nonlinear effects result when the background density profile is adjusted to match the average density profile in the presence of waves.

It is shown here that the nonlinear effects at the interface result in higher-frequency internal waves propagating throughout the fluid. These waves are harmonic to the incident wave only at the interface, where they contribute to steepen the wave. Away from the interface, the harmonics propagate at an angle to the horizontal that is different than the incident wave; the interface causes the harmonics to be scattered. For some parameter values, the linear solution gives an evanescent wave in the upper layer, meaning the vertical structure is not oscillatory. For all parameter values, the higher harmonic waves in both layers become evanescent when the effective frequency is greater than the buoyancy frequency.

Waves on the interface of two fluids have been extensively studied; they are reviewed in the book by Craik (1985). Two-layer fluids of constant but different densities can support waves on the interface, analogous to free surface waves. If the two layers also have uniform horizontal flow, then this phenomenon is known as the Kelvin–Helmholtz problem. The dynamics of the interfacial waves display many nonlinear phenomena, including self-interaction and three-wave resonance. The problem considered here has a continuous density profile. The density at the tropopause does not have a discontinuity, although there is a “kink,” meaning the first derivative of density has a discontinuity. If there is no discontinuity in the density at the interface, then purely interfacial waves cannot exist, and the vast results on the Kelvin–Helmholtz problem do not directly apply. The linear mode of importance in the present theory is an internal wave, not an interfacial wave.

Details of the governing equations and the interfacial conditions are given below in section 2. Section 3 develops the weakly nonlinear solution. Section 4 provides some general remarks on the results, and section 5 summarizes the paper.

2. Basic equations

The flow is assumed to be incompressible, inviscid, and two-dimensional. Stratification is present because of a nondiffusing quantity. The flow is then governed by the Euler equations, the continuity equation, and the equation of incompressibility. Long (1953) reduced these equations to a form now known as Long’s model. Long’s model assumes a horizontal reference flow \( u_0(z_0) \) with a density profile \( \rho_0(z_0) \), where \( z_0 \) is the vertical position in the reference flow. The streamlines may be deflected by a disturbance, often considered to be a barrier to the flow, such as a mountain. The derivation of Long’s equation is given by Long (1953) and will not be repeated here. The resulting equation is

\[ \nabla^2 \delta + \frac{1}{2} \frac{d \beta}{dz} \left[ \frac{\partial \delta}{\partial z} - (\nabla \delta)^2 \right] + \frac{N^2}{u_0^2} \delta = 0, \tag{3} \]

where \( \delta = z - z_0 \)

is the streamline displacement,

\[ N^2 = - \frac{g}{\rho_0} \frac{\partial \rho_0}{\partial z_0}, \quad \text{and} \tag{5} \]

\[ q = \rho_0 u_0^2. \tag{6} \]

If the Boussinesq approximation is assumed and \( u_0 \) is taken to be constant, then Long’s equation reduces to

\[ \nabla^2 \delta + \frac{g \beta}{u_0^2} \delta = 0, \tag{7} \]

\[ \beta = - \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial z_0}. \tag{8} \]

The mean density profile is chosen to be continuous, but it has a discontinuous first derivative such that the fluid exists in two semi-infinite layers, each layer having a unique value of the buoyancy frequency. The kinematic interfacial condition is that the normal velocity must be continuous across the interface. The dynamic interfacial condition is that the pressure must be continuous across the interface. As stated by Durrant (1992), these conditions can be met by choosing \( \delta \) to obey

\[ \delta_1 = \delta_2 \quad \text{and} \tag{9} \]

\[ \delta_{1z} = \delta_{2z}. \tag{10} \]
at the interface, where $\delta_1$ is the streamline displacement field in the lower layer and $\delta_2$ in the upper layer. A complete derivation of these conditions is provided in the appendix. Note that these are still the fully nonlinear interfacial conditions, and not just the linear condition. Of course, the condition must still be met on the actual interfacial position rather than the mean interfacial position, a feature that results in the important nonlinear effects.

The vertical position of the origin of the coordinate system is chosen to be at the mean position of the interface, where $\delta_1$ is the streamline displacement $z$ at the interface, so that the streamline that corresponds to the actual interfacial position rather than the mean interfacial position, a feature that results in the important nonlinear effects.

The vertical position of the origin of the coordinate system is chosen to be at the mean position of the interface, so that the streamline that corresponds to the interface is given by $z_0 = 0$. The displacement of the interface is then determined by

$$\delta(x, z) = z.$$ \hspace{1cm} (11)

Inverting this expression to obtain the interfacial shape, even for relatively simple expressions for $\delta$, is nontrivial. Assume such a solution to (11) is

$$z = \eta(x).$$ \hspace{1cm} (12)

Note that $\delta(x, 0)$ is not the same as $\eta(x)$, except in the linear case. The interfacial displacement $\eta$ is related to the velocity field by the familiar kinematic interfacial condition for each layer:

$$\eta_t + (1 - \delta_z) \eta_x = \delta_x$$ \hspace{1cm} (13)

on the interface.

Now expand (9), (10), and (13) in a Taylor series about the mean position of the interface and insert into the interfacial conditions to obtain

$$\eta_t + (1 - \delta_z) \eta_x = \delta_x$$ \hspace{1cm} (13)

3. Finite-amplitude waves

The governing equation in each layer is the linear Long’s equation,

$$\nabla^2 \delta_1 + \frac{g \beta_1}{u_1^2} \delta_1 = 0$$ \hspace{1cm} (18)

$$\nabla^2 \delta_2 + \frac{g \beta_2}{u_2^2} \delta_2 = 0,$$ \hspace{1cm} (19)

where $u_1$ and $u_2$ are background velocities for each layer. The expansion of $\delta$ in a power series in $\epsilon$, the ratio of the wave amplitude to horizontal wavelength, is

$$\delta = \epsilon \delta_{11} + \epsilon^2 \delta_{12} + \epsilon^3 \delta_{13} + \cdots,$$ \hspace{1cm} (20)

and similar expressions are chosen for $\delta_2$ and $\eta$. Note the subscript convention: the first index indicates the layer and the second indicates the order.

A coordinate system is chosen to be moving with the wave speed (as yet undetermined) to make the flow steady, and the wave speed must also be expanded in

the wave amplitude to suppress secular terms. Both features are treated with

$$u_1 = c_{10}(1 + \epsilon \beta_{11} + \epsilon^2 \beta_{12} + \cdots)$$ \hspace{1cm} (21)

$$u_2 = c_{20}(1 + \epsilon \beta_{21} + \epsilon^2 \beta_{22} + \cdots).$$ \hspace{1cm} (22)

Ultimately, $u_1$ and $u_2$ must be equal to the wave speed and to each other, so that the flow is steady. However it is convenient to maintain them separately for now and ultimately equate them later to determine the wave-number in the upper layer.

To expedite the correction to the background density profile, $\beta$ is also expanded in the wave amplitude:

$$\beta_1 = \beta_{10}(1 + \epsilon \beta_{11} + \epsilon^2 \beta_{12} + \cdots),$$ \hspace{1cm} (23)

and a similar expression can be written for $\beta_2$.

The first-order governing equations are

$$\nabla^2 \delta_{11} + \frac{g \beta_{10}}{c_{10}^2} \delta_{11} = 0$$ \hspace{1cm} (24)

$$\nabla^2 \delta_{21} + \frac{g \beta_{20}}{c_{20}^2} \delta_{21} = 0,$$ \hspace{1cm} (25)
and the interfacial conditions are
\begin{align}
\delta_{11} &= \delta_{21}, \\
\delta_{11z} &= \delta_{21z}, \quad \text{and} \\
\eta_z &= \delta_{11z} = \delta_{21z}
\end{align}
(26)–(28) on \( z = 0 \).

The solution to (24)–(28) is chosen to be an upwardly propagating incident wave in the bottom layer. The linear interfacial conditions require a reflected wave in the lower layer and a transmitted wave in the upper layer:
\begin{align}
\delta_{11} &= Ae^{inx} + \frac{(n_1 - n_2)}{n_1 + n_2} Ae^{inz} + A^* e^{-inx} + \frac{n_1 - n_2}{n_1 + n_2} e^{-inz}, \\
\delta_{21} &= \frac{2n_1}{n_1 + n_2} [A e^{imx - nz_2} + A^* e^{-imx + nz_2}],
\end{align}
(29)–(30) where \( n_1 \) and \( n_2 \) are the vertical wavenumbers in the lower and upper layers, respectively, and \( A^* \) is the complex conjugate of \( A \). Note that (29) and (30) are chosen to satisfy a radiation condition in each layer. The corresponding displacement of the interface is
\begin{equation}
\eta_1 = \frac{2n_1}{n_1 + n_2} (A e^{inx} - A^* e^{-inx}).
\end{equation}
(31) The dispersion relations are
\begin{align}
c_{10}^2 &= \frac{g\beta_{10}}{m^2 + n_1^2} \quad \text{and} \\
c_{20}^2 &= \frac{g\beta_{20}}{m^2 + n_2^2}.
\end{align}
(32)–(33) The dispersion relation for the upper layer must give the same wave speed and share the same horizontal wavenumber \( m \) as the lower layer, which determines the vertical wavenumber in the upper layer. The first approximation to this vertical wavenumber is
\begin{equation}
n_2^2 = m^2 \left( \frac{\beta_{20}}{\beta_{10}} - 1 \right) + n_1^2 \frac{\beta_{20}}{\beta_{10}}.
\end{equation}
(34) a. Second order

The second-order governing equations are
\begin{equation}
\nabla^2 \delta_{12} + \frac{g\beta_{10}}{c_{10}^2} \delta_{12} = -2c_{11} \nabla^2 \delta_{11} - \frac{g\beta_{11}}{c_{10}^2} \delta_{11}
\end{equation}
and (35) (36)
\begin{equation}
\nabla^2 \delta_{22} + \frac{g\beta_{20}}{c_{20}^2} \delta_{22} = -2c_{21} \nabla^2 \delta_{21} - \frac{g\beta_{21}}{c_{20}^2} \delta_{21}.
\end{equation}

Before proceeding further, the correction to the upstream condition must be considered to determine \( \beta_{11} \) and \( \beta_{21} \). This is achieved as described in the next section. The analysis shows that there is no correction at this order, matching the conclusions of Yih (1974), and the details will not be given:
\begin{equation}
\beta_{11} = \beta_{21} = 0.
\end{equation}
(37) The only secular term left in (24) and (25) is suppressed by choosing \( c_{11} = c_{21} = 0 \), making the second-order equations homogeneous. The second-order interfacial conditions are
\begin{align}
\delta_{12} - \delta_{22} &= (\delta_{21z} - \delta_{11z}) \eta_1, \\
\delta_{12z} - \delta_{22z} &= (\delta_{21xz} - \delta_{11xz}) \eta_1, \\
\eta_z - \delta_{12z} &= \frac{2}{\delta_{11z} \eta_1} (\delta_{21z} \eta_1), \quad \text{and} \\
\eta_z - \delta_{22z} &= \frac{2}{\delta_{11z} \eta_1} (\delta_{21z} \eta_1)
\end{align}
(38)–(41) on \( z = 0 \). Using (27) reduces (38) to
\begin{equation}
\delta_{12} - \delta_{22} = 0
\end{equation}
(42) on \( z = 0 \). Inserting (29), (30), and (31) into (39) then simplifies gives
\begin{equation}
\delta_{12z} - \delta_{22z} = \left( \frac{2n_1}{n_1 + n_2} \right)^2 (n_1^2 - n_2^2) \times (A^* e^{2mx} + A e^{-2mx} + 2AA^*)
\end{equation}
(43) on \( z = 0 \). The forcing terms in (43) are not resonant and thus require a solution of the form
\begin{equation}
\delta_{12z} = B \sin(2mx + n_{12z}) + C \sin(\gamma_{12z}),
\end{equation}
(44) where \( B \) and \( C \) still need to be determined. The vertical wavenumber in (44), \( n_{12} \), is not determined by the interfacial conditions; it is chosen to satisfy the governing equations in the bottom layer, reflected in the dispersion relation for the bottom layer:
\begin{equation}
n_{12}^2 = n_1^2 - 3m^2.
\end{equation}
(45) The term \( C \sin(\gamma_{12z}) \) balances the constant in (43) and represents a wave-induced mean flow, but one driven by the interfacial conditions. The governing equation determines the exponent:
\begin{equation}
\gamma_{12z}^2 = \frac{g\beta_{10}}{c_{10}^2} = m^2 + n_1^2.
\end{equation}
(46)
Similarly in the upper layer,
\[ \delta_{22} = B \sin(2mx - n_{22}z) + C \sin(\gamma_{22}z), \]  
(47)
\[ n_{22}^2 = n_1^2 - 3m^2, \]  
(48)
\[ \gamma_{22}^2 = \frac{g\beta_{20}}{c_{20}^2}, \]  
(49)
where the coefficients have already been chosen to satisfy (42). Note that the negative sign in front of \( n_{22} \) is chosen in the upper layer to obtain an upwardly propagating harmonic, meeting the radiation condition at the top.

Equation (39) determines \( B \) and \( C \):
\[ B = -4n_1 n_{21} - n_{22} A^2 \]  
(50)
\[ C = \left( \frac{2n_1}{n_1 + n_{22}} \right)^2 \left( n_{11}^2 - n_{22}^2 \right) \gamma_{12} + \gamma_{22} 2AA^n. \]  
(51)

b. Correction to \( \beta \)

The second-order solution must match the background conditions. This is achieved using the method of Yih (1847). The definition of \( \delta \) is rearranged to obtain
\[ z = z_0 + \delta(x, z). \]  
(52)
The inversion of this equation is
\[ z = \eta(x, \rho). \]  
(53)
An average over one wave period of (53) is the average displacement of a line of constant density in the presence of waves. A wave of permanent form must have this vertical displacement equal to zero. If it does not, then the upstream density profile must be adjusted. This adjustment is determined using
\[ \frac{1}{\rho_0} \frac{d\rho}{dz_0} = \frac{1}{\rho_0} \frac{d\eta}{\eta dz_0} = -\beta \frac{d\eta}{dz_0}. \]  
(54)

In practice, (52) is inverted using the method of successive approximations for each layer, as in Yih (1974) and Stokes (1847). The final result is
\[ \beta_{21} = 4AA^n n_{11} \left( \frac{n_{11} - n_{22}}{n_1 + n_{22}} \right) \left( \frac{\gamma_{12}}{\gamma_{11} + \gamma_{22}} \cos \gamma_{12} z_0 - 2 \cos 2n_1 z_0 \right) \]  
(55)
\[ \beta_{22} = 0. \]  
(56)

Note that the correction for the upper layer is zero because there is no reflection, only the incident wave. The correction for the lower layer is nonzero because of the interaction between the incident wave and its reflection from the interface.

c. Third order

The third-order solution is pursued just far enough to demonstrate uniform validity and thereby determine the second-order correction to the wave speed. The third-order governing equations are
\[ \nabla^2 \delta_{13} + \frac{g\beta_{10}}{c_{10}^2} \delta_{13} = -2c_{121} \nabla^2 \delta_{11} - \frac{g\beta_{12}}{c_{12}^2} \delta_{11} \]  
and  
(57)
\[ \nabla^2 \delta_{23} + \frac{g\beta_{20}}{c_{20}^2} \delta_{23} = -2c_{221} \nabla^2 \delta_{21} - \frac{g\beta_{22}}{c_{22}^2} \delta_{21}. \]  
(58)
The third-order interfacial conditions are
\[ \delta_{13} - \delta_{23} = (\delta_{211} - \delta_{111}) \eta_2 + (\delta_{222} - \delta_{122}) \eta_1 + \frac{1}{2}(\delta_{21zz} - \delta_{11zz}) \eta_1^2, \]  
(59)
\[ \delta_{13z} - \delta_{23z} = (\delta_{211z} - \delta_{111z}) \eta_2 + (\delta_{222z} - \delta_{122z}) \eta_1 + \frac{1}{2}(\delta_{21zzz} - \delta_{11zzz}) \eta_1^2, \]  
(60)
\[ \eta_3 + \eta_3 - \delta_{13z} = \frac{\partial}{\partial x} \left( \delta_{11zz} \eta_2 + \delta_{12zz} \eta_1 + \frac{1}{2} \delta_{11zzz} \eta_1^2 \right). \]  
(61)
\[ \eta_3 + \eta_3 - \delta_{23z} = \frac{\partial}{\partial x} \left( \delta_{21zz} \eta_2 + \delta_{22zz} \eta_1 + \frac{1}{2} \delta_{21zzz} \eta_1^2 \right). \]  
(62)
on \( z = 0. \)

The forcing terms in the interfacial conditions are not resonant, nor will they be at any order; hence, they do not contribute to the second-order correction to the wave speed. This is because the wave speed is not determined by the interfacial conditions; interfacial waves do not exist when the density is continuous. The interfacial forcing terms are still important, however, because they result in scattered harmonics of the incident waves (discussed in the next section).

The forcing terms in the governing equation for the lower layer are secular as a result of the nonzero value of \( \beta_{12} \). These terms are suppressed, again following Yih (1974), by multiplication with the expression for \( \delta_1 \) in the lower layer and integration over a single wave period. The resulting second-order correction to the wave speed in the lower layer is
whereas for the upper layer
\[ c_{12} = 2n_1^2AA^p \left\{ \left[ \frac{(n_1 - n_2)}{(n_1 + n_2)} \left( \frac{1}{\gamma_{12} + \gamma_{22}} \right) \right] \frac{2\pi \gamma_{12}}{n_1} + 1 \right\} \times \left\{ -2 + \left( \frac{1}{2n_1 - \gamma_{22}} \right) \right\} \frac{\sin(2n_1 - \gamma_{12})}{2\pi n_1} + \left( \frac{1}{2n_1 + \gamma_{12}} \right) \frac{\sin(2n_1 - \gamma_{12})}{2\pi n_1} \} \right\} \). (63)

The final stage is to determine \( n_2 \), which is found by equating the wave speeds in the two layers. Setting \( n_1 = u_2 \) in (21) and (22) and keeping terms to second-order results in
\[ \frac{g\beta_{10}}{m^2 + n_1^2} (1 + \varepsilon^2 c_{12}) = \frac{g\beta_{20}}{m^2 + n_2^2}. \] (65)

This expression, along with (63), represents two coupled algebraic expressions for \( n_2 \) and \( c_{12} \), which are determined numerically.

### 4. Discussion

Internal waves only exist for frequencies that are less than the buoyancy frequency \( N = g\beta \), as can be shown from the dispersion relation. Hence, the incident waves in the lower layer must have a frequency less than \( N = g\beta_{10} \). The buoyancy frequency above the interface may be larger or smaller than below: the buoyancy frequency at the stratopause decreases. For the case in which \( N_2 < N_1 \) (the stratopause), it is possible for the frequency of the linear incident wave to be greater than \( N_2 \). The linear transmitted wave will be evanescent in the upper layer, with no vertical oscillations in the upper layer.

For the case in which \( N_2 > N_1 \) (the tropopause and mesopause), the linear incident wave creates a wave in the upper layer that oscillates with the vertical. However, the harmonics that are created at the interface may be evanescent in either layer. The harmonic may be considered to have an effective frequency; for example, the second harmonic in the lower layer would have the effective frequency
\[ \frac{2m}{\sqrt{(2m)^2 + n_{12}^2}} N_1. \] (66)

If the effective frequency of a harmonic is greater than the Brunt–Väisälä frequency in either layer, then the harmonic will be evanescent in that layer. This transition occurs when the vertical wavenumber becomes purely imaginary; for example, if \( n_{12} > 0 \) in (45), then \( n_{12} \) is real and the harmonic is oscillatory. If \( n_{12} < 0 \), then the behavior is evanescent.

Away from the interface, the higher harmonics with vertical oscillation will not coincide with the primary mode. For example, the incident waves in the lower layer are traveling waves with wavenumbers \( m \) and \( n_1 \), as discussed previously. The lines of constant phase for this wave make an angle \( \theta_1 \) to the horizontal:
\[ \theta_1 = \arccos \frac{m}{\sqrt{m^2 + n_{12}^2}}. \] (67)

The second-order solution requires a harmonic with wavenumbers \( 2m \) and \( n_{12} \), where \( n_{12} \) is given by (45). This harmonic is a wave that makes an angle \( \theta_{12} \) with the horizontal:
\[ \theta_{12} = \arccos \frac{2m}{\sqrt{(2m)^2 + n_{12}^2}} = \arccos \frac{2m}{\sqrt{m^2 + n_{12}^2}}, \] (68)

where (45) has been used to simplify (68). Clearly, the angle of the incident wave is different from the angle of the second-order harmonic wave; the second-order harmonic has been scattered by the interface. The resulting pattern of waves for \( N_2 > N_1 \) is shown in Fig. 2 using contours of \( \phi \) using the second-order approximation assembled using (4), (20), (29), (30), (44), and (47). A second example with \( N_2 < N_1 \) is shown in Fig. 3. Note that the interface in Figs. 2 and 3 is at the vertical center of the diagram. Despite the complexity of these wave
patterns, the harmonic waves travel with the same horizontal wave speed as the incident wave, and this wave pattern does not evolve with time.

The third-order interfacial conditions will result in third-order harmonics with a higher effective frequency than the second-order harmonic. Depending on the parameter values of the incident wave, this frequency could be greater than the buoyancy frequency in either layer, resulting in nonoscillatory behavior and an evanescent mode. Furthermore, eventually there is a harmonic in the Stokes expansion that will result in a harmonic frequency that is greater than both $N_1$ and $N_2$; this harmonic will be evanescent in both layers, as will all higher harmonics.

The deflection of the interface due to the second-order harmonics has the same phase as the deflection of the interface due to the linear waves given by (31), whether the harmonic is oscillatory or not. The third-order harmonic will also have this phase. As a result, the nonlinear wave at the tropopause behaves as a Stokes wave, where the crest is sharpened and the trough broadened, as shown in Fig. 4.

However, the behavior of the waves away from the interface is significantly different. Only the oscillatory harmonics extend away from the interface and, depending on the wavenumbers for the incident wave, only the first few harmonics are likely to be oscillatory. The remainder of the harmonics will decay exponentially in each layer. Hence, a horizontal profile would consist of the sum of only a few sinusoidal components, quite different than the behavior at the interface.

The second-order correction to the wave speed and the wavenumber in the upper layer is determined by the two coupled nonlinear algebraic equations given in (63) and (65). Values of $c_{12}$ and $n_2$ have been determined numerically using the bisection method and are shown in Figs. 5 and 6. Figure 5 shows that $c_{12}$ is negative, meaning that larger-amplitude waves travel more slowly than infinitesimal waves. This result was demonstrated by Yih (1974) for internal waves bounded with rigid horizontal surfaces. The rigid surfaces in Yih (1974) cause complete wave reflection, which in turn leads to a nonzero wave-induced mean flow and a displacement of isopycnic lines. The present results have partial wave reflection from the interface, rather than complete reflection as in Yih (1974). Furthermore, there is an additional contribution to the wave-induced mean flow as a result of the interfacial conditions, given by the second term in (44). This interfacial mean flow weakens the...
effect but is not significant enough to overcome the effect found by Yih (1974); hence, the same mechanism that results in the negative wave speed correction is also at work here.

5. Conclusions

A variety of observations indicate that the tropopause and similar interfaces experience higher levels of turbulence than other altitudes. The recent statistical analysis of Wolff and Sharman (2008) indicates that turbulence at this altitude is common. The experimental results of Anderson (2004) show that a packet of internal waves breaks at the interface, suggesting that the turbulence at the tropopause is caused by breaking waves. However, internal waves can break at any altitude, thus begging the question: why do they break at the interface? The above analysis provides a plausible explanation. The accumulation of harmonics at the interface, which does not happen away from the interface, results in two features that promote breaking: 1) the wave amplitude is higher at the interface and 2) the wave crest at the interface is sharper, similar to free-surface waves. These features are present whether $N$ increases or decreases with altitude. As a result, enhanced levels of turbulence are expected at all interfaces of this type, including the stratopause and the mesopause.

Away from the interface the incident waves result in scattered harmonics. Hence, higher-frequency components of a wave spectrum measured during atmospheric observations in the vicinity of the tropopause may not be a result of wave forcing, as the spectrum would include these scattered components.

Finally, the second-order correction to the wave speed is negative, meaning that higher-amplitude waves propagate slower than infinitesimal waves. This result matches the previous result of Yih (1974) with rigid boundaries.

Acknowledgments. This work was supported by AFOSR under Grant FA9550-05-1-0439.

APPENDIX

The Interfacial Conditions

The kinematic interfacial condition is that the normal velocity must be continuous across the interface. In steady flow the interface is a streamline and the normal velocity is zero. Hence, the kinematic condition can be satisfied by choosing the two layers to share a streamline, resulting in

$$\delta_1 = \delta_2$$  \hspace{1cm} (A1)

at the interface.

The dynamic interfacial condition is that the pressure must be continuous across the interface. The pressure has been removed from the system of equations, but an equivalent boundary condition on $\delta$ is obtained from Bernoulli’s equation. Consider Bernoulli’s equation for the lower layer and choose the streamline to be the interface:

$$\frac{P_1}{\rho_1} + \frac{V_1^2}{2} + g\gamma_1 = B_1,$$  \hspace{1cm} (A2)

where $V_1$ is the magnitude of the velocity and $B_1$ is constant. Similarly, Bernoulli’s equation for the upper layer is

$$\frac{P_2}{\rho_2} + \frac{V_2^2}{2} + g\gamma_2 = B_2,$$  \hspace{1cm} (A3)

again choosing the streamline to be the interface. Because the two layers share the interface as a streamline, then $B_1 = B_2$ and $\gamma_1 = \gamma_2$ on the interface. The choice of continuity of density leads to $\rho_1 = \rho_2$ at the interface, whereas the dynamic interfacial condition dictates that $P_1 = P_2$ at the interface. The only remaining term in (A2) and (A3) is $V^2/2$, which therefore must also be continuous: $V_1^2 = V_2^2$ on the interface. The pressure boundary condition can now be stated as

$$\delta_1^2 + (1 - \delta_1 z_1)^2 = \delta_2^2 + (1 - \delta_2 z_2)^2$$  \hspace{1cm} (A4)

on the interface.

It is now proven that the continuity of $\delta$ and $(1 - \delta)z$ is equivalent to the continuity of $\delta$ and $(1 - \delta)z^2$. Furthermore, if the assumption is made that the sense is continuous across the interface, then continuity of $\delta z$ may be used. The final interfacial conditions are given by (9) and (10).

FIG. 6. Wavenumber in the upper layer.
**Theorem 1**

Continuity of $\delta$ and $[(\delta_x)^2 + (1 - \delta_z)^2]$ is equivalent to continuity of $\delta$ and $(1 - \delta_z)^2$ at the interface.

1) **Proof**

Start with

$$\delta_1 = \delta_2$$  \hspace{1cm} (A5)

and

$$(\delta_{1x})^2 + (1 - \delta_{1z})^2 = (\delta_{2x})^2 + (1 - \delta_{2z})^2$$  \hspace{1cm} (A6)

and show that this results in $(1 - \delta_{1z})^2 = (1 - \delta_{2z})^2$.

Let the interface be given by $F(x, z) = 0$. The normal to this surface is

$$n = \frac{\nabla F}{|\nabla F|}. \hspace{1cm} (A7)$$

The vertical position of the origin of the coordinate system is chosen to be at the upstream position of the interface so that the streamline that corresponds to the interface is given by $z_0 = 0$. The displacement of the interface is then determined by

$$\delta = z \quad \text{or} \quad F = z - \delta = 0. \hspace{1cm} (A9)$$

Therefore,

$$\bar{n} = \frac{-\delta_{1x}i + (1 - \delta_{1z})k}{\sqrt{\delta_{1x}^2 + (1 - \delta_{1z})^2}}. \hspace{1cm} (A10)$$

Note that $\delta_2$ can be used in place of $\delta_1$ in (A10) to define $\mathbf{n}$; however, the sense of $\mathbf{n}$ will depend on this choice and hence must be consistent throughout the proof. The normal velocity of the interface is then $\mathbf{u} \cdot \mathbf{n}$. A tangential vector $\mathbf{s}$ may be found using the fact that $\mathbf{n} \cdot \mathbf{s} = 0$, resulting in

$$\mathbf{s} = \frac{(1 - \delta_{1z})i + \delta_{1x}k}{\sqrt{\delta_{1x}^2 + (1 - \delta_{1z})^2}}. \hspace{1cm} (A11)$$

The kinematic boundary condition, $\delta_1 = \delta_2$ at the interface, may be replaced with

$$\frac{\partial \delta_1}{\partial s} = \frac{\partial \delta_2}{\partial s}. \hspace{1cm} (A12)$$

where $s$ is the distance along a streamline. The derivatives in (A12) may be evaluated by taking the $s$ component of the gradient, defined as $\mathbf{s} \cdot \nabla$, resulting in

$$\frac{\partial}{\partial s} = \frac{(1 - \delta_{1z}) \frac{\partial}{\partial x} + \delta_{1x} \frac{\partial}{\partial z}}{\sqrt{\delta_{1x}^2 + (1 - \delta_{1z})^2}}. \hspace{1cm} (A13)$$

Bear in mind that the gradient of $\delta$ does not give the velocity field: $\nabla \delta$ is used here only for geometric purposes. The condition given by (A12) is now

$$(1 - \delta_{1z})\delta_{1x} + \delta_{1x} \delta_{1z} = (1 - \delta_{1z})\delta_{2x} + \delta_{1x} \delta_{2z}$$  \hspace{1cm} (A14)

at the interface. The denominator cancels, and further reduction results in

$$\delta_{1x}(1 - \delta_{2z}) = \delta_{2x}(1 - \delta_{1z}). \hspace{1cm} (A15)$$

This result may be combined with (A6) and rearranged to obtain

$$(1 - \delta_{1z})^2 = (1 - \delta_{2z})^2. \hspace{1cm} (A16)$$

Note that this also means that

$$\delta_{1z}^2 = \delta_{2z}^2. \hspace{1cm} (A17)$$

2) **Reverse Proof**

Start with $\delta_1 = \delta_2$ and $(1 - \delta_{1z})^2 = (1 - \delta_{2z})^2$ and show that this results in

$$(\delta_{1x})^2 + (1 - \delta_{1z})^2 = (\delta_{2x})^2 + (1 - \delta_{2z})^2. \hspace{1cm} (A18)$$

Again, replace $\delta_1 = \delta_{2z}$ with $\delta_1 \delta_{1z} = \delta_2 \delta_{2z}$, with derivatives evaluated as before to obtain

$$\delta_{1x}(1 - \delta_{2z}) = \delta_{2x}(1 - \delta_{1z}). \hspace{1cm} (A19)$$

Square both sides of (A19), use $\delta_{1z}^2 = \delta_{2z}^1$, and simplify to obtain

$$\delta_{1z}^2 = \delta_{2z}^1. \hspace{1cm} (A20)$$

Because $\delta_{1z}^2$ and $\delta_{2z}^2$ are independently continuous, then the quantity $[(\delta_{1z})^2 + (1 - \delta_{1z})^2]$ is also continuous, and the theorem is proven.

REFERENCES


