Finite-Amplitude Wave Activity and Mean Flow Adjustments in the Atmospheric General Circulation. Part II: Analysis in the Isentropic Coordinate

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ABSTRACT

The finite-amplitude wave activity diagnostic developed for quasigeostrophic (QG) flows in Part I is extended to the global primitive equation system in the isentropic coordinate. The Rossby wave activity density $A$ is proportional to Kelvin’s circulation around the wavy potential vorticity (PV) contour minus that around the zonal circle that encloses the same isentropic mass. A quasi-conservative, eddy-free reference state flow $u_{\text{REF}}$ is constructed from the observed Kelvin’s circulation by zonalizing the PV contours conservatively while enforcing gradient balance. The departure of the observed zonal-mean flow of the atmosphere from the reference state $Du$ is defined as the net adjustment by the eddies. Then $Du$ is further partitioned into the direct eddy drag $2A$ and the residual impulse $Du_R$ consistent with the time-integrated transformed Eulerian mean (TEM) zonal-wind equation.

The analyzed climatological-mean wave activity in the 40-yr European Centre for Medium-Range Weather Forecasts Re-Analysis (ERA-40) is similar to that in Part I. The net adjustment $Du$ is mainly due to the direct eddy drag ($2A$) in the winter polar stratosphere and can reach approximately $260 \text{ m s}^{-1}$ in the Northern Hemisphere. In the extratropical troposphere $Du$ is a small residual ($Du_R$), yet it clearly reveals a 5–6 m s$^{-1}$ eddy driving of the Southern Hemisphere jet as well as a 7–8 m s$^{-1}$ eddy drag in the subtropical upper troposphere of both hemispheres. The local maxima in wave activity in the equatorial upper troposphere and the extratropical lower stratosphere found in Part I are undetected, while negative wave activity is found where the isentropes intersect the ground. As in the QG case, $u_{\text{REF}}$ exhibits significantly less transient and interannual variability than $u$, implying a better signal-to-noise ratio as a climate variable.

1. Introduction

The traditional eddy–mean flow interaction theory deals with the tendency of the zonal-mean flow in response to eddy forcing. The theory addresses how and how fast the mean flow is being modified by the eddies, but it does not necessarily answer how much the mean flow is already modified by the eddies at any given time. Quantifying this last aspect was the goal of Nakamura and Solomon (2010, hereafter Part I), who introduced a new diagnostic theory for finite-amplitude eddy–mean flow interaction in the quasigeostrophic (QG) framework based on the formalism of Nakamura and Zhu (2010, hereafter NZ10). Unlike most previous theories based on the Eulerian zonal mean (Andrews and McIntyre 1976; Andrews 1983; Killworth and McIntyre 1985; McIntyre and Shepherd 1987; Haynes 1988), the new formalism adopts a hybrid Eulerian-Lagrangian mean to define the amplitudes of eddies. Specifically, the density of wave activity (negative pseudomomentum) is defined as the net displacement of potential vorticity (PV) substance from zonal symmetry. For example, on the beta plane ($x, y, z$):

$$A_{\text{QG}}(Y, z, t) = \frac{2}{L_x} \left[ \int_{q_{\text{QG}} \geq Q(Y)} q_{\text{QG}} dS - \int_{y \geq Y} q_{\text{QG}} dS \right],$$

where $q_{\text{QG}}(x, y, z, t)$ is QG PV; $L_x$ is the zonal length of the domain; $dS = dx dy dz$; and $Y$ is equivalent latitude (Butchart and Remsberg 1986; Allen and Nakamura 2003) that associates a wavy contour of PV, $q_{\text{QG}} = Q$, to a latitude $y = Y$ such that they delimit the same area on either side. The first surface integral in Eq. (1) is performed over the domain “north” of the PV contour, whereas the

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second integral covers the domain north of latitude \( y = Y \). Here \( Q \) is assumed to be a monotonically increasing function of \( Y \) for a given \( z \) and \( t \). The right-hand side of Eq. (1) is readily evaluated from data by approximating integrals with conditional box counting.

As shown by NZ10, wave activity \( A_{\text{OG}} \) is nonnegative and it reduces to the familiar form \( q_{\text{OG}}^2/(2\partial q_{\text{OG}}/\partial y) \) in the conservative small-amplitude limit, where the overbar and prime denote the zonal mean and departure from it (“eddy”), respectively. Furthermore, \( A_{\text{OG}}(y, z, t) \) satisfies an exact Eliassen–Palm (E–P) relation:

\[
\frac{\partial A_{\text{OG}}}{\partial t} + \bar{v}q_{\text{OG}} = S,
\]

where \( Y \) is replaced by \( y \) as the descriptor of latitude, \( \bar{v}q_{\text{OG}} \) is the meridional eddy PV flux at \( y \), and \( S \) represents nonconservative sources–sinks of wave activity. Unlike the E–P relation for the Eulerian wave activity (e.g. Killworth and McIntyre 1985), Eq. (2) does not involve a cubic term in eddy amplitude. As discussed by Solomon and Nakamura (2011, manuscript submitted to J. Fluid Mech.), there is a close relationship between the above formalism and the generalized Lagrangian mean (GLM) of Andrews and McIntyre (1978a) and McIntyre (1980), but, unlike the GLM pseudomomentum density, \( A_{\text{OG}} \) is readily calculable from data even for eddies with arbitrarily large amplitudes. Moreover, Eq. (2) may be combined with the transformed Eulerian mean (TEM) zonal-wind \( \bar{u} \) equation (Andrews et al. 1987) to yield

\[
\frac{\partial^2}{\partial y^2} + \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \rho_0 e \frac{\partial}{\partial z} \right) \left( \frac{\partial \bar{u}}{\partial t} \right) = \frac{\partial^2}{\partial y^2} \left( \frac{\partial A_{\text{OG}}}{\partial t} - S - X \right) - \frac{\partial^2}{\partial y \partial z} \left( f_0 \bar{\varphi} \right) \frac{\partial}{\partial \theta_0/\partial z},
\]

where \( z \) is pressure pseudoheight; \( \rho_0(z), \theta_0(z), \) and \( e(z) \) are the density, potential temperature, and the square of the Prandtl ratio, respectively, of the background state; \( X \) and \( \bar{\varphi} \) represent the zonal mean frictional forcing and nonadiabatic heating, respectively; and \( f_0 \) is the constant Coriolis parameter. Together with suitable boundary conditions, Eq. (3) represents a finite-amplitude generalization to the Charney–Drazin nonacceleration theorem (Charney and Drazin 1961; Andrews and McIntyre 1978b): if \( S = X = \bar{\varphi} = 0 \) and \( \partial A_{\text{OG}}/\partial t = 0 \), then \( \partial \bar{u}/\partial t = 0 \).

To quantify modification to the mean flow due to eddies, one needs to define an eddy-free reference state. There can be an arbitrary number of such states, but NZ10 chooses a reference state flow \( u_{\text{REF}} \) that satisfies the following:

\[
\frac{\partial^2}{\partial y^2} + \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \rho_0 e \frac{\partial}{\partial z} \right) \left( \frac{\partial u_{\text{REF}}}{\partial t} \right) = \frac{\partial^2}{\partial y^2} \left( \frac{\partial A_{\text{OG}}}{\partial t} \right),
\]

\[
\frac{\partial^2}{\partial y^2} + \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \rho_0 e \frac{\partial}{\partial z} \right) \left( \frac{\partial u_{\text{REF}}}{\partial t} \right) = \frac{\partial^2}{\partial y^2} \left( S + X \right) - \frac{\partial^2}{\partial y \partial z} \left( f_0 \bar{\varphi} \right) \frac{\partial}{\partial \theta_0/\partial z}.
\]

Note that adding Eqs. (4a) and (4b) recovers Eq. (3) and that \( u_{\text{REF}} \) evolves only in response to nonconservative processes. In fact, NZ10 shows that Eq. (4b) is equivalent to

\[
- \frac{\partial}{\partial t} \left( \frac{\partial Q}{\partial y} \right) = - \frac{\partial \bar{Q}}{\partial y},
\]

where \( Q \) is PV in equivalent latitude, and \( \bar{Q} \) is the nonconservative sources–sinks of \( Q \). [See Eqs. (46)–(48) of NZ10; note that the heating term is inadvertently left out of those equations.] \( u_{\text{REF}} \) is invariant with time in the absence of nonconservative processes and that it is invertible from the instantaneous PV make it a fundamental choice for the reference state.

To define eddy-driven adjustment to the mean flow, we consider the time-integrated form of Eq. (4a):

\[
\frac{\partial^2}{\partial y^2} + \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \rho_0 e \frac{\partial}{\partial z} \right) \left( \Delta u \right) = - \frac{\partial^2 A_{\text{OG}}}{\partial y^2},
\]

where \( \Delta u = \bar{u}(t) - u_{\text{REF}}(t) \). If the dynamics is conservative, then \( u_{\text{REF}} \) is constant in time so \( \Delta u \) is the net “adjustment” to \( \bar{u} \) due to finite-amplitude eddies. One can further partition \( \Delta u \) into the direct eddy drag \( -A_{\text{OG}} \) and the residual impulse \( \Delta u_R = A_{\text{OG}} + \Delta u \). The residual impulse \( \Delta u_R \) satisfies

\[
\frac{\partial^2}{\partial y^2} + \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \rho_0 e \frac{\partial}{\partial z} \right) \left( \Delta u_R \right) = \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \rho_0 e \frac{A_{\text{OG}}}{\partial z} \right),
\]

and it is associated with the torque of residual circulation.

Let \( \rho_0 \approx e^{-z/H}, e = f_0^2 N_0^2 (H \text{ and } N_0 \text{ are constant}), \) and consider a Fourier mode of \( \Delta u, \Delta u_R, \) and \( A_{\text{OG}} \) in the form \( \exp\{i[(y/L) + (z/L)]\} \exp(z/2H) \). Substitution in Eqs. (6) and (7) leads to

\[
\Delta u = \frac{1}{1 + L_R^2 L^2} A_{\text{OG}}^\dagger,
\]

\[
\Delta u_R = \frac{1}{1 + L_R^2 L^2} A_{\text{OG}}^\dagger.
\]
where \( L_R = (N_0/f_0)((1/h)^2 + (1/4H^2))^{-1/2} \) is the Rossby radius and the dagger denotes Fourier coefficients. Thus, the projection of \( A_{OG} \) on \( \Delta u \) and \( \Delta u_R \) depends on the ratio of the horizontal scale of wave activity to the Rossby radius: \(|A_{OG}| \approx |\Delta u| \gg |\Delta u_R|\) if \( L \ll L_R \) and \(|A_{OG}| \approx |\Delta u_R| \gg |\Delta u|\) if \( L \gg L_R \). In Part I it is shown that the former is representative of the polar winter stratosphere, whereas the latter is more relevant to the extratropical troposphere.

Equations (6)–(8) hold even when nonconservative processes are present. However in that case \( \Delta u = \overline{u} - u_{REF} \) may not represent the adjustment to \( \overline{u} \) alone, since \( u_{REF} \) may also be altered. In particular, a statistical steady state requires a careful interpretation, as time average (denoted below by the angle bracket) of Eqs. (4b) and (5) consists only of the nonconservative terms:

\[
\frac{\partial^2}{\partial y^2} \langle S \rangle - \frac{\partial^2}{\partial y \partial z} \left( \frac{f_0}{\tau} \nabla^2 \right) \left( \frac{\overline{\psi}}{\tau} \right) - \frac{\partial}{\partial y} \left( \frac{\overline{\theta}}{\tau} \right) = 0.
\]

Assuming that \( \overline{\psi} \) represents relaxation toward radiative equilibrium with a uniform radiative damping time \( \tau_r \), Eq. (9) becomes

\[
\frac{\partial^2}{\partial y^2} \left( f_0 \overline{\psi} \right) - \frac{\partial}{\partial z} \left( \frac{\overline{\theta}}{\tau_r} \right) + \frac{1}{\tau_c \rho_0} \frac{\partial}{\partial z} \left( \rho_0 \overline{e} \frac{\partial}{\partial z} \right) \times \left( u_{REF} - u_e \right) = 0.
\]

where \( \overline{\psi} \) is the meridional component of the residual circulation and \( u_e \) is a time-independent field corresponding to radiative equilibrium [see appendix A for the derivation of Eq. (10)]. A special solution to Eq. (10) is

\[
\langle \overline{\psi} \rangle = \tau_r^{-1} \langle \Delta u_R \rangle, \quad \langle u_{REF} \rangle = u_e, \\
\langle \overline{\psi} \rangle = \langle u_{REF} \rangle + \langle \Delta u \rangle = u_e + \langle \Delta u \rangle
\]

with associated boundary conditions. This limit is realized if the radiative relaxation of \( \overline{u} \) is balanced entirely by the radiative damping of wave activity with no friction or mixing involved (i.e., \( \langle X \rangle = 0 \), \( \langle S \rangle = -\tau_r^{-1} \langle \Delta u_R \rangle \)). In this case \( \langle \Delta u_R \rangle \) is proportional to the meridional component of the residual circulation and \( \langle u_{REF} \rangle \) does not depart from radiative equilibrium. Hence, \( \langle \Delta u \rangle \) represents modification to the zonal-mean flow by the time-mean wave activity (not to be confused with stationary eddies—the time-mean wave activity includes contributions from transient eddies). Even in the presence of surface friction, gravity wave drag, mixing, and the variation in \( \tau \), with height, etc., when Eq. (11) does not hold exactly, \( \langle \Delta u \rangle \) represents nonconservative departure of the zonal-mean flow from the reference state due to finite-amplitude eddies.

In Part I we extended the above formalism to the spherical geometry and applied it to the 40-yr European Centre for Medium-Range Weather Forecasts (ECMWF) Re-Analysis (ERA-40) data (Uppala et al. 2004). The analysis revealed high wave activity associated with extra-tropical synoptic eddies in the troposphere and planetary waves in the winter/spring stratosphere, together with the mean flow modification that they induce. Although the analysis in Part I is global, the QG assumption breaks down in the tropics and the validity of the result in that region is questionable: for example, a pocket of suspiciously large negative \( \Delta u \) was found in the upper equatorial troposphere. In this article, we extend the QG diagnostic of Part I to the more accurate primitive equation system using the isentropic coordinate. While some theoretical advantages of the QG dynamics are lost, the isentropic formulation grants a simpler interpretation of wave activity in terms of Kelvin’s circulation and paints a fully Lagrangian-mean picture of general circulation (Hoskins 1991; Nakamura 1995).

The next section outlines the formalism and discusses the theoretical issues specific to the isentropic coordinate. Section 3 describes the numerical procedures of the diagnostics. Sections 4 and 5 present the analysis of the ERA-40 data and compare the results with the QG analysis in Part I. Section 6 provides a summary.

2. Theory

2a. Layer mass, circulation, and wave activity

In the QG theory, the area demarcated by the PV contour on the pressure surface serves as a quasi-material equivalent latitude. This works because the horizontal winds that advect PV are (to the lowest order in the Rossby number) nondivergent and hence area preserving. Since isentropic winds in the primitive equation system are not divergence free, we instead use layer mass demarcated by the PV contour on isentropic surface as a material meridional coordinate. The layer mass \( M \) (kg K\(^{-1}\)) that resides to the “south” of the instantaneous PV contour \( q = Q \) is

\[
M(Q, \theta, \tau) = \int_{q \leq Q} \sigma dS = \int_{q \leq Q} dM, \quad (12)
\]

where \( \theta \) is potential temperature, \( t \) is time, \( \sigma = -g^{-1} \partial \rho/\partial \theta \) is isentropic density (\( g \) is gravitational acceleration and \( p \) is pressure), \( q = \omega/\sigma \) is Ertel’s PV (\( \omega \) is the vertical component of absolute vorticity). \( Q \) prescribes a contour value of \( q \), and \( dS = a^2 \cos \phi dx d\phi \) is the area element (\( a \) is planetary radius, \( \lambda \) is longitude, \( \phi \) is latitude).

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1 With Eq. (8b) this implies \( \langle S \rangle = -(1/(1 + L_k L^{-1})) (A_{OG}^* / \tau_r) \), so that the damping rate of wave activity varies with its aspect ratio.
There is one-to-one relationship between \( Q \) and \( M \): the minimum \( Q \) (equivalent South Pole) corresponds to \( M = 0 \) and the maximum \( Q \) (equivalent North Pole) corresponds to \( M = M_{\text{max}} \) (the total mass of the layer). Hence, \( M \) may be used as the meridional coordinate for the PV contours. Note that we assume \( \sigma = 0 \) in Eq. (12) where the isentropic layer goes “underground” (Andrews 1983), effectively excluding those massless regions from the integrals. This avoids the difficulty in dealing with infinite \( q \) there but requires a special treatment for the isentropic surfaces that intersect the ground. For the isentropic layer that goes underground in the tropics, we separate the above-ground portion of the layer into regions with positive PV (Northern Hemisphere) and negative PV (Southern Hemisphere) and construct an \( M-Q \) relationship for each region, with \( M \) measured relative to the respective pole (see next section for further details).

Note that \( M \) is a quasi-constant of motion:

\[
\frac{d}{dt} M(Q, \theta, t) = \dot{M},
\]

where \( \dot{M} \) represents nonconservative sources and sinks. If there is no friction or nonadiabatic heating, the bounding surfaces of the layer mass (\( Q \) and \( \theta \)) become material and, hence, \( M \) is invariant (\( \dot{M} = 0 \)).

Another quasi-constant of motion associated with PV contour is Kelvin’s circulation \( C(Q, \theta, t) \):

\[
\frac{d}{dt} C(Q, \theta, t) = \dot{C}.
\]

Notice that there is no baroclinic production of circulation on the isentropic surface, since density becomes only a function of pressure. The source/sink on the right-hand side of Eq. (14) vanishes in the absence of friction and nonadiabatic effects, since in that case the PV contour becomes material (Kelvin’s theorem). From the definition of \( C \)

\[
C(Q, \theta, t) = \oint_{\theta = 0} v_{\theta} \cdot dl = -\int_{q \leq Q(M)} \omega \ dS - \int_{q \geq Q(M)} q \ dM,
\]

where the second identity uses Stokes’s theorem and the minus sign is due to the fact that the integral is defined on the south side of the PV contour. As in Eq. (12), we shall exclude the massless underground region from the integrals in Eq. (15). (This means that parts of the circuit are replaced by the intersection of the isentropic surface and the ground when the PV contour is terminated at the ground.) It is easy to show from Eqs. (12) and (15) that

\[
\frac{\partial C}{\partial Q} \frac{\partial Q}{\dot{M}} = \frac{\partial C}{\dot{M}} |_{\theta} = -\dot{Q}.
\]

where the partial derivatives are defined on the isentropic surface.

It is useful to define analogous quantities with respect to Eulerian latitude. First define the layer mass residing south of latitude \( \phi = \Phi \):

\[
\overline{M}(\Phi, \theta, t) = \int_{\phi \leq \Phi} dM = 2\pi a^2 \int_{-\pi/2}^{\phi} \sigma \cos \phi \ d\phi.
\]

Kelvin’s circulation around the zonal circle at \( \Phi \) is

\[
\overline{C}(\Phi, \theta, t) = -\int_{\phi \leq \Phi} \omega \ dS = -\int_{\phi \leq \Phi} q \ dM,
\]

which may be also written as

\[
\overline{C}(\Phi, \theta, t) = 2\pi a \cos \Phi [\overline{\pi}(\Phi, \theta, t) + \Omega a \cos \Phi],
\]

where \( \overline{\pi} \) is the zonal-mean zonal wind and \( \Omega \) is the angular velocity of the earth’s rotation.

Note that the integrals in Eqs. (17) and (18) are bounded by a zonal circle so that \( \overline{M} \) and \( \overline{C} \) are written completely in terms of the Eulerian-mean quantities. Unlike Eqs. (12) and (15), the integrals in Eqs. (17) and (18) include the underground region when the zonal circle intersects the boundary of the isentrope on the ground. Although this does not affect Eq. (17) since \( \sigma = \dot{M} = 0 \) in the underground region, the infinite \( q \) in the same region makes the contribution to Eq. (18) nonzero: \( q \ dM = \omega \ dS \neq 0 \).

Since both \( \overline{M} \) and \( \overline{M} \) vary monotonically between 0 and \( M_{\text{max}} \) in the isentropic layer, we can uniquely determine mass equivalent latitude \( \Phi_{M} \) at which \( \overline{M}(\Phi_{M}, \theta, t) = M(Q, \theta, t) \) for a given \( Q \). This allows one to specify \( Q \) as a function of \( (\Phi_{M}, \theta, t) \). For the isentropic layer that partially lies underground, the parts overlying the ground are separated into regions with positive and negative PV, and in each region we determine \( \Phi_{M} \) relative to the respective pole. It is not necessary to determine \( \Phi_{M} \) inside the underground region.

We define finite-amplitude Rossby wave activity density at \( \Phi_{M} \) as

\[
A(\Phi_{M}, \theta, t) = \frac{1}{2\pi a \cos \Phi_{M}} \{C[Q(\Phi_{M}, \theta, t)] - \overline{C}(\Phi_{M}, \theta, t)\}.
\]

Note that
\[ C[Q(\Phi_M, \theta, t)] - \overline{C}(\Phi_M, \theta, t) = -\int_{q \leq Q(\Phi_M)} q \, dM + \int_{\phi \leq \Phi_M} q \, dM \]
\[ = \int_{[q \leq Q(\Phi_M)]/(\phi \leq \Phi_M)} q \, dM - \int_{[q \leq Q(\Phi_M)]/(\phi > \Phi_M)} q \, dM. \]  \quad (21)

Since \( q \) in the first of the last two integrals is everywhere greater than \( q \) in the last integral, and since the domain of each integral has the same mass, Eq. (21) is nonnegative. Thus from Eq. (20)
\[ A(\Phi_M, \theta, t) \geq 0, \]  \quad (22)
and \( A \) vanishes if the PV contour coincides with the zonal circle \( \phi = \Phi_M \). The only exception to Eq. (22) is when a wavy isentrope intersects the ground, in which case wave activity near the ground becomes negative as we describe later. In appendix B it is shown that the conservative, small-amplitude limit of \( A \) is \((1/2)\sigma^2(dQ/d\phi)(\Delta \phi)^2\), where \( \Delta \phi(\lambda, \Phi_M, \theta, t) \) denotes the meridional displacement of the PV contour from \( \phi = \Phi_M \).

b. An exact \( E-P \) relation and nonacceleration theorem

Even if \( M(Q, \theta) \) is invariant with time (i.e., when the flow is conservative), the same is not true for \( \overline{M}(\Phi, \theta, t) \) because there can be an Eulerian mass flux through \( \phi = \Phi \). In other words, the mass equivalent latitude \( \varphi = \Phi(M, \theta, t) \) moves with respect to Eulerian latitude. This is essentially the Stokes drift. The rate of change of \( \Phi_M \) due to advection is related to the isentropic mass flux as
\[ \frac{\partial \overline{M}}{\partial \Phi} \frac{d\Phi_M}{dt} = 2\pi a \sigma(\Phi_M, \theta, t) \cos \Phi_M \frac{d\Phi_M}{dt} \]
\[ = 2\pi a \cos \Phi_M \sigma(\Phi_M, \theta, t), \]  \quad (23)
where the first identity uses Eq. (17) and \( v = a \, d\phi/dt \) is the meridional velocity on the isentropic surface. From the last identity
\[ \frac{d\Phi_M}{dt} = \frac{\sigma v}{\sigma} \big|_{\phi = \Phi_M} = \hat{v}^s, \]  \quad (24)
where
\[ \hat{v}^s = \frac{\sigma(\cdot)}{\sigma} \]  \quad (25)
denotes mass-weighted zonal average (Andrews 1983; Tung 1986; Andrews et al. 1987). Taking the time derivative of Eq. (20) times \( \cos \Phi_M \) and using Eq. (14), one obtains
\[ \frac{d(A \cos \Phi_M)}{dt} = -\frac{1}{2\pi a} \frac{d\overline{C}}{dt} + \frac{\dot{\Phi}}{2\pi a}, \]  \quad (26)
where \( d/dt = \partial/\partial t + (\hat{v}^s/a) \partial/\partial \Phi \). Equation (26) represents a finite-amplitude generalization of the nonacceleration theorem (Charney and Drazin 1961; Andrews and McIntyre 1978b): in the absence of nonconservative processes, the zonal circulation \( \overline{C} \) does not change following the movement of equivalent latitude if wave activity density \( A \cos \Phi_M \) does not change. With Eqs. (19) and (24), \( d\overline{C}/dt \) in Eq. (26) may be further rewritten as
\[ \frac{d\overline{C}}{dt} = \frac{\partial \overline{C}}{\partial t} + \frac{v^s}{a} \frac{\partial \overline{C}}{\partial \Phi} = 2\pi a \cos \Phi_M \frac{\partial \overline{C}}{\partial t} \big|_{\phi = \Phi_M} \]
\[ - 2\pi a \cos \Phi_M \hat{v}^s \sigma(\Phi_M, \theta, t), \]  \quad (27)
where
\[ \overline{\sigma}(\Phi_M, \theta, t) = 2\Omega \sin \Phi_M - \frac{1}{a \cos \Phi_M} \frac{\partial (\pi \cos \Phi)}{\partial \Phi} \big|_{\phi = \Phi_M} \]  \quad (28)
was used. Also note that
\[ \frac{\partial \overline{\sigma}}{\partial t} = \overline{\sigma} + \overline{N} = \overline{v^s \sigma} + \overline{\sigma \hat{v}^s} + \overline{N}, \]  \quad (29)
where \( \overline{\cdot} = \langle \cdot \rangle - \langle \cdot \rangle^s \) and \( \overline{N} \) represents zonal-mean friction plus cross-isentropic advection of zonal momentum.\(^2\) Substituting Eq. (29) into Eq. (27) and then into Eq. (26), one obtains
\[ \frac{d(A \cos \Phi_M)}{dt} = -\left( \cos \Phi_M \hat{v}^s \sigma \right) \big|_{\phi = \Phi_M} + \frac{\dot{\Phi}}{2\pi a} - \overline{N} \cos \Phi_M. \]  \quad (30)

\(^2\) An additional term appears on the right-hand side of Eq. (29) if the zonal circle intersects the boundary of the isentropes on the ground, assuming \( u = v = 0 \) on the underground parts of the circle. This term gives rise to the surface heat flux when Eq. (29) is integrated vertically (Schneider 2005; Koh and Plumb 2004).
In the absence of the two nonconservative terms, the material tendency of $A \cos \Phi_M$ is given by the negative of instantaneous eddy PV flux through the Eulerian latitude $\Phi_M$. Equation (30) defines a finite-amplitude E–P relation and corresponds to its QG version [Eq. (2)]. According to Tung (1986)

$$\cos \Phi_M \frac{d}{dt} [\cos \Phi_M (\Pi + \Omega a \cos \Phi_M + A)] = \frac{\partial}{\partial \Phi} \left( \cos^2 \Phi_M (\Pi) \right) + \frac{\partial}{\partial \Pi} \left( \frac{g}{\Phi_M} \cos \phi \frac{\partial \phi}{\partial \phi} - \frac{\partial}{\partial t} (\cos \Phi_M u') \right),$$

(31)

where $\psi$ is the Montgomery streamfunction. The first two terms on the right-hand side of Eq. (31) represent the divergence of the generalized E–P flux, whereas the last term represents exchange of pseudomomentum with gravity waves.

c. Reference state and mean flow adjustment

Following NZ10 and Part I [see Eq. (4) above], we seek a quasi-conservative, eddy-free $u_{\text{REF}}$ that satisfies

$$\frac{d}{dt} \left[ \cos \Phi_M (\Pi + \Omega a \cos \Phi_M + A) \right] = \frac{d}{dt} \cos \Phi_M (u_{\text{REF}} + \Omega a \cos \Phi_M),$$

(32a)

$$\frac{d}{dt} \left[ \cos \Phi_M (u_{\text{REF}} + \Omega a \cos \Phi_M) \right] = \frac{C}{2 \pi a}.$$  

(32b)

Note that Eq. (26) is recovered if one eliminates $u_{\text{REF}}$ from Eqs. (32a) and (32b) and uses Eq. (19). Since from Eqs. (20) and (19)

$$\cos \Phi_M (\Pi + \Omega a \cos \Phi_M + A) = \frac{C (\Phi_M, \theta, t)}{2 \pi a},$$

(33)

the left-hand side of Eq. (32a) is the tendency of Kelvin’s circulation around the wavy PV contour $C$ divided by $2 \pi a$, whereas the right-hand side is the tendency of the corresponding quantity for the “zonalized” PV contour at $\Phi$. The distinction between $\Phi_M$ and $\Phi$ arises from the fact that the zonalization of PV, while conserving mass and circulation, does not necessarily conserve area.

Consider a hypothetical evolution of $[\Pi, A]$ from $[u_{\text{REF}}(t_0), 0]$ to $[\Pi(t), A(t)]$ governed by Eq. (32a). Integrating Eq. (32a) in time from $t_0$ to $t$, one obtains

$$\cos \Phi_M (\Pi(\Phi_M, \theta, t) - \Pi(\Phi_M, \theta, t_0)) + \Omega a (\cos^2 \Phi_M - \cos^2 \Phi) + (\cos \Phi_M)A(\Phi_M, \theta, t) = 0.$$  

(34)

Assuming that $\Delta \Phi_M = \Phi_M - \Phi$ is small, Eq. (34) may be approximated as

$$\Delta u(\Phi, \theta, t) = \Pi(\Phi, \theta, t) - u_{\text{REF}}(\Phi, \theta, t)$$

$$\approx -A(\Phi, \theta, t) + \Delta u_R(\Phi, \theta, t),$$  

(35a)

$$\Delta u_R(\Phi, \theta, t) = a \Delta \Phi_M \cos \Phi_M(\Phi, \theta, t).$$  

(35b)

Equation (35a) corresponds to Eq. (9) of Part I. The left-hand side is the instantaneous departure of $\Pi$ from $u_{\text{REF}}$. As in Part I, we interpret this quantity as a net adjustment to the zonal-mean flow by the observed finite-amplitude eddies. Wave activity on the right-hand side represents direct eddy drag, whereas $\Delta u_R$ represents impulse arising from the torque of the residual circulation. This interpretation is consistent with the time-integrated TEM theory. Note that even if $\Delta \Phi_M$ is small $\Delta u_R$ is not necessarily negligible because of large $\Omega$. For example, for $a \Phi_M = 100$ km and $\Pi = 1 \times 10^{-4} \text{s}^{-1}$ one obtains $\Delta u_R = 10 \text{ m s}^{-1}$, comparable to $\Pi$ in the troposphere.

In the QG case one can readily eliminate $\Delta u_R$ from Eq. (35) and express $\Delta u$ in terms of $A$ [Eq. (6)], then compute $u_{\text{REF}}$ as $\Pi - \Delta u$. No such linear relationship exists between $A$ and $\Delta u$ in the isentropic formalism. To obtain $u_{\text{REF}}$ as a function of latitude and $\theta$, we require that $u_{\text{REF}}$ be in gradient balance (cf. Thuburn and Largau 1999; Magnusdottir and Haynes 1996):

$$f u_{\text{REF}} + u_{\text{REF}}^2 \tan \Phi = -\frac{1}{a} \frac{\partial \psi_{\text{REF}}}{\partial \phi},$$  

(36)

where $f = 2 \Omega \sin \Phi$ is the Coriolis parameter. By taking the derivative of Eq. (36) with respect to $\theta$ twice and using $\partial \psi_{\text{REF}}/\partial \theta = c_p p_{\text{REF}} p_{0}^{\kappa}$ ($c_p$ is specific heat at constant pressure, $p_0 = 1000 \text{ hPa}$, $\kappa = R c_p^{-1}$, $R$ is gas constant, $p_{\text{REF}}$ is the reference-state pressure) and noting

$$C_{\text{REF}} = 2 \pi a \cos \Phi(u_{\text{REF}} + \Omega a \cos \Phi),$$  

(37)

$$\frac{1}{2 \pi a^2 \cos \Phi} \frac{\partial M_{\text{REF}}}{\partial \Phi} = \sigma_{\text{REF}} = -g^{-1} \frac{\partial p_{\text{REF}}}{\partial \theta},$$  

(38)

one obtains

$$\frac{P_0}{2 \pi g R (1 - \mu^2)^2} \frac{\partial^2 C_{\text{REF}}}{\partial \phi^2} = \frac{\partial}{\partial \mu} \left( \frac{P_{\text{REF}}}{P_0} \right)^{\kappa - 1} \frac{\partial M_{\text{REF}}}{\partial \mu},$$  

(39)

where $\mu = \sin \Phi$. Using an alternate form of Eq. (16):

$$\frac{\partial M_{\text{REF}}}{\partial \mu} = -\frac{1}{Q_{\text{REF}}} \frac{\partial C_{\text{REF}}}{\partial \mu},$$  

(40)

Eq. (39) can be cast into the following equation for $C_{\text{REF}}$:
\[
\frac{\partial}{\partial \mu} \left[ \frac{1}{Q_{\text{REF}}} \left( \frac{p_{\text{REF}}}{p_0} \right)^{\frac{1}{\kappa-1}} \frac{\partial C_{\text{REF}}}{\partial \mu} \right]
+ \frac{p_0}{2\pi g R (1 - \mu^2)^{\frac{3}{2}}} \frac{\partial^2 C_{\text{REF}}^2}{\partial \theta^2} = 0.
\] (41)

Thus, once \(Q_{\text{REF}}\) and \(p_{\text{REF}}\) are given (since \(Q_{\text{REF}}\) is obtained by zonalizing the PV contours, it is equal to the observed PV as a function of equivalent latitude), \(u_{\text{REF}}\) may be calculated by solving Eq. (41) for \(C_{\text{REF}}\) and by using Eq. (37). However, in practice it proves difficult to solve Eq. (41) globally because \(Q_{\text{REF}}^{-1}\) (the reciprocal of PV) diverges across the equator. An alternative method for calculating \(u_{\text{REF}}\) is described in the next section.

3. Data and numerical procedures

As in Part I, we use the 23-yr segment (1979–2001) of the daily (1200 UTC) ERA-40 reanalysis (Uppala et al. 2004) for the following diagnostic study. The dataset covers the troposphere and stratosphere with 23 pressure levels (1000–1 hPa) and uniform longitude–latitude grids with a resolution of 2.5°. [Only the zonal and meridional wind components (\(u, v\)) and temperature are used.]

a. Vertical interpolation

First we interpolate \(u\) and \(v\) from ERA-40’s pressure surfaces to a set of prescribed isentropic surfaces using a simple linear interpolation at each longitude–latitude grid. (More sophisticated methods such as cubic spline did not improve the results because of highly inhomogeneous nature of the data.) Absolute vorticity \(\omega\) is then computed on each isentropic surface from the interpolated winds with finite differencing. The isentropic density \(\sigma\) is calculated first in the original pressure levels by finite differencing and then interpolated to isentropic surfaces. (Performing the finite difference before the interpolation reduces numerical noise.) To allow the isentropes to sample the atmosphere approximately evenly in height, we introduce a pseudoheight \(z(\theta)\) as follows:

\[
\theta(z) = [1 - W(z)](\theta_0 + Az) + W(z)\theta_1 \exp[\alpha(z - z_1)],
\] (42a)

\[
W(z) = 0.5[1 + \tanh((z - z_1)/d)],
\] (42b)

\[
\theta_0 = 288\text{ K}, \quad \theta_1 = 320\text{ K}, \quad z_1 = 9\text{ km},
\] (42c)

\[
d = 1\text{ km}, \quad \Lambda = 3\text{ K km}^{-1}, \quad \alpha = 0.044\text{ km}^{-1}.
\] (42c)

The \(\theta-z\) relation in Eq. (42) mimics the crossover of linear and exponential potential temperature profiles in height between the troposphere and the stratosphere. It is important to include sufficiently low levels of potential temperature to resolve the surface boundary. The top isentrope should remain within the data domain throughout the analysis period. For these reasons, we choose our vertical domain as:

\[
z_B = -25 \leq z \leq 40\text{ km} = z_T \quad \text{or}
\]

\[
\theta_B = 213 \leq \theta \leq 1252\text{ K} = \theta_T.
\] (43)

Note that \(z = 0\) does not coincide with the ground due to baroclinicity of the flow: in fact \(z\) at the surface is negative in the polar region and positive in the equatorial region. We specify isentropic levels with equal spacing in \(z\) with a resolution of 500 m. The corresponding resolution in \(\theta\) is about 1.5 K near the surface and about 55 K at the top of the domain. Where the grid point lies below the surface (i.e., \(\theta\) falls below the temperature at 1000 hPa), winds are assumed to be zero and pressure is set to 1000 hPa, thus \(\omega = f\) and \(\sigma = 0\), following Schneider (2005). On all grids above the surface, PV is computed as \(q = \omega \sigma^{-1}\).

b. Wave activity

To compute wave activity according to Eq. (20), we must first compute \(C[Q(\Phi_M, \theta, t)]\) and \(C(\Phi_M, \theta, t)\). On each isentropic level, we establish \(M(Q)\) and \(C(Q)\) by evaluating Eqs. (12) and (15) with area-weighted box counting. We use 90 equally spaced PV bins between the maximum and minimum values on each surface. In doing so, isolated points at which \(\sigma < 0\) are excluded as they tend to contribute to spurious maxima and minima in PV. When parts of an isentropic surface lie underground, the grid points above ground are separated into two groups according to the sign of \(q (q \geq 0\) and \(q < 0\), and \(M(Q)\) and \(C(Q)\) are constructed for each group. [For \(q \geq 0\) (i.e., the “Northern Hemisphere”) it is convenient to reverse the inequality in Eqs. (12) and (15) (and also the sign of \(C\) so that \(M\) and \(C\) are defined relative to the “North Pole.”]

Next, at each data latitude \(\Phi\), we evaluate \(M(\Phi, \theta, t)\) and \(C(\Phi, \theta, t)\) according to Eqs. (17) and (19). Even if the isentropic surface lies underground in parts of the zonal circle where \(u = v = \sigma = 0\), averaging is performed over the entire zonal circle. Then from \(M(Q)\) and \(C(Q)\) constructed earlier, we find the PV contour \(Q\) for which \(M = M\) and compute the corresponding \(C = C(Q(\Phi))\) by interpolation. The difference between \(C[Q(\Phi)]\) and \(C(\Phi)\) is used to calculate wave activity at \(\Phi\) according to Eq. (20).

c. Reference state

To calculate \(u_{\text{REF}}\) from Eq. (39), first the equation is rewritten using \(z\):

\[
\frac{\partial}{\partial \mu} \left[ \frac{1}{Q_{\text{REF}}} \left( \frac{p_{\text{REF}}}{p_0} \right)^{\frac{1}{\kappa-1}} \frac{\partial C_{\text{REF}}}{\partial \mu} \right]
+ \frac{p_0}{2\pi g R (1 - \mu^2)^{\frac{3}{2}}} \frac{\partial^2 C_{\text{REF}}^2}{\partial \theta^2} = 0.
\] (41)
for $\Delta M$ using the direct method due to Lindzen and Kuo (1969) with the boundary conditions:
\[ \Delta M = 0 \text{ at } \mu = \pm 1, \quad z = z_T, z_B. \] (48)

Note that the boundary condition at $z = z_B$ is purely computational: we also need a physical boundary condition at the surface. In Part I, a reference state that closely traces the actual climate is obtained with a no-slip boundary condition. This boundary condition is difficult to impose in the present problem because the location of the boundary in the $(\mu, z)$ space (or the surface temperature distribution in the reference state) is not known a priori. Instead, we prescribe the surface temperature of the reference state to be the zonal-mean surface temperature of the observed state. The choice is based on the assumption that the amplitude of the surface temperature perturbation remains reasonably small due to thermal damping, so the observed zonal mean does not depart too far from the reference state. This allows us to specify the location of the surface in the $(\mu, z)$ space. Then we require $Q = 0, C = 2\pi\Omega^2 \cos^2\phi$ at the grid points below the surface. Although this might seem contradictory to $Q = 0$ when the layer mass vanishes, Eq. (46b) makes it clear that in the linearized problem $Q = 0$ is equivalent to $\Delta C = 0$, consistent with the prescribed $C (\mu = v = 0)$. This keeps the surface wind in the reference state close to zero. The obtained $\Delta M$ is used to update $M_{REF}$ and $C_{REF}$ with Eq. (46). Potential vorticity is then updated by taking the finite difference of the obtained above. We assume $p_{REF} = \overline{p}$, consistent with Eq. (45a). Since $M_0$ and $C_0$ do not satisfy Eq. (44) exactly, we introduce correction terms:
\[ M_{REF} = M_0 + \Delta M, \] (46a)
\[ C_{REF} = C_0 + \Delta C \approx C_0 + \frac{\partial C_0}{\partial M_0} \bigg|_{\theta} \Delta M = C_0 - Q_0 \Delta M, \] (46b)
where the last identity in Eq. (46b) uses Eq. (16) and $Q_0 = Q(M_0, z, t)$. By substituting Eq. (46) into Eq. (44) and by ignoring terms on the order of $(\Delta M)^2$, one obtains
\[ \frac{\partial}{\partial \mu} \left[ \left( \frac{p_{REF}}{p_0} \right)^{\kappa-1} \frac{\partial M_{REF}}{\partial \mu} \right] + \frac{p_0}{gR} \left[ \frac{\mu}{(1 - \mu^2)^2 \gamma(z)} \frac{\partial}{\partial z} \left( C_0 Q_0 \Delta M \right) \right] \]
\[ = - \frac{\partial}{\partial \mu} \left[ \left( \frac{p_{REF}}{p_0} \right)^{\kappa-1} \frac{\partial M_0}{\partial \mu} \right] + \frac{p_0}{2\pi R} \left[ \frac{\mu}{(1 - \mu^2)^2 \gamma(z)} \frac{\partial}{\partial z} \left( C_0 \Delta M \right) \right]. \] (47)

We solve the finite-difference version of this linear PDE for $\Delta M$ using the direct method due to Lindzen and Kuo (1969) with the boundary conditions:

4. Wave activity in ERA-40

a. Seasonal climatology

Figures 1a–d depict the seasonal climatology of wave activity density $A$ as a function of latitude and pseudo-height. These figures correspond to Figs. 2a–d of Part I. The upper white curves in each panel indicate the approximate locations of the extratropical tropopause [$\pm 2.5$ potential vorticity unit (PVU) contours of Ertel’s PV ($1$ PVU = $10^{-6} \text{ m}^2 \text{s}^{-1} \text{ kg}^{-1}$)], whereas the lower white curve defines the climatological-mean, zonal-mean surface temperature. Since in the presence of eddies some air parcels along a latitude circle carries potential temperatures lower than the zonal-mean value, a small amount of mass exists below the mean surface and we include
this sublayer in the analysis. The region not sampled by 
the atmosphere at all ($s = 0$) is masked out in white.

The structure of wave activity is similar to the QG 
analysis in Part I. In the stratosphere, the amplitude is 
large in the Northern Hemisphere winter, while it is 
smaller but significant in the Southern Hemisphere spring. 
The large, persistent tropospheric wave activity in the 
extratropics is also familiar and it is associated with mobile 
baroclinic eddies. Noticeable differences from Part I in-
clude 1) the present result shows no maximum in the 
equatorial upper troposphere; and 2) the summertime 
maxima at the top of the lowermost stratosphere (LMS), 
a rather unexpected finding in Part I, are also absent.

Since the QG approximation breaks down at low lati-
tudes, the absence of an equatorial maximum in the pres-
ent study strongly suggests that such feature in the QG 
analysis is spurious. Besides, an equatorial maximum re-
quires that the PV contours cross the equator—an inertially 
unstable situation. On the other hand, the discrepancy in 
wave activity in the summer LMS (where the QG as-
sumption should be valid) is more curious. In Part I, the 
maxima are attributed to the weak meridional gradients 
of QG PV (Fig. 3 of Part I). However, the gradients of 
Ertel PV in the present study are also small in the same 
region yet wave activity does not exhibit maxima. Figure 2 
shows the climatology of the meridional gradients of PV 
in equivalent latitude multiplied by $\sigma$ (a quantity analogous 
to the gradients of QG PV in the pressure coordinate). 
The PV gradients are maximal along the extratropical 
tropopause and in the polar troposphere, as well as at the 
edge of the winter stratospheric vortices. There is also 
a weak maximum along the equator in the stratosphere. 
In contrast, the gradients are weak in the equatorial tro-
posphere and summer stratosphere, particularly the 
extratropics of the LMS. Like the QG case, PV in the 
extratropical LMS is well stirred, but the variance of eddy 
PV in the present analysis is much smaller and thus fails 
to generate significant wave activity. The discrepancy can 
be a result of subtle difference between the isentropic and 
isobaric analyses if the layer of wave activity is very thin 
as suggested in Part I, but at least it calls into question the 
robustness of such feature against vertical resolution and 
interpolation of data.

In addition to the positive wave activity in the interior, 
there is a layer of negative wave activity in the sub-
surface layer of Fig. 1. The negative wave activity arises 
where a wavy isentrope intersects the ground. To see 
why this is the case, consider the lowest limiting latitude 
of the isentropic layer where $\sigma$ vanishes, namely the 
edge of the white area in Fig. 1. At this latitude the zonal 
circle circumscribes the wavy isentrope on the ground 
(Fig. 3a), and it encloses an isentropic mass $\mathcal{M}$ equal to 
the total mass of the isentropic layer $M_{\text{total}}$ because the 
shaded area outside the wavy boundary has no mass. Yet 
Kelvin’s circulation around the latitude circle $\mathcal{C}$ is 
greater than the circulation around the wavy boundary 
$C$, because the latitude of the boundary is higher than 
the zonal circle everywhere except where they touch.
each other. The difference between $C$ and $\overline{C}$ is the surface integral of the Coriolis parameter over the shaded area. Hence using Eq. (20), $A < 0$.\(^3\)

Wave activity takes a minimum (the greatest negative) value at the limiting latitude of an isentropic layer. (In Fig. 1, the minimum appears halfway between the mean latitude of the surface and the limiting latitude; this is because the latter fluctuates and the time averaging blurs its location.) For a higher latitude ($\overline{M} < M_{\text{total}}$) the zonal circle can either cross the wavy boundary of the isentrope or resides completely inside of it. In the latter situation (Fig. 3b) the circulation around the zonal circle is greater than that around the PV contour enclosing the same mass just as the standard case, so the wave activity is positive. This implies that wave activity vanishes at some latitude whose circle crosses the boundary of the isentrope at the ground.

The negative wave activity is analogous to the surface wave activity in the QG formalism associated with the decreasing surface temperature with latitude (Shepherd 1989; NZ10). That its sign is opposite from the interior wave activity is a necessary condition for baroclinic instability (Pedlosky 1964; Charney and Stern 1962; Shepherd 1989; NZ10). In Fig. 1 the strongest negative wave activity is found in the high latitudes of northern winter where both the surface temperature gradients and eddy forcing are strong. However, unlike the interior wave activity, the negative surface wave activity exists through all latitudes and even seeps into the interior in

\(^3\) The negative wave activity may be eliminated, as one of the reviewers pointed out, if we exclude the underground region from the definition of $\overline{C}$ in Eq. (18). However, doing so requires Lagrangian information of wavy isentropes on the ground, and $\overline{C}$ will no longer be an Eulerian-mean quantity. The distinct sign of the near-surface wave activity arises from an additional term in Eq. (29) when parts of the zonal circle reside underground ($\mu = \nu = \sigma = 0$; Schneider 2005). When Eq. (29) is integrated vertically this term becomes approximately proportional to the surface heat flux, analogous to the QG formalism. Hence, the negative wave activity is analogous to the surface wave activity in the QG case.
the (sub)tropics of the Northern Hemisphere during summer. Careful examination reveals that very high temperatures over summer continents cause isentropic surfaces to plunge to the ground locally, bringing negative wave activity above the zonal-mean surface temperature.

b. Daily wave activity at high latitudes

Figure 4 depicts daily wave activity at 65°N and 65°S between 1996 and 2001 (see also Fig. 4 of Part I). The seasonal and interannual variability in the stratospheric wave activity closely follow those of the OG analysis, as do the higher-frequency fluctuations in the tropospheric wave activity. In addition, Fig. 4 reveals the annual cycle in the surface temperature and the negative wave activity in the subsurface region. During winter the surface temperatures are comparably low at both locations but in summer they are significantly higher at 65°N than at 65°S. As a result of this, the tropopause-to-surface potential temperature difference (or the mean tropospheric static stability) during summer is significantly less at 65°N. Negative wave activity below the mean surface grows stronger during winter, with the amplitude at 65°N appreciably greater than at 65°S. Although during summer a thin layer of negative wave activity still remains at 65°N, it almost disappears at 65°S.

5. Mean flow modification and reference state in ERA-40

To evaluate the relative importance of the terms in Eq. (35a), we show in Fig. 5 the seasonal climatologies of $\pi$, $u_{\text{REF}}$, $\Delta u$, and $\Delta u_R$ in the meridional plane. The last quantity is computed as a residual in Eq. (35a) (i.e., $\Delta u_R = \pi + A - u_{\text{REF}}$).

As in the OG analysis with the no-slip lower boundary condition in Part I, $\bar{u}$ and $u_{\text{REF}}$ share salient features (Figs. 5a–h). Their difference $\Delta u$ reveals the structure of the net adjustment to the mean flow due to eddies (Figs. 5i–l). A very large negative $\Delta u$ (up to $-60$ m s$^{-1}$) dominates high latitudes of the Northern Hemisphere stratosphere during winter (and to a lesser extent, during fall and spring), reflecting the frequent deceleration associated with sudden warming events. A similar pattern is observed in the stratosphere of the Southern Hemisphere but it is much weaker (up to $-30$ m s$^{-1}$) and significant only during winter and spring. As in Part I, $\Delta u$ in the stratosphere closely echoes $-A$ (Fig. 1); that is, the left-hand side of Eq. (35a) is balanced mostly by the first term on the right-hand side.

In the extratropical troposphere, there is a marked interhemispheric asymmetry in $\Delta u$. A nearly barotropic, positive tower exists around 30°–60°S throughout the year, reaching 5–6 m s$^{-1}$ at the surface and the tropopause. This feature is sandwiched by zones of negative $\Delta u$ ($-7$–$8$ m s$^{-1}$) on both sides. The pattern is consistent with the arrangement of the zonal-mean momentum by baroclinic eddies (Simmons and Hoskins 1978; Held and Hoskins 1985). According to Fig. 4, the wave forcing in the troposphere is indeed dominated by transient synoptic events. The positive $\Delta u$ clearly provides an evidence for an eddy driving of the extratropical jet. However in the Northern Hemisphere, such evidence is weak at best: $\Delta u$ is barely positive at the surface and at the core of the subtropical jet but otherwise negative. This asymmetry is probably due to the greater surface friction in the Northern Hemisphere preventing the eddies from accelerating the surface winds.

The main difference in $\Delta u$ from the OG analysis is found in the tropics. In Part I, a pocket of very large negative $\Delta u$ was identified in the equatorial upper troposphere, related to the local maximum in wave activity in the same region. As discussed earlier, the equatorial wave activity is absent in the present study, which eliminates the spurious negative peak in $\Delta u$. Although $\Delta u$ is still negative in the equatorial upper troposphere, the local minima in the region are located in the subtropics (20°–30° off the equator) in both hemispheres ($-7$–$8$ m s$^{-1}$). These are the regions in which the transient eddies emanating from the storm tracks encounter critical latitudes, and the negative $\Delta u$ is consistent with the deceleration of the flow by the breaking waves (Edmon et al. 1980; Held and Hoskins 1985).

There are also tongues of positive $\Delta u$ on both sides of the equator in the tropical lower troposphere, which are absent in the OG analysis. These tongues stem from the similar structures in $u_{\text{REF}}$ with opposite sign (Figs. 5e–h), which in turn reflect the negative wave activity that occupies the same region (Fig. 1). As pointed out earlier, the negative wave activity in the tropics arises from strong heating over continents. Therefore, this feature is created nonadiabatically, primarily through $C/(2\pi a)$ in Eqs. (30) and (32b).

The structure of $\Delta u_R$ in the troposphere is very similar to that of $A$ in Fig. 1 (Figs. 5m–p). This means that the predominant balance in Eq. (35a) is between the two right-hand-side terms and $\Delta u$ is a small residual in the troposphere. Indeed, $\Delta u_R$ and $A$ exceed 100 m s$^{-1}$ near the tropopause, whereas $|\Delta u|$ in the troposphere is on the order of 10 m s$^{-1}$ or less. In contrast, $\Delta u_R$ is generally smaller than $A$ or $\Delta u$ in the stratosphere. As discussed in section 1 and also previously by Pfeffer (1987), the difference is due to the difference in the aspect ratio of wave activity between the two regions.

If friction is negligible, the time-averaged $\Delta u_R$ is proportional to the meridional component of the residual circulation [Eq. (11)]. Therefore the predominantly
FIG. 4. Daily wave activity during 1996–2001 of ERA-40 at (left) 65°N and (right) 65°S as functions of time and height. The abscissa spans one calendar year from 1 January to 31 December. The upper and lower white curves denote the locations of the 2.5 PVU (−2.5 PVU at 65°S) contour and the zonal-mean surface potential temperature, respectively. The region that has zero zonal-mean mass is filled in white. The contour interval is 15 m s$^{-1}$. This figure is to be compared with Fig. 4 of Part I.
Seasonal climatology of $\pi$, $u_{REF}$, $\Delta u$, and $\Delta u_R$ in ERA-40 (1979–2001) as functions of latitude and height: (a)–(d) $\bar{u}$, contour interval is 5 m s$^{-1}$ and negative values are dashed; (e)–(h) $u_{REF}$, contour interval is 5 m s$^{-1}$; (i)–(l) $\Delta u$, contour interval is 2 m s$^{-1}$; and (m)–(p) $\Delta u_R$, contour interval is 15 m s$^{-1}$. (from left to right) December–February, March–May, June–August, and September–November. Regions below the zonal-mean surface temperature are filled in white.
positive $\Delta u_R$ in the troposphere suggests that there is a significant poleward drift of air associated with eddies. On the other hand, $\Delta u_R$ does not include a response to a zonally symmetric forcing such as the Hadley circulation.

Figure 6 shows time–height cross sections of $\bar{u}$ and $u_{\text{REF}}$ for the year 1996 in ERA-40. The contour interval is 10 m s$^{-1}$ and the negative values are dashed. (a) $\bar{u}$ at 65°N. (b) $u_{\text{REF}}$ at 65°N. (c) $\bar{u}$ at 65°S. (d) $u_{\text{REF}}$ at 65°S. This figure is to be compared with Fig. 8 of Part I.

Figure 7 shows daily $\bar{u}$ and $u_{\text{REF}}$ at 65°N, $z = 25$ km (Arctic lower stratosphere), both $\bar{u}$ and $u_{\text{REF}}$ show large interannual variability during winter months, but the variability of $u_{\text{REF}}$ is appreciably less. Also $u_{\text{REF}}$ shows a much greater seasonal variation. The mean standard deviations about the seasonal climatology, computed from the 23 samples for each day and averaged over November–April, are 12.1 m s$^{-1}$ for $\bar{u}$ and 9.8 m s$^{-1}$ for $u_{\text{REF}}$. The Antarctic counterparts (65°S, $z = 25$ km, May–October, figure not shown) are 5.7 m s$^{-1}$ for $\bar{u}$ and 4.2 m s$^{-1}$ for $u_{\text{REF}}$. At 55°S, $z = 9$ km (extratropical storm track), the seasonal variability is weak but day-to-day fluctuation associated with synoptic eddies exists throughout the year. The amplitude of fluctuation is again much less in $u_{\text{REF}}$ than in $\bar{u}$, whereas the annual mean $\bar{u}$ is significantly higher than $u_{\text{REF}}$ as we have seen in Fig. 5. The annual-mean climatological-mean wind and the annual-mean standard deviation based on the interannual variability are 24.9 ± 4.7 m s$^{-1}$ for $\bar{u}$ and 19.4 ± 2.5 m s$^{-1}$ for $u_{\text{REF}}$. At 55°N, $z = 9$ km, the above comparison becomes 12.8 ± 4.5 m s$^{-1}$ for $\bar{u}$ and 16.9 ± 2.1 m s$^{-1}$ for $u_{\text{REF}}$. While here the mean $\bar{u}$ is smaller than $u_{\text{REF}}$, its standard deviation is more than twice as large.

These results reinforce our argument in Part I: $u_{\text{REF}}$ eliminates much of the fluctuation associated with
advective effects of eddies and hence likely possesses a better signal-to-noise ratio as a climate variable.

6. Summary

The finite-amplitude wave activity diagnostic developed in Part I for QG flows has been extended to the primitive equation system using the isentropic coordinate. The isentropic formulation (like the barotropic theory of NZ10) allows wave activity density to be written in terms of Kelvin’s circulation around an instantaneous PV contour minus that around the zonal circle at equivalent latitude. The finite-amplitude nonacceleration theorem arises naturally with the application of Kelvin’s theorem. The formalism allows one to integrate the TEM set in time and quantify the instantaneous departure of the observed state of the atmosphere from an eddy-free reference state. This eliminates “fluxes” from the diagnostic: the E–P flux and its divergence are implicit and not diagnosed at all. The reference-state flow $u_{\text{REF}}$ is a balanced zonal flow that would ensue if one removed wave activity from the observed state, while conserving mass and circulation about the PV contour and requiring gradient balance. This entails the solution of a nonlinear elliptic equation, which is linearized and solved by iteration with a prescribed surface temperature profile. The distribution of isentropic mass and circulation with respect to PV is identical in the observed and reference states, so the latter is tightly constrained to the actual climate, albeit eddy free.

Overall, the isentropic analysis reinforces the results of the QG isobaric analysis. The structure of wave activity density diagnosed from the ERA-40 reanalysis is similar in both analyses. Eddies directly alter the mean flow in the stratosphere, whereas they mainly drive the residual circulation in the troposphere. The derived reference state varies much less than $u$ both at synoptic and interannual time scales, suggesting that it has a better signal-to-noise ratio as a climate variable.

Some features identified in Part I are absent, including the wave activity maxima in the upper equatorial troposphere and at the top of the LMS during summer. In addition, the present diagnosis reveals a layer of negative wave activity in the region where isentropes intersect with the ground. This last feature is analogous to the surface wave activity in the QG analysis associated with temperature anomalies, and it is strongest during the winter extratropics.

Since there is no requirement for a small Rossby number, the present analysis affords a greater accuracy.
than the QG analysis particularly at low latitudes. Indeed, the reference state in the tropical upper troposphere is visibly improved from Part I: instead of straddling the equator, maximum negative values of }\Delta u\text{ in the upper troposphere are now found in the subtropics in both hemispheres, consistent with the breaking of synoptic waves at critical latitudes (Edmon et al. 1980; Held and Hoskins 1985). Of course the added accuracy comes at a cost. Since the inversion relation [Eq. (44)] is nonlinear, the numerical procedure is more involved than the QG case, and the theoretical advantage of the linear dependence of }\Delta u\text{ on }A\text{ is lost. However, the overall similarity of the present analysis to the QG analysis in the extratropics suggests that the effect of nonlinearity is sufficiently weak.

Regardless of which analysis one adopts, the strength of the finite-amplitude wave activity diagnostic is that it quantifies the net adjustment by the eddies without calculating fluxes. For example, it shows not only that the relationship between the variabilities }\Delta u\text{ and }u_r\text{ using reanalysis data in an attempt to shed further light on the dynamics of annular modes; and the hydrodynamic stability of }u_{\text{REF}}\text{.

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APPENDIX A

Derivation of Eq. (10)

The time-averaged QG TEM equation for the zonal mean zonal wind is

\[
0 = f_0\langle \bar{\mathbf{v}}^y \rangle + (u'q'_{\text{QG}}) + \langle \mathbf{S} \rangle. \tag{A1}
\]

Combining Eq. (A1) with the time average of Eq. (5) we have

\[
f_0\langle \bar{\mathbf{v}}^y \rangle = -\langle u'q'_{\text{QG}} \rangle - \langle \mathbf{S} \rangle = -\langle S \rangle - \langle \mathbf{X} \rangle. \tag{A2}
\]

We substitute Eq. (A2) in Eq. (9) and assume that the heating term in Eq. (9) represents relaxation toward radiative equilibrium. This leads to

\[
\frac{\partial^2}{\partial y^2}(f_0\langle \bar{\mathbf{v}}^y \rangle) + \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \frac{\rho_0 \beta}{\tau_r} \frac{\partial}{\partial z} (\langle \mathbf{u} \rangle - u_r) \right) = 0, \tag{A3}
\]

where the thermal wind balance,

\[
f_0\frac{\partial u}{\partial z} = -g \frac{\partial \theta}{\partial y}
\]

was used. If }\tau_r\text{ is uniform, by substituting }<\mathbf{u}>_r\text{ in Eq. (A3) and by using Eq. (6),

\[
\frac{\partial^2}{\partial y^2} \left( f_0 \langle \bar{\mathbf{v}}^y \rangle \right) - \frac{\langle A_{\text{QG}} \rangle}{\tau_r} - \frac{\langle \Delta u \rangle}{\tau_r} + \frac{1}{\tau_r \rho_0} \frac{\partial}{\partial z} \left( \rho_0 \beta \frac{\partial}{\partial z} (\langle u_{\text{REF}} \rangle - u_r) \right) = 0. \tag{A4}
\]

Since }\Delta u_R = \Delta u + A_{\text{QG}}\text{ by definition, Eq. (10) follows immediately.

APPENDIX B

Conservative, Small-Amplitude Limit of }A

To derive the conservative, small-amplitude form of wave activity density }A\text{, first consider a zonally symmetric state in which a PV contour is specified as }q(\lambda, M, t) = Q(M),\text{ where the layer mass }M\text{ is used as the meridional coordinate. Suppose this contour is slightly displaced in a conservative fashion to }q(\lambda, M + \Delta M, t + \Delta t),\text{ where }\Delta M(\lambda)\text{ is the meridional displacement of the contour and }\Delta M = 0\text{, so that the contour encloses the same }M\text{ before and after the displacement. Let Kelvin’s circulation about the }q\text{ contour after the displacement be }C(M),\text{ and the circulation about the original zonal circle but at time }t + \Delta t, C(\Phi_{\lambda t}).\text{ The difference between }C\text{ and }\overline{C}\text{ is given by Eq. (21), and because of the small-amplitude assumption,

\[
C - \overline{C} = -\int_M^{M + \Delta M(\lambda)} q(\lambda, M') dM' - \int_0^{\Delta M(\lambda)} [q(\lambda, M + m) - Q(M)] dm - Q(M)\Delta M
\]

\[
-\int_0^{\Delta M(\lambda)} \frac{dQ}{dM} m dm \approx \frac{1}{2\Delta M} \frac{dQ}{dM} \Delta M^2. \tag{B1}
\]

From the first to the second line, we have used }\overline{\Delta M} = 0\text{ and }q(M + m) - Q(M) \approx -(dQ/dM)m.\text{ Substitute

\[
\Delta M \approx 2\pi a^2 \cos \Phi_M \sigma \Delta \phi, \tag{B2}
\]
where $\Delta \phi$ is the meridional displacement of the contour in latitude (radians) and

$$
\frac{dQ}{dM} = \frac{1}{2\pi a^2 \cos \Phi_M \sigma} \frac{dQ}{d\phi}.
$$

(B3)

And Eq. (B1) becomes

$$
C - \mathcal{C} \approx 2\pi a^2 \cos \Phi_M \sigma \frac{dQ}{d\phi} \frac{(\Delta \phi)^2}{2}.
$$

(B4)

Substituting Eq. (B4) in Eq. (20), we obtain the conservative, small-amplitude form of wave activity density:

$$
A \approx \frac{1}{2} a \sigma \frac{dQ}{d\phi} (\Delta \phi)^2.
$$

(B5)

Note that this result does not hold when parts or all of the zonal circle reside in the massless layer outside the boundary of the isentropic surface. In that case the last term in the first line of Eq. (B1) does not necessarily vanish even if $\Delta M = 0$ because $|Q| = \infty$ in the massless layer. For example, in the situation depicted in Fig. 3a, $\Delta M = 2\pi a^2 \cos \Phi_M \sigma \Delta \phi = 0$ because $\sigma = 0$, but $\Delta \phi \neq 0$ ($\Delta \phi$ depicts the separation of the zonal circle and the boundary of the isentropic surface) and $-Q \Delta M = -4\pi a^2 \Omega \sin \Phi_M \cos \Phi_M \Delta \phi$. Thus, in this case the leading order of small-amplitude wave activity becomes linear in $\Delta \phi$, namely, $A \approx -2\Omega \sin \Phi_M a \Delta \phi$.

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