A Compact Model for the Stability Dependency of TKE Production–Destruction–Conversion Terms Valid for the Whole Range of Richardson Numbers

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ABSTRACT

Recently, observational and numerical evidence has accumulated against the concept of a critical Richardson number $Ri_c$, beyond which too-stable stratification would extinguish turbulence. It also appeared that the characteristics of the “weak turbulent regime” where the Prandtl number $\sigma_t$ increases proportionally to the Richardson number $Ri$ can be explained via the conservation of total turbulent energy in a strongly anisotropic flow. Having a “No $Ri_c$” situation together with due consideration of the anisotropy thus leads to the correct asymptotic behavior at high stabilities in several recent proposals [revisit of the Mellor–Yamada basic system, non-Reynolds-type quasi-normal scale elimination (QNSE) theory, and energy and flux budget (EFB) theory leading to a fully self-consistent hierarchy of increasingly prognostic schemes]. The present work derives a simple unique analytical framework for these various alternatives, simplifying, in two complementary but surprisingly converging ways, the revisited Mellor–Yamada formulation and emulating with high accuracy the relevant solutions within QNSE and EFB. The simplification or emulation steps differ from one case to the next, but the obtained common framework is very compact, valid for $Ri$ going from $2'$ to $1'$, depending only on four free parameters and on three “functional dependencies.” Each functional dependency corresponds either to a constant value or to a regular function of the flux Richardson number $Ri_f$ depending on the complexity of the considered hypotheses. Four realizations of this codification are representative of all related possibilities, the analytical scheme thus possessing high transversal validity. Extension toward higher-order solutions and/or moist turbulence can be envisaged in such a unified framework.

1. Introduction

a. Recent advances in the study of dry turbulence

The study of dry-air turbulence has made big progress in the past 13 years or so with recognitions in various scientific interests, namely (among others, less relevant here)

(i) Formulations for surface drag coefficients and stability dependency functions [linking turbulent kinetic energy (TKE) and vertical exchange coefficients] can be made compatible across the whole range of possible stabilities (without critical Richardson number $Ri_c$, on the stable side), when using the equilibrium solution in the analytical derivation of the exchange coefficients of a system with prognostic TKE (Redelsperger et al. 2001, hereafter RMC01).

(ii) A more complex version of the pressure correlation formulations may be used (Lauder et al. 1975) in the level-2.5 system of Mellor and Yamada (MY; Mellor 1973; Mellor and Yamada 1974, 1982) and still get a turbulence scheme of similar shape (Cheng et al. 2002, hereafter CCH02). For the comparison of both differing sets of hypotheses, see Galperin and Kantha (1989). An extension of CCH02 avoiding the appearance of a critical Richardson number in the more complex CCH02 system was later added (Canuto et al. 2008, hereafter CCHE08), albeit valid for stable stratifications only. Furthermore, the appendix of CCH02 indicates a path for linking with a level-3.0 closure for the leading term representing the conversion between turbulent kinetic and potential energies.
The absence of Ri (i.e., the maintenance of turbulence in very stable stratifications) is explained via the velocity shear (Zilitinkevich et al. 2007, 2008) and the associated residual momentum flux. This corresponds to a sustained weak vertical diffusivity through the effect of internal waves in such conditions (Sukoriansky et al. 2005, hereafter SGS05). The behavior of stability dependency functions in stable conditions can then be described by the juxtaposition of a strongly turbulent and a weakly turbulent regime (Zilitinkevich et al. 2008) or equivalently a weakly stable, a transitional (not mentioned in the previous case), and a very stable regime (Sukoriansky and Galperin 2013).

The concept of stability dependency functions can still be used under some reasonable assumptions when prognostic aspects are encompassed not only in TKE but also in turbulent potential energy (TPE) or equivalently in the sum of both, turbulent total energy (TTE = TKE + TPE). Under the energy and flux budget turbulence closure theory (EFB) (Zilitinkevich et al. 2007, 2008, 2009, 2013, the latter hereafter ZEKRE13) a prognostic TKE–TPE(TTE) system is seen as the direct extension of the classical formulation (where only TKE is prognostic).

The most complete version of the EFB turbulence closure theory is based on budget equations for both turbulent energies (TKE and TPE) and for the turbulent fluxes of momentum and heat. Besides the usual downgradient transport term, the heat flux equation contains an additional term describing essentially positive (countergradient in stable stratification) contributions to the heat flux caused by temperature fluctuations. This additional flux, disregarded in conventional theories, is an inherent feature of the EFB theory ensuring no critical Richardson number. L’vov et al. (2008) also follow this approach for the heat flux equation.

For stable (or weakly unstable) stratifications, the Reynolds-averaging technique is not the only one leading to a compact formulation for both stability dependency functions (momentum $S_M$ and heat $S_H$). A “wave range by wave range” stand-alone computation of the cumulated effects of anisotropy and of wave–turbulence interactions also delivers such functions (SGS05). This result is part of the wider quasi-normal scale elimination (QNSE) framework.

Although fundamentally distinct in the methodology from previous competing theories, QNSE data can be transformed from the wavenumber space to one of stability-representing entities (e.g., by using equilibrium assumptions). In addition, QNSE then confirms the absence of any critical Richardson number (Galperin et al. 2007). The asymptotic properties (in strongly stable stratifications) of the stability dependency functions, in agreement with the Osborn–Cox mixing model (Osborn 1980; Galperin and Sukoriansky 2010), that is,

$$\lim_{Ri \to \infty} S_H = 0, \quad \lim_{Ri \to \infty} S_M = \text{const} > 0,$$

and of their ratio—the turbulent Prandtl number $\sigma = \sigma_t = \sigma_r \approx Ri$, all as obtained from the QNSE theory, are similar to the equivalent properties later derived in the EFB theory or in CCHE08.

For higher-order closure, there exists an heuristic relatively simple way to express third-order-moment (TOM) contributions to the second-order-moment (SOM) budgets (by simplifying the outcome of a complete reduction starting from the fourth order), with only three additional terms to consider (Canuto et al. 2007, hereafter CCH07). The link with the previous enumeration is that the general analytical solution (e.g., appendix of CCH02) for the “conversion term” is compatible with the CCH07 formulation. This allows applying the reduced complexity method (valid for the whole range of stabilities) to other systems than CCH02.

It transpires from the above that recent advances have been very rapid, in several partly interconnected directions, without major incompatibilities between them, but with no attempt (to our best knowledge) to bridge some gaps. This probably follows from some restricted conditions of application in most cases. The differing choices for the basic free parameters from one paper to the next might also be hampering the search for bridges between the formulations, although it should not matter, if equivalences are correctly expressed.

### b. Aims of the paper

The aim of the present paper is thus to try and make some analysis toward the wider goal of a unique concept encompassing all the above (if ever possible) and to propose an intermediate analytical framework, aiming at “taking the best of each proposal while contradicting as little as possible the rest of it.” More precisely, we would like to do the following:
propose a solution valid for all stability conditions and practically compatible with numerical weather prediction (NWP) applications;

(ii) account for the impact of turbulence anisotropy on both momentum- and heat-related terms;

(iii) identify a suitable set of free parameters in our framework of stability dependency functions, in line with solutions proposed following the work of Schmidt and Schumann (1989, hereafter SS89);

(iv) solve the analytical problem of removing the critical Richardson number and find the most compact formulation;

(v) emulate as much as feasible the QNSE and EFB systems on the basis of an adaptive ensemble consisting of MY-type analytical stability dependency formulations and of adjustable parameters and prepare a continuous extension of these emulations toward unstable stratification;

(vi) keep all this compatible with a TOM-type extension of the system, according to the CCH07 proposal; and

(vii) prepare a “natural evolution” of such a set of solutions with an additional prognostic equation for TPE(TTE), in the spirit of ZEKRE13.

We shall show below that these goals are all compatible. We therefore hope to convince the reader of the value of this approach to seek generality and modularity without giving up consistency.

In practice, we shall stick (implicitly or explicitly) to our closure-discretization method, which could be classified as somewhere between the level-2.0 and level-2.5 systems of MY. It uses stability dependency functions as in equilibrium conditions, making them dependent only on the Richardson number $R_i$, but relaxing this assumption for the evolution of TKE (see section 2), contrary to a more sophisticated use of equilibrium conditions in “quasi equilibrium” modeling (Galperin et al. 1988). For details about this difference, see section 8b.

Given its analytical orthogonal character with respect to the above, we shall treat the problem of the specification of the length scale only marginally. This and other aspects of the dry turbulent framework omitted here shall be addressed in a forthcoming paper.

Nonetheless, justifying the efforts described in the present paper only on the basis of the above arguments may still appear somewhat arbitrary. But there is a more important issue linked to such efforts. The extension from the dry-only case (i.e., even without water vapor) to the moist case (be it without or with saturation) destroys one of the basic pillars upon which all the above-mentioned “dry turbulence” theories are built, namely the double role of the potential temperature $\theta$ as a tracer of buoyancy and of entropy. This identity is lost as soon as water is present (because water vapor has a higher gas constant than dry air, because of latent heat release, or both). For NWP applications, it is a challenge to accurately predict both the partial cloud cover and the averaged buoyancy flux. This is a reason why, in our opinion, simple analytical solutions did not progress as much, in the past 13 years, as those for the dry case only. It is our belief that exploring separately the links between the moist case and all above-itemized “dry avenues of progress” would lead to a rather chaotic NWP situation. Hence we advocate first going through the above guidelines for streamlining the modern situation in the dry case, before going to the complex “moist” task with a minimum of modifications. The inference of the stability dependencies on the basis of the sole flux Richardson number $R_i$ is at the heart of this preparation for a moist extension in our system. This extension shall be described in another forthcoming paper.

c. Constraints and scope of the work

With regard to NWP requirements, we note the Louis (1979, hereafter L79) scheme used in the Action de Recherche Petite Echelle Grande Echelle [ARPEGE; the global realization at Météo-France of the joint Integrated Forecast System (IFS)/ARPEGE common development with the European Centre for Medium-Range Weather Forecasts (ECMWF)] and Aire Limitée Adaptation Dynamique Développement International (ALADIN; the limited-area counterpart of the latter, developed and used by a community of 16 European National Meteorological Services) models. The last evolution of this scheme leads (J.-F. Geleyn et al. 2001, unpublished manuscript) to a formulation in terms of stability dependency functions having all the correct asymptotic behaviors (see above) required by our work plan. We could thus prepare, as an operational intermediate step and as a basis for further development, a “p-TKE” solution where the last version of the L79 scheme is complemented by advection and self-diffusion of a TKE value computed under the assumption that the level-2.0 basic Louis-type scheme should just be the stationary solution of the resulting system (Geleyn et al. 2006). On this basis, all proposals from the ensuing sections could be coded and pretested in a realistic NWP-type environment.

Owing to the restriction of our analysis to the hypothetical fully dry case, the results of such tests will not be presented here. The paper will only outline analytical developments and compare them with (laboratory and atmospheric) experimental, direct numerical simulations (DNS), and large-eddy simulation (LES) data and with the corresponding outcomes of the various schemes, which we try emulating and assembling at the same time.
The paper is organized in the following way. As “boundary conditions” for the central effort, section 2 describes our closure-discretization method and length-scale aspects and section 3 treats the problem of selecting the suitable set of free parameters and functional dependencies, which determine the whole set of model equations within constraints ensuring “No Ri(є)”. Section 4 explains the first development leading to our basic system and section 5 analyzes in more detail the surprising result that a second and completely independent path can lead to the same reduction of complexity for the main stability dependency functions of the CCH02 system. Sections 6 and 7 study the bridges between the resulting formulation and the stability dependency functions delivered respectively by the QNSE and EFB theories. Section 8 attempts to synthesize the various findings from a practical point of view. Section 9 briefly treats outlook aspects concerning higher-order terms (diagnostic and prognostic) and the extension to mesoscale NWP models with short time steps often reaching τ/|time step| > 1 (see Fig. 1). Indeed, auto-diffusion and generation–dissipation of TKE cannot then be treated separately.

The stability of the above-described system is easy to control. Diffusion of momentum and heat is stable provided that the (Deleersnijder et al. 2008, hereafter DHBD08) criterion for the corresponding level-2.0 model is satisfied (see section 8b). Stability of the implicit TKE solver is then ensured by diagonal dominance of its matrix.

The roles of ₋ (also free-stream value; see below) and C3 are obvious. The evolution of TKE is still influenced by advection and vertical diffusion, but a relaxation of TKE toward its equilibrium value for the current time step ē replaces the source and dissipation terms in Eq. (2):

\[
\frac{∂e}{∂t} = \text{Advection}(e) + \frac{∂}{∂z} \left( K_E \frac{∂e}{∂z} \right) + \frac{1}{τ} (\dot{e} - e),
\]

where e = 0.5(\bar{u}'\bar{u}') is TKE, L is the turbulence length scale, K_E is the TKE vertical exchange coefficient, K_M and K_H are respectively the momentum and heat vertical exchange coefficients, N is the Brunt–Väisälä frequency (BVF; note that N^2 is negative for unstable stratifications), δ is the wind shear, and C3 is the “freestream value” of the constant controlling the intensity of the turbulent dissipation term.

In our NWP-oriented closure-discretization method, the vertical exchange coefficients for momentum and heat are computed from stability dependency functions \( \chi_3 \) and \( \phi_3 \) derived for equilibrium conditions as functions of the Richardson number \( \text{Ri} = N^2/δ^2 \) and from an independent prognostic TKE value:

\[
K_M = C_K L \chi_3(\text{Ri}) \sqrt{\dot{e}},
\]

\[
K_H = C_3 C_K L \phi_3(\text{Ri}) \sqrt{\dot{e}}.
\]

The relaxation time scale \( τ_e \), can then be related to the dissipation time scale \( τ \) so that in the time-continuous limit the relaxation term is equal to the difference between source terms for prognostic TKE and TKE dissipation:

\[
τ_e = \frac{\dot{e} - e}{K_M S^2 - K_H N^2 - (C_e/L)e^{3/2}} = \frac{τ}{2},
\]

\[
τ = \frac{2L}{C_e \sqrt{\dot{e}}},
\]

with \( τ_e \) (and \( K_E \)) computed from past time-step values of \( e \) and also via \( L \) when appropriate (see below).

This makes Eq. (5) in the time-continuous case identical to the prognostic TKE equation having source terms computed from Ri-dependent \( K_M \) and \( K_H \) in Eqs. (3) and (4) (like in a level-2.0 model) and from independent \( S^2 \) and \( N^2 \) (like in a level-2.5 model), hence the difficulty classifying our choice for closure and time discretization. The computation of \( \phi_3 \) for equilibrium conditions ensures a balance between its anisotropy part \( \phi_Q \) (downgradient) and its TPE conversion part \( \phi_{\text{conv}} \) (countergradient in stable stratification), which is one of the main factors for avoiding Ri(є) in our framework (see section 3). Given all these properties, we shall call this closure-discretization method “stability dependent adjustment for turbulent energy modeling.”

The above method is less sophisticated than that of a level-2.5 model but, contrary to the simpler level-2.0 case, it enables a fully consistent treatment of the prognostic TKE. This is an important aspect for present-day mesoscale NWP models with short time steps often reaching \( τ_e/(\text{time step}) > 1 \) (see Fig. 1). Indeed, auto-diffusion and generation–dissipation of TKE cannot then be treated separately.

2. Model closure and discretization

Our starting point is the basic version of the prognostic TKE equation:

\[
\frac{∂e}{∂t} = \text{Advection}(e) + \frac{∂}{∂z} \left( K_E \frac{∂e}{∂z} \right) + K_M S^2
\]

\[
- K_H N^2 - C_3 e^{3/2}/L,
\]

where e = 0.5(\bar{u}'\bar{u}') is TKE, L is the turbulence length scale, K_E is the TKE vertical exchange coefficient, K_M and K_H are respectively the momentum and heat vertical exchange coefficients, N is the Brunt–Väisälä frequency (BVF; note that N^2 is negative for unstable stratifications), δ is the wind shear, and C3 is the “freestream value” of the constant controlling the intensity of the turbulent dissipation term.

In our NWP-oriented closure-discretization method, the vertical exchange coefficients for momentum and heat are computed from stability dependency functions \( \chi_3 \) and \( \phi_3 \) derived for equilibrium conditions as functions of the Richardson number \( \text{Ri} = N^2/δ^2 \) and from an independent prognostic TKE value:

\[
K_M = C_K L \chi_3(\text{Ri}) \sqrt{\dot{e}},
\]

\[
K_H = C_3 C_K L \phi_3(\text{Ri}) \sqrt{\dot{e}}.
\]
achieved by limiting the value of $K_E \cdot \tau_c$ above $3/8$ of the square of the vertical local spacing.

The length scale is a crucial quantity in parameterization of turbulence. In our particular framework, it can be chosen quasi independently from the stability-dependency functions. We would nevertheless recommend the use of the length scale from Bougeault and Lacarrere (1989), with its nonlocal properties. Specifically, in our implementation, we combine this choice with the length-scale treatment allowing continuity with the surface layer’s handling via the mixing length from similarity laws (with notation $l_m$). This is done by following RMC01 and matching equilibrium expressions for TKE and for the momentum flux with similarity laws in the surface layer [see RMC01 just after their Eqs. (21b) and (21c)]. Then a unique length scale in the neutral surface layer is computed from

$$L = \frac{C_0}{\nu^2} l_m, \quad (9)$$

$$\nu = (C_r C_K)^{1/4}, \quad (10)$$

instead of assuming $L = l_m$. Note that the expression for $L$, valid in the surface layer, is expressed in terms of $C_r$ and $C_K$, which is also appropriate for free-stream turbulence.

We plan to generalize the above choice of length scale by adapting its computation to make it dependent on the (moist) squared BVF (Váňa et al. 2011). We also consider introducing memory when determining this generalized length scale by using a prognostic approach (following ZEKRE13, but replacing the time scale by the length scale as additional prognostic quantity).

3. Model framework

a. Free parameters

The following discussions rely on a combination of the RMC01 and CCH02 papers, complemented by a small key extract from CCHE08. This is indeed a sufficient starting point for the goals set in the introduction. Separately, the said papers cannot, however, cover all our needs. RMC01 does not account for the anisotropy of turbulence, while CCH02 falls short of eliminating the critical Richardson number in equilibrium conditions. Since CCH02 is the more complete of both papers, our notations (enumerated in appendix A) will mainly be taken from it, with additions either taken from RMC01 [C_r, C_K, C_3, C_4, $\phi_3(Ri)$] or added for our purpose [$P, R, Q, \alpha_3, \chi_3(Ri)$].

For the reader unfamiliar with the MY-type derivations of the CCH02 paper, the main relevant equations are reproduced in appendix B. Note that for the sake of simplicity, we transversally assume that the $\alpha_3$ coefficient of CCH02 is equal to zero [CCH02 authors also advocate such a step in the transition between their Eqs. (21b) and (21c)].

In RMC01 the basic set of free parameters $C_r$, $C_K$, $C_3$, and $C_4$ [see Eq. (22)] is essentially the same as the one originating from SS89. We want to stay with four basic free parameters in our system but (i) choose them on targeted physical principles and (ii) get a clearer view of their interdependencies.

Considering the importance of $\nu$ [implicitly $C_K$ via Eq. (10)] and $C_r$ both in the TKE evolution [see Eqs. (3), (4), and (6)] and in our length-scale evaluation [Eq. (9)], we choose both of them as free parameters. Note that, unlike $C_r$ and $C_K$, $\nu$ is independent of the hypothesis made in the choice of the length scale in the surface layer [Eq. (9) or its alternative]. It is hence a good dimensionless measure of the overall intensity of turbulence. For $C_r$, we follow the recommendation of SS89 and (considering our choice for $L$) link it to $\nu$ via

$$C_r = \pi \nu^2, \quad (11)$$

but a direct setting with a similar order of magnitude could equally well have been chosen.

It is obvious from Eqs. (4) and (6) that $C_3$, the physically important inverse of the Prandtl number at neutrality, has to be chosen as our third free parameter.

Parameters $C_3$ and $\nu$ are useful as “tuning” parameters for weighting the relative importance of terms contributing to the energy budgets. But, in CCH02, most of the computations go back to $\lambda$ (the ratio of the time scales of pressure–velocity correlations $\tau_{p,v}$ and of TKE dissipation $\tau$) and $F$ [a coefficient influencing mean
shear–turbulence interactions for pressure correlations, via $\lambda_2$ and $\lambda_3$; see Eq. (B27)]. The relevant (invertible) relationships are

$$\nu = \left(\frac{4R^l_\text{f} \lambda}{15}\right)^{1/4},$$

with

$$R^l = 1 - \frac{88F + 320\sqrt{F} - 50}{375} \lambda; \quad \text{and}$$

$$C_3(1 + C_3) = \frac{Q^l}{\lambda R^l},$$

where

$$Q^l = 1 - \frac{4\sqrt{F}}{5} \lambda.$$

This introduces the constants $R^l$ and $Q^l$ as one of the possible realizations of more general entities of our target system named $R$ and $Q$ (precisely defined at the end of this section and in sections 4 and 5). These variables play a key role for modulating the stability dependence under the influence of anisotropy.

Our last-selected free parameter is linked to the influence of TPE on the heat flux [see Eq. (A2) in CCH02 and/or ZEKRE13]:

$$\frac{w'\theta'}{\tau \lambda_5} = -w'^z \frac{\partial \theta}{\partial z} + 2(1 - \gamma_1)\text{TPE} \frac{\partial \theta}{\partial z},$$

$$\text{TPE} = \frac{1}{2} \frac{g}{\theta} \frac{\left(\frac{\partial \theta}{\partial z}\right)^{-1}}{\theta'^z},$$

with $\lambda_5$ influencing the pressure-correlation dissipation time scale for the heat flux while $\tau_{p,\theta} = \tau \lambda_5^{-1}$ and $\gamma_1$ control the pressure-correlation term depending on TPE. In equilibrium conditions (see ZEKRE13), the conversion term in Eq. (16) can be expressed in terms of TKE via the ratio of the dissipation time scales for TKE ($\tau$) and TPE ($\tau_{\theta}$; see Eq. (B8)):

$$\frac{w'\theta'}{\tau \lambda_5} = -w'^z \frac{\partial \theta}{\partial z} + 2(1 - \gamma_1)\tau_{\theta} \frac{\partial \theta}{\partial z},$$

It is apparent that the conversion term in Eq. (20) is obtained via the expression $(1 - \gamma_1)(\tau_{\theta}/\tau)$, but, owing to our links to RMC01 (see below), we shall use as our last free parameter $O_\lambda$ given by

$$O_\lambda = C_3(1 - \gamma_1) \frac{\tau_{\theta}}{\tau}.$$  

In RMC01, $\gamma_1$ is implicitly set to $\frac{1}{3}$ and their last free parameter is thus related only to $\tau_{\theta}/\tau$.

$$C_4 = 2C_3 \frac{\tau_{\theta}}{\tau}$$

and is recommended to be 2.0. In CCH02, $C_4$ is implicitly set to 2.0 by choosing $\tau = C_3 \tau_0$ and $\gamma_1$ is recommended to be $\frac{1}{3}$. In our framework we enable modification of both $C_4$ and $\gamma_1$ even when only the product $O_\lambda$ matters—that is, as long as we do not extend the model to a prognostic TPE treatment (for details see section 9).

In case of the model extension toward prognostic TPE or TOMs parameterization (for details, see section 9), we also need to identify the downgradient anisotropy part of $\phi_3$ [$\phi_Q$, obtained by using Eqs. (20), (4), (8) and (10)]:

$$\frac{w'\theta'}{\tau \lambda_5} = -e\tau C_3 \frac{\nu}{2} \frac{\phi_Q}{\phi_{\text{conv}}} \frac{\partial \theta}{\partial z} = -e\tau C_3 \frac{\nu}{2} \phi_3 \frac{\partial \theta}{\partial z},$$

where

$$\phi_Q = \frac{2}{C_3 \nu} \frac{w'^z}{\lambda_5 e} \quad \text{and} \quad \phi_{\text{conv}} = \phi_Q / \phi_3.$$  

The above relationship actually describes the balance between downgradient transport [first term in Eq. (20)] and TPE conversion [second term in Eq. (20)] by means of the $\phi_3$ decomposition. This balance resulting in $\phi_3$ going toward zero as inverse function of $\text{Ri}$ for infinite stability is one of the main factors for having No $\text{Ri}(cr)$ in turbulence schemes (Zilitinkevich et al. 2008).

The second required property for the absence of a $\text{Ri}_{cr}$ is then $\chi_3$ going asymptotically toward a nonzero value for infinite stability. Provided both these requirements are met, the flux Richardson number asymptotically approaches to a nonzero value at infinite $\text{Ri}$, named the “critical flux Richardson number” and written $\text{Ri}_c$.

Note that in our framework $\text{Ri}_c$ [determined from Eqs. (20) and (21) for $\text{Ri} \to \infty$] is not an independent physical quantity but is given by the set $\nu$, $C_3$, and $O_\lambda$ or by the set $\lambda$, $F$, and $(\gamma_1 \wedge C_4)$ (see sections 4 and 5).
The asymptotic properties of \( \chi_3, \phi_3, \) and \( \phi_{Q} \) for the absence of \( \text{Ri}_{ic} \) also correspond well with the asymptotic behavior of the Osborn–Cox (Osborn 1980; Galperin and Sukoriansky 2010) model. Once we assume equilibrium conditions for obtaining the coefficient \( \Gamma \) in the Osborn–Cox relationship for \( K_{eff} \), that is,

\[
K_{eff} = \Gamma \frac{\epsilon}{N^2},
\]

(25)

where \( \epsilon \) is the dissipation rate of TKE, we get for \( \Gamma \)

\[
\Gamma = \frac{\tau_{TPE}}{\tau_{p,ee}} = \frac{\text{Ri}_f}{1 - \text{Ri}_f},
\]

(26)

which asymptotically approaches, for \( \text{Ri} \to \infty \), to

\[
\Gamma_{\infty} = \frac{\text{Ri}_{fc}}{1 - \text{Ri}_{fc}}.
\]

(27)

The above properties and thus \( \text{No Ri(cr)} \) are reached in the RMC01 formulation [via \( f(\text{Ri}) = \chi_3(\text{Ri})(1 - \text{Ri}_f) \), with their \( f \) function used as a filter ensuring the equilibrium condition], but this outcome is somewhat biased by their enforcement of isotropy \((R = 1)\) and \((Q = 1)\). The correct asymptotic behaviors also appear (in a more straightforward way) in the QNSE and EFB frameworks, and we shall try to make them general by enforcing their appearance in all our CCH02-related derivations.

Looking back at CCH02, there remains one value, which is independent from the quartet \((C, \nu, C_3, O_4)\): namely, the parameter \( \lambda_4 \) controlling the direct influence of the heat flux on the momentum flux [see Eq. (B9)]. Implicitly, \( \lambda_4 \) is set to zero in the RMC01 system. This is one of the main factors for absence of \( \text{Ri}_{ic} \) in RMC01 and it inspired the derivation of model I in our framework (see section 4). But the hypothesis \( \lambda_4 = 0 \) is rather restrictive (e.g., Shih and Shabbir 1992; Canuto 1994). This issue will be extensively discussed in sections 4 and 5 (in rather contrasted terms). In the analytical path developed in section 5, we shall keep \( \lambda_4 \) to its CCH02 formulation [first part of Eq. (B28)].

Concerning orders of magnitude, the basic recommended setup of CCH02 \((\lambda = 0.4, F = 0.64, \gamma_1 = 1/3, \) and \( C_4 = 2)\) leads here to \( \nu = 0.526, C_3 = 1.18, \) and \( O_4 = 0.667 \), which are all physically rather reasonable values.

b. Core equations

We now go to the determination of the analytical shape of the stability dependency functions. As we shall show in sections 4 and 5, the stability dependency functions \( \chi_3, \phi_3, \) and \( \phi_Q \) for the turbulent scheme without \( \text{Ri}_{ic} \) can be written as functions of \( \text{Ri}_f \) in the following shape:

\[
\phi_3(\text{Ri}_f) = \frac{1 - (\text{Ri}_f/P)}{1 - \text{Ri}_f},
\]

(28)

\[
\chi_3(\text{Ri}_f) = \frac{1 - (\text{Ri}_f/R)}{1 - \text{Ri}_f},
\]

(29)

\[
\phi_Q(\text{Ri}_f) = \frac{1 - (\text{Ri}_f/Q)}{1 - \text{Ri}_f},
\]

(30)

Assuming that (i) \( \chi_3, \phi_3, \) and \( \phi_Q \) are nonnegative and monotonously decreasing with \( \text{Ri}_f \) in the whole range of Richardson numbers and (ii) that the properties required for the absence of \( \text{Ri}_{ic} \) in our turbulent scheme (see above) are fulfilled, we get asymptotic properties for the variables \( P, R, \) and \( Q \):

\[
0 < \lim_{\text{Ri} \to \infty} P = \text{Ri}_{ic} < 1,
\]

(32)

\[
\text{Ri}_{fc} < \lim_{\text{Ri} \to \infty} R = R_{\infty} \leq 1,
\]

(33)

\[
\text{Ri}_{fc} \leq \lim_{\text{Ri} \to \infty} Q = Q_{\infty} \leq 1.
\]

(34)

Logically the variables \( P, R, \) and \( Q \) are determined by the sets of parameters \( \nu, C_3, \) and \( O_4 \) or by the set \( \lambda, F, \) and \( (\gamma_1 \wedge C_4) \). Depending on the characteristics of the scheme that we want either to make more compact (CCH02) or to emulate (QNSE or EFB), they may be either constants or stability dependent functions. Table 1 gives an anticipation of what we shall find in the next four sections for this key aspect. Our proposed scheme is compactly written, but it adapts itself in this way to various and rather differing situations.

Since \( R \) may not be known a priori for emulation extension purposes, we may derive from Eqs. (19) and (28) a linking relationship between \( \chi_3 \) and \( \phi_3 \), without dependency on \( R \):

\[
C_3 \text{Rii} \phi^2_3(\text{Ri}) - \left[ \chi_3(\text{Ri}) + \frac{C_3 \text{Ri}}{P} \right] \phi_3(\text{Ri}) + \chi_3(\text{Ri}) = 0.
\]

(35)

This equation will be valid for all realizations of CCH02 and will become an excellent way to fit the stability dependency functions in our emulation extension of QNSE.

4. Equations: The basic system

In this section, we develop a modification of the CCH02 system by setting \( \lambda_4 \) to zero, thus canceling the direct
influence of the heat flux on the momentum flux. Such an approach leads to a simplified scheme without \( \text{Ri}_{\text{se}} \), which we shall name model I. Because of the rough nature of the simplification, we do not recommend the use of model I in practical applications, but it still represents a basic element in our “family” of models that avoid \( \text{Ri}_{\text{se}} \) by construction.

Furthermore, we shall see later that it is an important step in the emulation extension of the EFB system.

On top of the information already given, we recall that

\[
G_M = \tau^2 S^2 \quad \text{and} \quad G_H = G_M \text{Ri} = \tau^2 N^2. \tag{36}
\]

Starting from Eqs. \((B14)–(B25)\) and \((B9)\) we can derive under our \( \lambda_4 = 0 \) (and \( \alpha_3 = 0 \Rightarrow \lambda_6 = \lambda_7 \)) conditions the following expressions for the stability dependency functions \( S_M, S_H, \) and the relative vertical component of TKE \( A_z: \)

\[
S_M = \frac{s_0}{1 + d_2 G_M}, \tag{37}
\]

\[
S_H = \frac{s_4 + s_6 G_M}{(1 + d_1 \text{Ri} G_M)(1 + d_2 G_M)}, \tag{38}
\]

\[
A_z = \frac{w_H^2}{2 e^3} = \frac{1}{3} \left( 1 - \frac{1 - Q^1}{2} S_M G_M \right). \tag{39}
\]

The stability dependency function \( S_M \) (or its scaled version \( \chi_3 \)) can be rewritten as function of \( \text{Ri}_f \) (no direct \( \text{Ri}_{\text{se}} \)) when we express \( G_M \) from the equilibrium TKE equation [see Eq. \((B31)\)]

\[
G_M = \frac{2}{S_M(1 - \text{Ri}_f)} \tag{40}
\]

and use it in Eq. \((37)\):

\[
\chi_3 = \frac{S_M}{S_M,0} = 1 - \frac{(\text{Ri}_f/Q^1)}{1 - \text{Ri}_f}, \tag{41}
\]

\[
S_{M,0} = S_M(\text{Ri} = 0) = s_0 - 2d_2, \tag{42}
\]

\[
R^I = \frac{S_{M,0}}{s_0} = 1 - \frac{2d_2}{s_0}. \tag{43}
\]

The stability dependency function \( S_H \) (or its scaled version \( \phi_3 \)) has a more complicated shape. We derive \( S_H \) using its relationship to the already known \( S_M \) [from Eqs. \((37), (38), \) and \((40)\)]:

\[
S_M = \frac{\text{Ri}}{\text{Ri}_f} = \frac{p^l(R^I - \text{Ri}_f)}{C_S R^l(P^l - \text{Ri}_f)}, \tag{44}
\]

\[
p^l = \text{Ri}_f c = \frac{Q^1}{1 + (3O_3/C_3)} = \frac{2s_6 + (s_0 - 2d_2)s_4}{s_0 (s_4 + 2d_1)}, \tag{45}
\]

\[
C_3 = \frac{S_H(\text{Ri} = 0)}{S_{M,0}} = \frac{s_0 s_4 R^I + 2s_6}{s_0^2 R^l}. \tag{46}
\]

This, together with the relationship for \( \chi_3 \) [Eq. \((41)\)], results in an expression of \( \phi_3 \) also depending only on \( \text{Ri}_f: \)

\[
\phi_3 = \frac{S_H}{S_{M,0}}(\text{Ri} = 0) = 1 - \frac{(\text{Ri}_f/Q^1)}{1 - \text{Ri}_f}. \tag{47}
\]

To complete the set of stability dependency functions with the anisotropy part of \( \phi_3 (\phi_Q) \), we use its definition [Eq. \((24)\)], the expression for \( A_z \) [Eq. \((39)\)], and the equilibrium relationship [Eq. \((40)\)]:

\[
\phi_Q = \frac{1}{1 - \text{Ri}_f/Q^1}. \tag{48}
\]

\[
Q^1 = 1 - 3\chi_3 + \lambda_2. \tag{49}
\]

As written in section 3a [Eqs. \((12)–(15)\)], \( R^I \) and \( Q^1 \) can be directly computed from \( \lambda \) and \( F \) or from \( \nu \) and \( C_3 \). As seen from Eq. \((45)\), \( P^l \), which in the case of model I is equal to \( \text{Ri}_f \), additionally depends on the combination of the coefficients \( C_4 \) and \( \gamma_1 \) or, equivalently, on the joint parameter \( O_3 \).

The compact set of equations for model I [Eqs. \((41), (47), (48), \) and \((44)\)] corresponds to the anticipated equations [Eqs. \((28)–(31)\)]. This characteristic is linked to the absence of influence of the vertical heat flux on the vertical momentum flux via the partly arbitrary assumption \( \lambda_4 = 0 \), in disagreement, for instance, with experimental data from Shih and Shabbir (1992) where \( \beta_5 \) is surely nonzero. However, such an approximation also appears in RMC01, inspiring our model I derivation. In the more complex EFB scheme, the vertical heat flux also has no influence on the vertical momentum flux [see Eq. \((34)\) in ZEKRE13], but it is then associated with a specific set of relationships for the TKE components. This discrepancy in model I with respect to the more complete EFB solution can be corrected by introducing a variation with respect to stability of the otherwise constant (see Table 1) variables \( P \) and \( R \) (see section 7).
Note also that assuming $\lambda_s = 0$ in model I yields an unrealistic isotropic $A_t = \frac{1}{3}$ value in a convective flow without shear [see Eq. (39)]. However, in our basic closure-discretization method, the determination of TKE components has a purely diagnostic purpose. The direct evaluation of $A_t$ influences the computation of turbulent fluxes only when we go to an extension with TOMs parameterization (see section 9). In such a case, the anisotropy properties of convective flows should automatically be improved owing to the contribution of plume-type TOMs terms.

5. Equations: A system with higher initial complexity

Unlike in section 4 where we used $\lambda_s = 0$, we follow here the idea proposed in CCH08 in order to eliminate $R_i$. This alternative approach is based on assuming linearity between the inverse of the dissipation time scale $\tau_{p,0}$ and the Prandtl number $\sigma_t$ [Eq. (9e) in CCH08]

$$\frac{\tau}{\tau_{p,0}} = \lambda_5 \sim c_i^{-1}(1 + \sigma_t).$$

which is a relationship that is approximated in CCH08 by assuming a Prandtl number proportional to $R_i$ [their Eq. (11)]:

$$\lambda_5 = \lambda_5(R_i = 0)(1 + R_i).$$

This indeed leads to a system without $R_i$, but with two drawbacks: (i) the system becomes more complicated and (ii) it is then only valid for the stable case.

Instead of approximating $\sigma_t$ in the damping factor, we will use here Eq. (50) directly, which leads to simply set

$$\lambda_5 = 5C_3(1 + \sigma_t)$$

with

$$\lambda_5(R_i = 0) = 5(1 + C_3)$$

in the CCH02 system—let us tag this system as model II.

Concerning this choice, it could be argued that the coefficients of the basic Reynolds decomposition system should ideally be constants (Lewellen 1977). But in the same work (p. 247) a model is still considered valid, if such coefficients vary, provided it is on the sole basis of dimensionless parameters, such as gradient Richardson number $R_i$ (or, by extension, $\sigma_t$). A similar approach was also used since the CCH08 publication in Kantha and Carniel (2009) and in Kitamura (2010).

After applying Eq. (52), the system acquires an implicit character

$$\sigma_t = \frac{R_i}{R_i_f} \frac{S_M[G_M, R_i, \lambda_5(R_i, R_i_f)]}{S_H[G_M, R_i, \lambda_5(R_i, R_i_f)]}$$

and at first sight can hardly be expected to lead to a compact shape of the stability dependency functions. We can however obtain explicit relationships for $S_M$ and $S_H$ from Eq. (54):

$$S_M = \frac{s'_0 + s'_1 R_i G_M + s'_2 G_M}{D},$$

$$S_H = \frac{s'_4 + s'_5 R_i G_M + s'_6 G_M}{D},$$

$$D = 1 + d'_1 R_i G_M + d'_2 G_M + d'_3(R_i G_M)^2$$

$$+ d'_4 R_i G_M^2 + d'_5 G_M^2,$$

where the primed variables $s$ and $d$ are now different from the original ones in CCH02 (see appendix C). Furthermore, if we now assume equilibrium conditions we can find simpler shapes of $S_M$ and $S_H$, indeed surprisingly resembling the ones of model I, Eqs. (37) and (38):

$$S_M = \frac{s''_0}{1 + d''_2 G_M},$$

$$S_H = \frac{s''_4 + s''_5 G_M}{(1 + d''_1 R_i G_M)(1 + d''_2 G_M^2)},$$

where double primed variables $s$ and $d$ are also listed and explained in appendix C.

Following now the steps earlier used for obtaining Eqs. (41), (47), and (44) from Eqs. (37) and (38), we automatically get stability dependency functions $\phi_3$ and $\chi_3$ in the same shape as in case of model I:

$$\chi_3 = \frac{1 - (R_i/R_f)^{\text{II}}}{1 - R_f},$$

$$\phi_3 = \frac{1 - (R_i/P_f)^{\text{II}}}{1 - R_f},$$

with

$$R_f^{\text{II}} = \frac{15 \lambda_1 + 20 \lambda_2^2 - 60 \lambda_3^2 C_3}{(60 \lambda_3 + 20 \lambda_2) \lambda_4 + 15 \lambda_1 C_3 + 12 \lambda_4},$$

$$P_f^{\text{II}} = R_i c_f = 30 C_1^2 S_{M,0} \left[ 4 \left( \frac{3 \lambda_4}{C_3} + \lambda_4 + 1 \right) - 5[(12 \lambda_3 + 4 \lambda_2) \lambda_4 + 3 \lambda_1 C_3]^{-1} \right].$$

This result is surprising (since obtained for constant values of $R$ and $P$ on both sides) and simpler than the one of CCH08 while also valid for the unstable range, even if derivation is rather tailored to our closure discretization.
The \( \phi_Q \) anisotropy part of \( \phi_3 \) has in model II different characteristics than in model I. First, there is the influence of the heat flux on the vertical component of TKE in model II due to \( \lambda_3 \neq 0 \) [Eq. (B9)]:

\[
A_z = \frac{1}{3} \left( 1 - \frac{Q^I}{2} S_M G_M - 2 \lambda_4 S_H G_H \right)
\]

(64)

and second \( \lambda_5 \) is a stability dependent function [Eq. (52)]. In equilibrium conditions, we can write the following for \( \phi_Q \) [from Eqs. (24), (64), and (52)]:

\[
\phi_Q = \frac{1 - (R_{ij}/Q^I)}{1 - R_{ij}}.
\]

(65)

where \( Q^I \) is a stability dependent function (see Fig. 2) asymptotically approaching \( Q^I = R_{i_k} \) for \( R_i \to \infty \), thus causing \( \phi_Q^I \) to asymptotically approach zero.

This property is the most apparent difference between model II and model I, where, by contrast, \( Q^I \) is a constant larger than \( R_{i_k} \) resulting in \( \phi_Q^I \) asymptotically approaching a nonzero positive value:

\[
\lim_{R_i \to \infty} \phi_Q^I = \frac{1 - (R_{ij}/Q^I)}{1 - R_{ij}}.
\]

(66)

Other differences between models I and II are less obvious, owing to the identical shape of the stability-dependency functions \( \chi_3 \) and \( \phi_3 \). While \( C_3 \) and \( \nu \) indeed depend on \( \lambda \) and \( F \) in the same way in both systems [see Eqs. (12) and (14)], the dependency of constant \( R \) and \( P = R_{i_k} \) on \( \lambda, F, \) and \( Q_3 \) differ [see Eqs. (43) and (45) in comparison with Eqs. (62) and (63)].

6. Emulation of QNSE: Bridge with the outcome of a spectral analysis of turbulence and waves

QNSE is a theory of turbulence based on a spectral analysis of the flow that uses a radically differing approach from Reynolds stress modeling (RSM). QNSE delivers general spectral-type results about turbulence activity, which may be converted to a Reynolds-averaging Navier–Stokes (RANS) framework with the help of some linking hypotheses between scales and Richardson numbers [Eqs. (182)–(189) of SGS05]. In such a case, one directly gets “data” allowing linking of \( \chi_3 \) and \( \phi_3 \) (separately) with \( R_i \). The challenge is to see whether the specific framework built above [RSM on the basis of CCH02 and RMC01; Eqs. (28)–(31)] can be used for emulating QNSE data in stable stratification and possibly extending such an emulation (for practical applications) toward unstable stratification.

We first set \( C_3 = 1.39 \) and \( R_{i_k} = C_j/(C_3 + 2.3) \), 2.3 being the opposite of the derivative of \( \phi_3 \) with respect to \( R_i \) at neutrality, all this according to QNSE publications.

We can then verify that the separate datasets of \( \chi_3 \) and \( \phi_3 \) in QNSE seem to agree (cf. the second and third columns of Table 2) with the equation linking \( \phi_3 \) and \( \chi_3 \) in our framework [Eq. (35)] when we assume a constant \( P = R_{i_k} \). This enables us to fit only one of \( \chi_3 \) and \( \phi_3 \) and to keep consistency between them. At first sight, it would seem simpler to deduce \( \chi_3 \) from \( \phi_3 \) than the reverse. But we chose to start with \( \chi_3 \) owing to the difficulties of assessing \( \phi_3 \) at high \( R_i \) values [data only approximately correct and need to somewhat tune them down—progressively with increasing \( R_i \); see section 5.1 of Sukoriansky et al. (2006)].

In the stable case, knowing that \( \chi_3 \) asymptotes at high \( R_i \) values around 0.232 (from the QNSE data) and looking at the functional shape, we postulate

\[
\chi_3^\text{QNSE,fit} = \frac{1 + 0.75R_i(1 + 13R_i)}{1 + (0.75 + 2.48)R_i(1 + 13R_i)}.
\]

(67)

We still have to verify whether this way of retrieving \( \phi_3 \) gives a good quality fit with respect to the corresponding corrected QNSE data. Going back to Table 2 and comparing this time the second and fifth columns, we see that the quality of the \( \phi_3 \) fit nearly matches the very good one for \( \chi_3 \) (first and fourth columns), as a confirmation of the validity of this approach.

Since the QNSE theory allows going only in the weak unstable regime (down to \( R_i \approx -0.35 \)) we need to postulate a more complete extension of our QNSE emulation. For this, we rely on the simple asymptotic regime of \( \chi_3 \) and \( \phi_3 \) in model I and model II (the functions tend toward \( R^{-1} \) and \( P^{-1} \) for high instability). Using then the QNSE data available in the weakly unstable situation [see Fig. 3 in Sukoriansky and Galperin (2008)], and ensuring continuity of the \( \chi_3 \) derivative with respect to \( R_i \) at neutrality (−2.48 as given by the QNSE publications), we propose
TABLE 2. Comparison of QNSE data with their emulation. Read $\chi^\text{QNSE}_3$ and read-corrected $\phi_3^\text{QNSE}$; QNSE data; $\phi_3^\text{linked}$, $\phi_3$ computed from $\chi^\text{QNSE}_3$ and Eq. (35); $\chi^\text{eeQNSE}_3$, fit of $\chi^\text{QNSE}_3$ [see Eq. (67)]; $\phi_3^\text{eeQNSE}$, $\phi_3$ computed from $\chi^\text{eeQNSE}_3$ and Eq. (35).

<table>
<thead>
<tr>
<th>Ri</th>
<th>$\chi^\text{QNSE}_3$</th>
<th>$\phi_3^\text{QNSE}$</th>
<th>$\phi_3^\text{linked}$</th>
<th>$\chi^\text{QNSE}_3$</th>
<th>$\phi_3^\text{eeQNSE}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
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<td>0.988</td>
<td>0.987</td>
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<td>0.977</td>
</tr>
<tr>
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<td>0.945</td>
<td>0.952</td>
<td>0.942</td>
<td>0.952</td>
</tr>
<tr>
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<td>0.850</td>
<td>0.873</td>
<td>0.838</td>
<td>0.871</td>
</tr>
<tr>
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<td>0.714</td>
<td>0.723</td>
<td>0.672</td>
<td>0.714</td>
</tr>
<tr>
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<td>0.430</td>
<td>0.463</td>
<td>0.428</td>
</tr>
<tr>
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<td>0.291</td>
<td>0.143</td>
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</tr>
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<td>0.249</td>
<td>0.065</td>
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</tr>
<tr>
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<td>0.031</td>
</tr>
<tr>
<td>5</td>
<td>0.233</td>
<td>0.012</td>
<td>0.013</td>
<td>0.233</td>
<td>0.013</td>
</tr>
</tbody>
</table>

$$\chi^\text{ext,uns}_3 = \frac{1 - 4.16R_i}{1 - (4.16 - 2.48)R_i}$$ (68)

As in the stable case, we compute in unstable stratification $\phi_3$ from $\chi^\text{ext,uns}_3$ via Eq. (35). Let us tag the resulting system valid for whole range of Ri as emulation extension of QNSE (eeQNSE). It is a different entity resulting system valid for whole range of Ri as emulation extension of QNSE (eeQNSE). It is a different entity from QNSE but sharing its $\phi_3$ and $\chi_3$ functional dependencies in the stable range.

It remains to be verified which quality the extension of the QNSE emulation delivers with respect to the asymptotic “free convection” case. For this, we go back to the equivalence (in equilibrium conditions) of our RANS model with the one of L79, via [for the derivation, see Bašták (2009)]

$$F_H = \phi_3 \sqrt{\chi_3 (1 - R_i)}.$$ (69)

The $\phi_3$ and $\chi_3$ functions having finite asymptotic values for the very unstable case, one indeed gets $F_H$ correctly proportional to $(R_i)^{1/2}$.

Developing $F_H$ on both sides (the fit and L79) then delivers for the $C_H^b$ constant of the Louis-type free-convective case [see Eqs. (16)–(20) and (24) in L79] the following expression and its associated “QNSE value”

$$C_H^b = \sqrt{\frac{(3P_0 - \omega)^3 C_3}{\kappa^2}} = 6.37,$$ (70)

where $\kappa$ is the von Kármán constant, while L79 indicates $C_H^b = 5.3$ as the best fit to data. This about 20% discrepancy seems rather acceptable in view of the difficulties to “observe” $C_H^b$.

The main difference between eeQNSE and models I and II is the stability dependent $R$ in eeQNSE. Even though $\rho_0^\text{eeQNSE}$ varies with stability (see Fig. 3), it is mostly $\pm 10$% around a 0.4 value, supporting our attempt to emulate and extend the QNSE behavior on the basis of our framework.

To complement eeQNSE with values for the free parameters $\nu$ and $\Omega$, which are not delivered by the QNSE framework, we use (for such purpose only) a constant $R = R_v = 0.44$ and assume that the internal links between $\nu$, $C_3$, and $\Omega$ and $R$ and $P$ in eeQNSE are the same ones as in model I or II [see Eqs. (12), (14), (43), (45), (62), and (63)].

The solution for the model I and II setups is then

$$(C_3^\text{eeQNSE} = 1.39, R_v^\text{eeQNSE} = 0.44, R_e^\text{eeQNSE} = 0.677) \Rightarrow
\nu^\text{eeQNSE,I} = 0.464, \quad \Omega^\text{eeQNSE,I} = 0.248;$$ (71)

$$(C_3^\text{eeQNSE} = 2.04, \quad \nu^\text{eeQNSE,II} = 0.504, \quad \Omega^\text{eeQNSE,II} = 0.324.$$ (72)

The QNSE theory delivers no decomposition of $\phi_3$ into its anisotropy ($\phi_Q$) and conversion parts. We may however achieve this decomposition in eeQNSE on the basis of parameters computed from a constant $R^\text{eeQNSE}$ [see Eqs. (71) and (72)]. The properties of $\phi_Q$ will of course correspond to the chosen model I or II. From now on we differentiate between both emulation extensions.

7. Emulation of EFB: Generalization of the framework

EFB is an RSM scheme based on the budget equations for TKE and TPE and for turbulent fluxes. Consideration of the TPE conversion term in the heat flux budget and an appropriate stability dependent parameterization of TKE components ensure the absence of $R_{\text{cr}}$ in the EFB scheme. The aim of this section is to compare models I and II with the “minimal prognostic model” variant of the EFB scheme (the one corresponding best to our
closure-discretization choices) and to explore whether we may put both realizations under a common hat.

It is straightforward to identify the values of both free parameters $\nu$ and $C_3$ in EFB. Comparing the relationships for exchange coefficients in CCH02 [their Eq. (16c)] with those in ZEKRE13 [their Eqs. (100) and (101)], we get

$$\nu_{\text{EFB}} = [(2A_{\zeta}^{(0)}C_\gamma)_{\text{ZEKRE13}} = 0.532, $$

$$C_3^{\text{EFB}} = \left(\frac{C_\zeta}{C_\gamma}\right)_{\text{ZEKRE13}} = 1.25, $$

as well as the basic shape of the $\chi_3$ and $\phi_3$ functions

$$\chi_3^{\text{EFB}} = \left[\frac{A_{\zeta}(Ri)}{A_{\zeta,0}}\right]_{\text{ZEKRE13}}, $$

$$\phi_3^{\text{EFB}} = \left\{\frac{A_{\zeta}(Ri)}{A_{\zeta,0}} - \frac{C_\psi C_\gamma Ri}{A_{\zeta}(Ri)(1 - Ri)}\right\}_{\text{ZEKRE13}}. $$

It is also clear that $C_4$ [see Eq. (22)] in the EFB scheme is related to their $C_\gamma$ [ratio of the dissipation time scales for TKE and for TPE; their Eq. (19)]:

$$C_4^{\text{EFB}} = 2C_3^{\text{EFB}}(C_\rho)_{\text{ZEKRE13}} = 2.15. $$

Furthermore, $\gamma_1$ [see Eq. (16)] can be identified as $1 - C_\theta$ in ZEKRE13 [their Eqs. (36), (52), and (55)], this giving

$$\gamma_1^{\text{EFB}} = 1 - C_\theta_{\text{ZEKRE13}} = 0.895. $$

Both $C_4^{\text{EFB}}$ and $\gamma_1^{\text{EFB}}$ can be joined into $O_{\alpha}^{\text{EFB}}$:

$$O_{\alpha}^{\text{EFB}} = 0.113. $$

Until now EFB appears very similar to the models I and II with different values of $\nu$, $C_3$, and $O_{\alpha}$. Let us now look at the shapes of stability dependency functions characterized by the variables $P$, $R$, and $Q$.

Implicit values of $P$ and $R$ in EFB (see Fig. 4) are stability dependent, contrary to the case of models I and II: $P$ ranges from $p_{0}^{\text{EFB}} = 0.377$ at neutrality to its asymptotic value at $Ri \to +\infty$:

$$Ri_{\text{lc}}^{\text{EFB}} = 0.25 $$

and $R$ varies significantly from $Ri_{0}^{\text{EFB}} = 0.455$ at neutrality to $Ri_{\text{lc}}^{\text{EFB}} = 0.282$ at $Ri \to +\infty$. Variation of both $R^{\text{EFB}}$ and $P^{\text{EFB}}$ are actually linked via [derived from Eqs. (28), (29), (75), and (76)]

$$p^{\text{EFB}} = \frac{A_{\zeta,0}C_\gamma R^{\text{EFB}}}{O_{\alpha}^{\text{EFB}} + A_{\zeta,0}C_3}. $$

Variable $Q$ can be deduced from the split of the $\phi_3$ stability dependency function, which we get by comparing Eq. (37) in ZEKRE13 with Eqs. (16) and (24):

$$\phi_3^{\text{EFB}} = \frac{\chi_3^{\text{EFB}}}{\phi_3^{\text{EFB}}}, $$

$$\phi_{\text{conv}}^{\text{EFB}} = \frac{\chi_3^{\text{EFB}}}{\phi_3^{\text{EFB}}}, $$

which gives

$$Q^{\text{EFB}} = R^{\text{EFB}}. $$

The differences between EFB and models I and II in $R$, $P$, and $Q$ are connected to a different parameterization of the pressure correlation terms. While EFB decouples the momentum flux from the heat flux [in pressure correlation term for momentum; Eq. (34) in ZEKRE13] in an equivalent way to model I ($\lambda_4 = 0$), pressure correlation terms in the budgets for TKE components are different. Namely, EFB proposes a partly independent model following results from simulations and atmospheric data measurements [Eqs. (48a)–(48c) in ZEKRE13].

The above properties of $R$, $P$, and $Q$ mean that links between $\nu$, $C_3$, $O_{\alpha}$ and $P$, $R$, $Q$ in models I and II are not valid in the EFB scheme. The specific emulation of EFB within our framework must take this into account by directly fitting one of the stability
dependencies while preserving its analytical links with the other ones.

On the basis of all the above, we elect to now emulate EFB in the framework of model I and with varying $R_{\text{EFB}}$ and $P_{\text{EFB}}$—Eq. (83) forbids that it could be fitted in model II with $\phi_Q$ going to zero for an infinite stability. Since those are linked [Eq. (82)], we may fit only one of them—$R_{\text{EFB}}$.

$$R_{\text{EFB.fit}}^\text{EFB} = \frac{R_{\text{EFB}}^0 + 0.64R_i + 28.48R_{\text{EFB}}^0 R_i^2}{1 + 2.92R_i + 28.48R_i^2},$$  

(86)

where the three numerical values have to evolve together with $R_{\text{EFB}}^0$ and $R_{\text{EFB}}^0$ if $R_{\text{i.e.}}$, $C_3$, $A_{\tau,0}$, or $O_{\Lambda}$ are modified.

The quality of the fit of $R_{\text{EFB}}$ and of the indirect fit of $P_{\text{EFB}}$ [via Eq. (82)] can be assessed by going back to Fig. 4. As soon as we know both $P$ and $R$, we can compute $R_{\text{fit}}$ from Eq. (31) and this delivers then indirect fits of $\chi_3$ and $\phi_3$ (the quality of such fits can be assessed from Fig. 5).

Finally, in order to enable a continuous prolongation of the emulated EFB scheme toward the unstable stratification case, we use stability dependency functions of model I with the variables set as

$$P_{\text{uns}} = P_{\text{EFB}}^0 = 0.377,$$  

(87)

$$R_{\text{uns}} = R_{\text{EFB}}^0 = 0.455,$$  

(88)

(i.e., getting exactly what would have been the model I results with these constant values). One then also obtains $C_{\tau}^\text{fit} = 6.72$. Let us tag the resulting system valid for whole range of $R_i$ as emulation extension of EFB (eeEFB). It is a different entity from the equivalent level within EFB but sharing all its functional dependencies in the stable range.

8. Practical aspects

a. Numerical realizations and intercomparison of schemes

Comparison of our models I and II with QNSE and EFB schemes showed that there exists a way of integrating all four occurrences (or very good approximations of them) in one single more general framework (three free parameters $\nu$, $C_3$, and $O_{\Lambda}$ and three variables $P$, $R$, and $Q$).

Table 3 shows a comparison of numerical values of the schemes with activated links (internal or taken over from models I and II in case of eeQNSE).

All schemes are relatively close to each other concerning $C_3$ and especially $\nu$, but they differ more in $O_{\Lambda}$ (lower values especially for eeEFB, but even for eeQNSE), which has an indirect influence on $R_{\text{i.e.}}$ via Eq. (45) or (63). Models I and II are, however, not restricted to basic setups of the $O_{\Lambda}$ value and can be tuned toward eeQNSE or eeEFB as anticipated in the third column of Table 3. Such targeted realizations cannot provide stability dependent $R(R_i)$ and/or $P(R_i)$ values (like in eeQNSE and eeEFB), but they represent possible model I and model II setups with via $O_{\Lambda}$ modified intensity of the TPE $\leftrightarrow$ TKE conversion.

We used such an adjusted model II setup in order to fit experimental, DNS, and LES data for relative vertical TKE component ($A_{\zeta}$, dimensionless squared momentum flux $[(\bar{u}'\bar{w}')^{2}]$, and dimensionless squared heat flux $[w'\theta'^{2}/(1/2\theta'^{2})]$ in stable stratification conditions. In this setup we keep $C_3$ and $\nu$ at their CCH02 values and modify only $O_{\Lambda}$ to

$$O_{\Lambda}^{\text{II}} = 0.29 \Rightarrow R_{\text{i.e.}} = 0.334.$$  

(89)

A comparison of models (adjusted model II, basic models I and II, eeQNSE, eeEFB, CCH02, and CCHE08) is displayed in Figs. 6–8 together with smoothed scatterplots generated from various available sources of verification data, using the same choices as CCHE08 and/or ZEKRE13. The results show that the adjustment of $O_{\Lambda}$ in the tuned model II significantly improves the anisotropy properties, especially concerning the momentum flux. For the vertical TKE component, there is a clear split between the eeEFB specific choice (see section 7) on the one side (a bit closer to the bulk of the verifying data) and the rest of the choices on the other side (among which the tuned model II behaves relatively well). The results for the heat flux confirm the already clearly seen problems against any basic use of model I and the need of tuning $O_{\Lambda}$ in model II, as a complement to using the basic relationship in Eq. (52). The differences between the various solutions can also be visualized in the 3D space of $R_{\infty}$, $P_{\infty} = R_{\text{i.e.}}$, and $Q_{\infty}$.
Table 3. Values of free parameters $n$, $O_a$, and $C_R$ for different schemes: basic model I and II—modified CCH02 scheme with $\alpha = 0.4$, $F = 0.64$, and $O_x = 0.25$. Modified CCH02 scheme with $l = 0.64$, and $O = \frac{2}{3}$.

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Basic model I</th>
<th>Basic model II</th>
<th>Tuned model II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_k$</td>
<td>$P_k(\nu, C_s, O_x) = 0.526$</td>
<td>$P_k(\nu, C_s, O_x) = 0.23$</td>
<td>$P_k(\nu, C_s, O_x) = 0.30$</td>
</tr>
<tr>
<td>$P$</td>
<td>$P(C_s, O_x) = 0.113$</td>
<td>$P(C_s, O_x) = 0.34$</td>
<td>$P(C_s, O_x) = 0.32$</td>
</tr>
<tr>
<td>$R_i$</td>
<td>$R_i = 0.72$</td>
<td>$R_i = 0.98$</td>
<td>$R_i = 0.71$</td>
</tr>
<tr>
<td>$C_R$</td>
<td>$C_R = 0.87$</td>
<td>$C_R = 0.79$</td>
<td>$C_R = 0.70$</td>
</tr>
<tr>
<td>$O_a$</td>
<td>$O_a = \frac{2}{3}$</td>
<td>$O_a = \frac{2}{3}$</td>
<td>$O_a = \frac{2}{3}$</td>
</tr>
<tr>
<td>$C_3$</td>
<td>$C_3 = 0.51$</td>
<td>$C_3 = 0.53$</td>
<td>$C_3 = 0.50$</td>
</tr>
</tbody>
</table>

Fig. 6. Scaled (with values at $R_i = 0$) dimensionless squared momentum flux $\left(\frac{\langle \nu^2 + \nu' \nu' \rangle}{U^2}\right)$ in stable stratification. Comparison of CCH02 and CHCHE08 schemes (both with $\alpha = 0$), of basic models I and II ($O_x = 2/3$), of tuned model II ($O_x = 0.29$), of eeQNSE, and of eeEFB with a smooth scatterplot (converted to a gray scale) aggregating laboratory experiments (Ohye 2001), LES (Zilitinkevich et al. 2007, 2008), and meteorological observations (Mahrt and Vickers 2005; Uttal et al. 2002; Poulos et al. 2002; Banta et al. 2002).

$C_3$ (the variables that directly influence the computation of exchange coefficients $K_M$ and $K_{ii}$). In Fig. 9, the eeQNSE and eeEFB single points are outside the basic model I and II isosurfaces but are closer to the adjusted $O_x = 0.29$ isosurface on which lays the tuned model II point.

A comparison of the resulting stability dependency functions $\phi_3$, $\chi_3$, and also of $R_i(\nu, C_s, O_x, \nu, C_s, O_x) = 0.187$ for eeQNSE and eeEFB in Figs. 10–12. In the latter two, the superposition of the eeQNSE and eeEFB curves in the unstable case is the result of a pure coincidence $R_i^{eeQNSE} = R_i^{eeEFB} = 0.377$.

The most significant difference between models I and II can of course be seen on the internal split of $\phi_3$ into $\phi_Q$ and $\phi_{conv}$ (for this, see Fig. 13). It is clear that eeEFB has the same asymptotic behavior as a model I; in the eeQNSE case the split of $\phi_3$ can equally well happen in model I or model II mode.

b. Realizability and stability analysis

Owing to the targeted binding of potentially independent $N^2$ and $S^2$ stability parameters into $R_i$ (or $R_{ii}$) via the equilibrium TKE equation (see section 2), our closure-discretization method ensures not only the absence of $R_{ik}$ but also always getting physically sound solutions in numerical simulation conditions.
An excellent tool for identifying potential spurious space oscillations (owing to the dependency of exchange coefficients on the gradients of the diffused variables) in turbulence schemes is the stability criterion developed by DHBD08. To avoid the associated risk of model divergence, it is required, in the case of our closure-discretization method, that the real part of each eigenvalue of the matrix $C$ [see section 3.1 and Eq. (5) in DHBD08]

$$
C = \begin{pmatrix}
\chi_3 - 2\text{Ri} \frac{\partial \chi_3}{\partial \text{Ri}} & \frac{\partial \chi_3}{\partial \text{Ri}} & 1 \\
-2C_3\text{Ri} \frac{\partial \phi_3}{\partial \text{Ri}} & C_3 \left[ \phi_3 + \text{Ri} \frac{\partial \phi_3}{\partial \text{Ri}} \right]
\end{pmatrix}
$$

should be positive for any stratification.

As seen from Fig. 14, all chosen model realizations systematically have positive real parts of $C$ eigenvalues. Thus our models are numerically stable in this respect.

More sophisticated models influenced by independent $N^2$ and $S^2$ increase the theoretical accuracy of the parameterization, but at the price of needing protection against the risk of unphysical results in extreme cases. The “quasi equilibrium” model requires a bounding of the $G_M$ value and the original level 2.5 of MY additionally requires a protection to be applied to the positive $G_M$ value.

Even if level-2.5 model closure is out of the main scope of our paper, it is possible to use Eqs. (37)–(38) and (55)–(57) as basis for level-2.5 closures of model I and model II, respectively. Both such potential schemes are suitable realizations at level 2.5 since they are unaffected by any TKE equilibrium approximation.

Both schemes are stable according to the stability criterion given by DHBD08 for a level-2.5 closure (see section 3.2 in DHBD08) provided $G_M$ and $G_H$ are adequately limited (see above). In case of model I, only an upper limitation of $G_M$ by a 134.0 value is required. For the basic and tuned ($O_\Delta = 0.29$) setups of model II, it is necessary not only to limit $G_M$ by the same maximum value of 134.0 but also to limit $G_H$ by minimal respective values of $-10.8$ and $-26.0$.

9. Outlook toward higher order terms and moist turbulence

The variety of the stability dependencies from one case to the next shown in the previous section indicates that our common framework, far from forcing an artificial convergence of the solutions proposed in the literature, just makes their intercomparison as unbiased as possible. Such comparisons may touch, for example, the $L$ specification, the choice of $C_e$, the extensions toward diagnostic TOMs and/or prognostic TPE, and the consideration of moist turbulent aspects. Conversely, such extensions, in principle necessary for the best possible use in NWP applications, can be derived from one of the considered theories and then safely translated to any emulation or simplification of the other ones. This will be briefly outlined in this section.

Concerning the transition from local “downgradient only” turbulent fluxes to those encompassing the
nonlocal effects of skewness in the distribution of fluctuations, we use the CCH07 proposal for extension of CCH02 with simplified diagnostic TOM terms (computed from SOMs):

\[
\overline{w'\theta'} = -K_H \left( \frac{\partial \theta}{\partial z} + \frac{O}{C_3} T_h(R_i) \right) \frac{\tau}{e} \left\{ 0.6 \frac{\partial \theta}{\partial z} \frac{\partial w'\theta'}{\partial z} + C_1 \frac{g}{\theta} \frac{\partial eA_z(R_i)}{\partial z} \frac{\partial w'\theta'}{\partial z} - \frac{g}{\theta} \left( \tau \frac{w'\theta'}{\partial z} \right) \right\},
\]

(91)

where \(C_1\) and \(C_2\) are tuned to 0.06 and 0.3 in the original paper (our specific closure-discretization method may require their further tuning). The derivation of Eq. (91) is based on the heat flux budget [Eq. (16)], this time with TPE expressed from Eq.(B8) as function of the heat flux and of the \(w'\theta'^2\) TOM term [third term in the curly brackets in Eq. (91)]. Extension to the eeQNSE and eeEFB cases concerning all the new terms is obtained by using their relationships for \(T_h(R_i)\) and \(A_z(R_i)\) in line with their \(P, R,\) and \(Q\). The advantage of this type of TOMs formulation is that \(K_H\) remains in factor of all terms.

The second of our potential extensions, the prognostic treatment of TPE (guideline taken this time from ZEKRE13), is also connected to the heat flux budget [Eq. (16)], but this time TPE is computed from the prognostic TKE and TTE equations (with the link \(TPE = TTE - TKE\)) and is thus influenced by the tendency term \(\frac{\partial w'\theta'^2}{\partial t}\) (a step then mostly orthogonal to the parameterization of TOMs):

\[
\overline{w'\theta'} = -\varepsilon C_3 S_{M,0} \frac{\partial \theta}{\partial z} \left[ 1 - \frac{2(1 - \gamma_1)(TTE - e)}{A_z(R_i)e} \right].
\]

(93)

This can be directly transformed, via a second steady-state approximation linking stability parameters [see Eq. (75) of ZEKRE13], into

\[
\overline{w'\theta'} = -K_H(\Pi) \frac{\partial \theta}{\partial z},
\]

(94)

\[
\Pi = \frac{TTE}{e} - 1.
\]

(95)

The time-stepping organization then looks as follows. In the p-TKE spirit, Eq. (2) is solved first, but with flux values from the previous time step used to replace the handling of Eqs. (3) and (4) by a direct computation of the shear and buoyancy terms. Such a choice implicitly solves the problem of the incorporation of the TOMs contributions into the conversion term. A prognostic equation for TTE (from which the buoyancy term vanishes) is solved in a parallel way. The new values of TKE\(^+,\) TTE\(^+\), and \(\Pi^+\) are then used to compute the vertical exchange coefficients \(K_M\) and \(K_H\). Those are then used for independent computations of the momentum and heat fluxes. In the latter case, factorization of the whole Eq. (91) by \(K_H\) is similar to the EFB corresponding level-3.0 proposal—the crucial ingredient of the algorithm.

The extension to the moist case can best be considered with the two previous extensions already granted. In such a case, apart from the classical “Betts’ transform” change of diffused heat and moisture variables, there remain three problems: how to compute the buoyancy flux in case the cloud cover is neither zero nor one, how to define an equivalent of TPE where the total moisture plays a symmetric role to that of the heat variable, and how to modify \(K_H\) assumed to be the same for heat and for moisture? The work reported here does not help for the first and the third issues. As all our stability dependencies can be expressed in terms of \(R_i\) (and hence if needed of \(\Pi\)) we are able to reduce the solution of the second problem to that of the first one. The formulation is, however, rather complex and hence out of the scope of the present paper.

10. Conclusions

We showed that, despite their sometimes widely diverging basic hypotheses, the dry turbulent schemes described in the CCH02 MY-type solution, in the QNSE theory, and in the relevant configuration of the EFB approach can all be adapted to our equilibrium-oriented closure-discretization method. The resulting compact and transversal framework [Eqs. (28)–(31) and Tables 1 and 3] matches each of the schemes’ basic behavior but helps to better understand their respective “signatures.” Essential properties of the new framework are (i) validity
for all stability conditions with absence of $\text{Ri}_{cr}$ and (ii) accounting for turbulence anisotropy in both momentum- and heat-related terms. The RMC01 paper inspired our discretization, as well as the related choice for the length-scale specification.

In the CCH02 case, as already identified by the same authors in CCHE08, some treatment was needed to obtain a No $\text{Ri}(cr)$ behavior. On the basis of the slight simplification $\alpha_3 = 0$, we found out that strictly following (i.e., without the initial approximation advocated there) the CCHE08-proposed path delivers the same stability-dependency functions as an unrelated method directly inspired by RMC01 (but with anisotropy effects duly accounted). This surprising result can hardly be the fruit of a coincidence. We believe that our resulting system indeed possesses some basic properties that must be easily transferable to any set of equations valid for the whole range of stabilities and leading to both a behavior of the No $\text{Ri}(cr)$ type and a separation between the $\chi_3$ and $\phi_3$ dependencies on $\text{Ri}$ (or $\text{Ri}_I$).

This belief is reinforced by the fact that the obtained system, once codified in its more general shape (i.e., with three free parameters and three variables determining the shape of stability dependency functions), is able to accurately fit the relevant functional dependencies derived in the independent QNSE and EFB theories for the range of positive $\text{Ri}$ values. Here no simplification or analytical manipulation is needed, just a consistent fitting procedure, on the basis of only one of the truly independent entities in the QNSE case, as well as in the EFB case. The strength of this approach is demonstrated (in both cases) by the quality of the indirect fit(s) of the other entities.

Despite the sometimes complex nature of the derivations leading to it, our scheme remains analytically very simple. This helps devising, with a minimum of arbitrariness, extensions to the unstable case, which neither QNSE nor EFB proposals possess in their original versions.

![Figure 9](image9.png)

**FIG. 9.** Points for basic models I and II ($\lambda = 0.4, F = 0.64, O_I = 2/3$) for tuned model II ($\lambda = 0.4, F = 0.64, O_{II} = 0.29$), for eeQNSE ($R_e = 0.44, P_e = 0.377, C_3 = 1.39$), and for eeEFB ($R_e = 0.282, P_e = 0.25, C_3 = 1.25$) and $O_I$ isosurfaces for models I and II ($O_I = O_{II} = 2/3$ and $O_{II} = 0.29$) in the space spanned by $C_3, R_e,$ and $P_e$. Projections of the eeQNSE, eeEFB, and tuned model II points are on the basic model II isosurface.

![Figure 10](image10.png)

**FIG. 10.** Stability dependency function for basic model I and II setups, tuned model II, emulation extension of QNSE, and emulation extension of EFB (set text) for (top) the whole range of $\text{Ri}$ and (bottom) stable stratification.

![Figure 11](image11.png)

**FIG. 11.** As in Fig. 10, but for the stability dependency function.
Identity and simplicity of the analytic shapes does not, however, mean uniformity of the resulting functional dependencies. On the contrary, each of the derived or emulated extended solution possesses its own way to condition the setup of the general shape of our formulation. Comparison between the values of free parameters (Table 3) and of stability dependency functions shapes shows for instance the signature of the wave–turbulence aspects of QNSE (Fig. 9) or indicates the impact of an independent model describing the anisotropy of the turbulent flow in EFB (Fig. 13).

We studied in detail five cases of our framework realization (basic model I and II, tuned model II, eeQNSE, and eeEFB). The comparison with experimental, DNS, and LES data shows the flexibility of our framework. The DHBD08 analysis confirms that these realizations of our framework are numerically stable.

The above five cases represent a progressive gradation (first $Q$, then $R$, and finally $P$) in transitions of variables from constant-value status to dependency on $R_i$. This is why we are confident in the representativeness of the present study’s outcome. However, we do not claim to have explored all issues related to this “codifying effort.”

Finally, owing to its full compatibility with the solutions proposed in CCH07 and ZEKRE13, our simplified framework can be extended without additional constraints both to the complementing of the conversion term by nonlocal TOM terms and to the prognostic handling of TTE or to their combination. The path toward a full moist turbulence extension is not as clear cut, since all that we reported here is, for instance, orthogonal to the shallow-convection closure problem. But the fact of being able to express all stability dependencies in function of the flux Richardson number is a nice asset for the formulation of moist source–sink terms of TKE. The fact that our set of equations systematically covers the whole range of possible stabilities might be here a big advantage, owing to the push toward instability created...
by the water vapor–induced change of density and/or by the latent heat release.

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APPENDIX A

List of Symbols and Acronyms

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\cdots)</td>
<td>Averaging operator</td>
</tr>
<tr>
<td>(\cdots)'</td>
<td>Fluctuation: (X = \overline{X} + X')</td>
</tr>
<tr>
<td>(\cdots)|</td>
<td>Model I</td>
</tr>
<tr>
<td>(\cdots)\II</td>
<td>Model II</td>
</tr>
<tr>
<td>(\cdots)_0</td>
<td>Value at (R_i = 0)</td>
</tr>
<tr>
<td>(\cdots)_\infty</td>
<td>Value for (R_i \to \infty)</td>
</tr>
<tr>
<td>(A_e)</td>
<td>Ratio of vertical component of TKE to TKE itself</td>
</tr>
<tr>
<td>(\alpha_3)</td>
<td>Coefficient determining (\lambda_6) and (\lambda_7) (see appendix B)</td>
</tr>
<tr>
<td>(\beta_5)</td>
<td>Coefficient controlling the direct influence of heat flux on momentum flux (see appendix B)</td>
</tr>
<tr>
<td>(C_{1-2})</td>
<td>Coefficients in TOM terms</td>
</tr>
<tr>
<td>(C_3)</td>
<td>(= 1/\sigma_3(\text{neutral})); inverse Prandtl number at neutrality, free parameter</td>
</tr>
<tr>
<td>(C_4)</td>
<td>(= 2C_3(\tau_\theta/\tau)_\nu); coefficient</td>
</tr>
<tr>
<td>(C_e)</td>
<td>Free parameter, linked to the choice of (L) and controlling the intensity of turbulent dissipation</td>
</tr>
<tr>
<td>(C_K)</td>
<td>Coefficient: (K_M = C_K L \sqrt{\epsilon} \chi_3), (K_H = C_3 C_K L \sqrt{\epsilon} \phi_3)</td>
</tr>
<tr>
<td>(C_H^b)</td>
<td>Coefficient of the Louis-type free-convective case</td>
</tr>
<tr>
<td>(\Gamma)</td>
<td>Coefficient of the Osborn–Cox model</td>
</tr>
<tr>
<td>(\gamma_1)</td>
<td>Coefficient controlling the influence of TPE on heat flux (see appendix B)</td>
</tr>
<tr>
<td>(\delta_{1-5})</td>
<td>Coefficients in CCH02 (see appendix B)</td>
</tr>
<tr>
<td>(D)</td>
<td>Denominator in (S_M) and (S_H) (see appendix B)</td>
</tr>
<tr>
<td>DNS</td>
<td>Direct numerical simulations</td>
</tr>
<tr>
<td>EFB</td>
<td>Energy and flux budget</td>
</tr>
<tr>
<td>(e)</td>
<td>((1/2)(\overline{u'w'} + \overline{v'w'})); turbulent kinetic energy</td>
</tr>
<tr>
<td>(\dot{e})</td>
<td>Current time-step equilibrium value of (e)</td>
</tr>
<tr>
<td>(\epsilon)</td>
<td>Coefficient influencing mean shear–turbulence interactions for pressure correlations (see appendix B)</td>
</tr>
<tr>
<td>(F)</td>
<td>Stability dependency function for heat in L79: (F_H = l_m h SF_{HF}), where (l_m h) represents Prandtl-type mixing lengths</td>
</tr>
<tr>
<td>(f(R_i))</td>
<td>(= \chi_3(\text{neutral})); function used as a filter ensuring the equilibrium condition in RMC01</td>
</tr>
<tr>
<td>(G_M)</td>
<td>(= \tau^2 S^2)</td>
</tr>
<tr>
<td>(G_H)</td>
<td>(= \tau^2 N^2)</td>
</tr>
<tr>
<td>(\theta)</td>
<td>Potential temperature</td>
</tr>
<tr>
<td>(K_E)</td>
<td>Vertical turbulent exchange coefficient for TKE: (\overline{w'\theta'} = -K_E(\partial \theta/\partial z))</td>
</tr>
<tr>
<td>(K_M)</td>
<td>Vertical turbulent exchange coefficient for momentum (see also (S_M)): (\overline{\text{u}'\text{w}'} = -K_M(\partial \theta/\partial z, \partial \text{u}/\partial z))</td>
</tr>
<tr>
<td>(K_H)</td>
<td>Vertical turbulent exchange coefficient for the thermodynamic quantity (see also (S_H)): (\overline{w'\theta'} = -K_H(\partial \text{w}/\partial z, \partial \theta/\partial z))</td>
</tr>
<tr>
<td>(\kappa)</td>
<td>Von Kármán constant</td>
</tr>
<tr>
<td>(L)</td>
<td>Length scale in the prognostic TKE formalism</td>
</tr>
<tr>
<td>LES</td>
<td>Large-eddy simulation</td>
</tr>
<tr>
<td>(l_m)</td>
<td>Mixing length from similarity laws</td>
</tr>
<tr>
<td>(\lambda)</td>
<td>(= (\tau_{p,\nu}/\tau)_\nu); ratio between pressure–velocity correlation time scale and TKE dissipation time scale</td>
</tr>
<tr>
<td>(\lambda_4)</td>
<td>Parameter controlling the direct influence of the heat flux on the momentum flux [see Eq. (B9)]</td>
</tr>
<tr>
<td>(\lambda_5)</td>
<td>(= (\tau/\tau_{p,\theta})); ratio between TKE dissipation time scale and pressure–temperature correlation time scale</td>
</tr>
<tr>
<td>(N)</td>
<td>Brunt–Väisälä frequency (BVF)</td>
</tr>
<tr>
<td>NWP</td>
<td>Numerical weather prediction</td>
</tr>
<tr>
<td>(\nu)</td>
<td>(= (C_1 C_3)^{1/4} = (2S_{MB})^{1/4}); free parameter (controlling the overall intensity of turbulence)</td>
</tr>
<tr>
<td>(O_\lambda)</td>
<td>(= <a href="%5Ctau_%5Ctheta/%5Ctau">C_3(1 - \gamma_1)</a>_\nu); free parameter</td>
</tr>
<tr>
<td>(P)</td>
<td>(= <a href="%5Ctau_%5Ctheta/%5Ctau">C_3(1 - \gamma_1)</a>_\nu); free parameter</td>
</tr>
<tr>
<td>(P)</td>
<td>Variable describing the joint effect of flow’s anisotropy and TPE conversion on the turbulent heat exchange with (\lim_{R_i \to \infty} P = R_iC)</td>
</tr>
<tr>
<td>(Q)</td>
<td>Variable describing the effect of the flow’s anisotropy on the turbulent heat exchange</td>
</tr>
<tr>
<td>QNSE</td>
<td>Quasi-normal scale elimination</td>
</tr>
</tbody>
</table>
\[ R = \text{Variable describing the effect of the flow's anisotropy on the turbulent momentum exchange} \]

**RANS**

Reynolds-averaging Navier–Stokes

**Ri**

Gradient Richardson number

**Ri,c**

Critical Richardson number

**Ri,f**

\( = \text{Ri}(K_H/K_M); \) flux Richardson number

**Ri,k**

\( = \lim \text{Ri}_f; \) critical flux Richardson number

**RSM**

Reynolds stress modeling

**S**

\( = [(\partial \bar{u} / \partial z)^2 + (\partial \bar{v} / \partial z)^2]^{1/2}; \) wind shear

**SOM**

Second-order moment

**TPE**

TOM Third-order moment

\[ \delta_{H} = \text{Stability dependency function for } (u, v); \quad K_M = \varepsilon \tau S_M \]

\[ \delta_{H0} = \text{Stability dependency function for } \theta; \quad K_H = \varepsilon \tau S_H \]

**σ_i**

\( = K_M/K_H; \) Prandtl number

**TKE**

\( = \varepsilon; \) turbulent kinetic energy

**TOM**

Third-order moment

**TPE**

\( = \text{TPE} + \text{TKE}; \) turbulent total energy

\[ T_H = \text{Stability dependency function in TOM terms} \]

\[ \tau = 2L/(C_\nu \varepsilon); \] TKE dissipation time scale

\[ \tau_l = \text{relaxation time scale} \]

\[ \tau_\theta = \text{Temperature variance dissipation time scale} \] (see Eq. (B8))

\[ \tau_{p,\theta} = \text{Pressure–temperature correlation time scale} \]

\[ \tau_{p,v} = \text{Pressure–velocity correlation time scale} \]

\[ u = u_z; \] zonal wind component

\[ v = u_y; \] meridional wind component

\[ w = u_z; \] vertical wind component

\[ \phi_3 = S_H/S_{H0}; \] scaled stability dependency function for \( \theta \)

\[ \phi_{\text{conv}} = \phi_3^2; \] conversion part of \( \phi_3 \) stability-dependency function

\[ \phi_Q = \text{Anisotropy part of } \phi_3 \text{ stability dependency function} \]

\[ \chi_3 = S_M/S_{M0}; \] scaled stability dependency function for \( (u, v) \)

**APPENDIX B**

**Relationships in CCH02 Paper**

This appendix lists the main relationships from CCH02 frequently referred to in the main text, arranged by type of equation.

**a. Heat flux**

\[ A_{ij} = \lambda_2 \delta_{ij} + \lambda_6 \tau S_{ij} + \lambda_7 \tau R_{ij} \] (B2)

\[ S_{ij} = \frac{1}{2} \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \] (B3)

\[ R_{ij} = \frac{1}{2} \left( \frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i} \right) \] (B4)

\[ \beta = (0, 0, \frac{g}{\theta}) \] (B5)

**b. Heat flux by neglecting \( d\theta^2/dt \) and TOM in TPE equation**

\[ A'_{ij} = -\tau \bar{u}_i \bar{u}_j \] (B6)

\[ A'_{ij} = \lambda_5 \delta_{ij} + \lambda_6 \tau S_{ij} + \lambda_7 \tau R_{ij} + \lambda_8 \tau^2 \beta_{ij} \] (B7)

**c. Temperature variance equation–TPE equation**

\[ \frac{d\theta^2}{dt} + \lambda_0 \tau \theta^2 = -2 \tau \bar{u}_{ij} \frac{\partial \theta}{\partial x_i} - \frac{\tau \theta^2}{\tau_\theta} \] (B8)

**d. Momentum flux**

\[ b_{ij} = -\lambda_1 \varepsilon \tau S_{ij} - \lambda_2 \tau S_{ij} - \lambda_3 \tau Z_{ij} + \lambda_4 B_{ij} \] (B9)

\[ b_{ij} = \bar{u}_i \bar{u}_j - \frac{2}{3} \bar{u}_l \bar{u}_l \] (B10)

\[ \Sigma_{ij} = b_{ik} S_{kj} + S_{ik} b_{kj} - \frac{2}{3} \delta_{ij} b_{km} S_{mk} \] (B11)

\[ Z_{ij} = R_{ik} b_{kj} - b_{ik} R_{kj} \] (B12)

\[ B_{ij} = \beta_i \bar{u}_j \theta + \beta_j \bar{u}_i \theta - \frac{2}{3} \delta_{ij} \beta_k \bar{u}_k \theta \] (B13)

**e. Stability dependency functions**

\[ S_M = \frac{s_0 + s_1 G_H + s_2 G_M}{D} \] (B14)

\[ S_H = \frac{s_1 + s_2 G_H + s_3 G_M}{D} \] (B15)

\[ D = 1 + d_1 G_H + d_2 G_M + d_3 G_H^2 + d_4 G_H G_M + d_5 G_M^2 \] (B16)
\[ d_1 = \lambda_5^{-1} \left( \frac{7}{3} \lambda_4 + \lambda_8 \right) \]
\[ d_2 = \left( \lambda_3^2 - \frac{1}{3} \lambda_2^2 \right) - \frac{1}{4} \lambda_5^2 (\lambda_6^2 - \lambda_7^2) \]
\[ d_3 = \frac{1}{3} \lambda_4 \lambda_5^2 (4 \lambda_4 + 3 \lambda_8) \]
\[ d_4 = \frac{1}{3} \lambda_4 \lambda_5^2 [\lambda_2 \lambda_6 - 3 \lambda_3 \lambda_7 - \lambda_5 (\lambda_6^2 - \lambda_7^2)] + \lambda_5^2 \lambda_8 \left( \frac{\lambda_3^2}{3} - \frac{1}{3} \lambda_2^2 \right) \]
\[ d_5 = -\frac{1}{4} \lambda_5^2 \left( \lambda_3^2 - \frac{1}{3} \lambda_2^2 \right) (\lambda_6^2 - \lambda_7^2) \]
\[ s_0 = \frac{1}{2} \lambda_1 \]
\[ s_1 = \frac{1}{3} \lambda_4 \lambda_5^2 (\lambda_6 + \lambda_7) + \frac{2}{3} \lambda_4 \lambda_5^2 (\lambda_6 - \lambda_3) \]
\[ s_2 = \frac{1}{2} \lambda_1 \lambda_5^2 \lambda_8, \]
\[ s_3 = -\frac{1}{8} \lambda_1 \lambda_5^2 (\lambda_6^2 - \lambda_7^2), \]
\[ s_4 = \frac{2}{3} \lambda_5^2, \quad s_5 = \frac{2}{3} \lambda_4 \lambda_5^2 \]
\[ s_6 = \frac{2}{3} \lambda_5^2 \left( \lambda_3^2 - \frac{1}{3} \lambda_2^2 \right) - \frac{1}{2} \lambda_1 \lambda_5^2 \lambda_8 \]
\[ + \frac{1}{4} \lambda_4 \lambda_5^2 (\lambda_6 - \lambda_7) \]
\[ \lambda_0 = 1 - \gamma_1, \quad \lambda_1 = \frac{4}{15} \lambda_1, \]
\[ \lambda_2 = \frac{5 \alpha - 6 \sqrt{\bar{F}_a}}{25}, \quad \bar{F}_a = \frac{5 \lambda + 14 \sqrt{\bar{F}_a}}{75}, \]
\[ \lambda_3 = \sqrt{\frac{2 \lambda + 3 \sqrt{\bar{F}_a}}{15}} \lambda_5 \]
\[ \lambda_4 = \frac{1}{2} \beta_3 \lambda_1, \quad \lambda_5 = \frac{\tau}{\rho \theta} = 5 (1 + \bar{C}_3), \]
\[ \lambda_6 = 1 - \frac{3}{4} \alpha_3, \quad \lambda_7 = 1 - \frac{5}{4} \alpha_3, \]
\[ \lambda_8 = (1 - \gamma_1) \frac{\tau}{\tau}. \]

**f. TKE equation in equilibrium condition**

\[ S_M G_M - S_H RiG_M - 2 = 0 \]

**APPENDIX C**

**Derivation of Model II**

We start from the general equation for \( \sigma_1 \), (54) and get

\[ Ri_f = \{ Ri[(60 \lambda_3^2 - 45 \lambda_1 \lambda_3 - 20 \lambda_2^2 + 15 \lambda_1 \lambda_2) \bar{C}_3 \lambda_4 \]
\[ - 225 \lambda_1 \bar{C}_3 + 60 \bar{C}_3] \} \times (Ri[(45 \lambda_1 \lambda_8 + (-60 \lambda_3 - 20 \lambda_2 + 60 \lambda_1 \lambda_4) \bar{C}_3 - 12 \lambda_4 \lambda_4 \bar{C}_3] + 225 \lambda_1 \bar{C}_3 \bar{C}_3)^{-1}. \]

\[ (C1) \]

When we insert Eq. (C1) back into the relationships for \( S_M \) and \( S_H \) [see Eqs. (B14) and (B15)], we get an explicit system:

\[ S_M = \frac{s_0 + s_1 RiG_M + s_2 G_M}{D}, \]
\[ S_H = \frac{s_4 + s_6 RiG_M + s_7 G_M}{D}, \]
\[ D = 1 + d_1^2 RiG_M + d_2^2 G_M + d_3^2 (RiG_M)^2 \]
\[ + d_4^2 RiG_M^2 + d_5^2 G_M^2, \]
\[ s_0 = \frac{1}{2} \lambda_1 \]
\[ s_1 = \frac{1}{3} \lambda_4 \lambda_5^2 (\lambda_6 + \lambda_7) + \frac{2}{3} \lambda_4 \lambda_5^2 (\lambda_6 - \lambda_3) \]
\[ + \frac{1}{2} \lambda_1 \lambda_5^2 \lambda_8, \]
\[ s_2 = -\frac{1}{8} \lambda_1 \lambda_5^2 (\lambda_6^2 - \lambda_7^2), \]
\[ s_4 = \frac{2}{3} \lambda_5^2, \quad s_5 = \frac{2}{3} \lambda_4 \lambda_5^2 \]
\[ s_6 = \frac{2}{3} \lambda_5^2 \left( \lambda_3^2 - \frac{1}{3} \lambda_2^2 \right) - \frac{1}{2} \lambda_1 \lambda_5^2 \lambda_8 \]
\[ + \frac{1}{4} \lambda_4 \lambda_5^2 (\lambda_6 - \lambda_7) \]
\[ \lambda_0 = 1 - \gamma_1, \quad \lambda_1 = \frac{4}{15} \lambda_1, \]
\[ \lambda_2 = \frac{5 \alpha - 6 \sqrt{\bar{F}_a}}{25}, \quad \bar{F}_a = \frac{5 \lambda + 14 \sqrt{\bar{F}_a}}{75}, \]
\[ \lambda_3 = \sqrt{\frac{2 \lambda + 3 \sqrt{\bar{F}_a}}{15}} \lambda_5 \]
\[ \lambda_4 = \frac{1}{2} \beta_3 \lambda_1, \quad \lambda_5 = \frac{\tau}{\rho \theta} = 5 (1 + \bar{C}_3), \]
\[ \lambda_6 = 1 - \frac{3}{4} \alpha_3, \quad \lambda_7 = 1 - \frac{5}{4} \alpha_3, \]
\[ \lambda_8 = (1 - \gamma_1) \frac{\tau}{\tau}. \]

**Using equilibrium conditions, the stability dependency functions \( S_M \) and \( S_H \) in Eqs. (C2) and (C3) can be simplified to get the same shape as \( S_M \) and \( S_H \) in model I**
[see Eqs. (37) and (38)]. We can namely modify $D$ [see Eq. (C4)] in model II by the following operation:

$$D + K_1(S_M G_M - S_H \text{Re}G_M - 2) = K_2(1 + d'' G_M + d'' G_M + d'' \text{Re}G_M^2),$$

(C12)

$$K_1 = \frac{2 \lambda_4[5C_4(3 \lambda_4 + \lambda_2) + 3]}{[(60 \lambda_3 + 20 \lambda_2) \lambda_4 + 15 \lambda_1] C_3 + 12 \lambda_4},$$

(C13)

$$K_2 = \frac{15 \lambda_1 C_3}{[(60 \lambda_3 + 20 \lambda_2) \lambda_4 + 15 \lambda_1] C_3 + 12 \lambda_4},$$

(C14)

$$d'' = \frac{9 \lambda_1 \lambda_2 + (12 \lambda_1 - 12 \lambda_3 - 4 \lambda_2) \lambda_4}{45 \lambda_1 C_3} - \frac{12 \lambda_4}{225 \lambda_1 C_3^2},$$

(C15)

$$d'' = \frac{1}{3}(3 \lambda_3 + \lambda_2) \lambda_4 + 3 \lambda_3^2 \lambda_2^2 - \frac{3 \lambda_4}{15 C_3},$$

(C16)

$$d'' = d' d''.$$  

(C17)

Using the right-hand side from Eq. (C12) as $D$ in the relationship for $S_M$ [see Eq. (55)] gives

$$s'' = s_0' \frac{[(60 \lambda_3 + 20 \lambda_2) \lambda_4 + 15 \lambda_1] C_3 + 12 \lambda_4}{30 C_3},$$

(C18)

$$s'' = s_0' \frac{1}{K_2},$$

(C19)

leading, together with Eq. (C17), to a simpler shape of $S_M$.

We do the same operation for $S_H$ [see Eq. (C3)] and get the resulting system, which has the same shape as model I, with differing $s$ and $d$ notations:

$$S_M = \frac{s'' G_M}{1 + d'' G_M},$$

(C20)

$$S_H = \frac{s'' + s'' G_M}{(1 + d'' G_M)(1 + d'' G_M)},$$

(C21)

$$s'' = (4 - 15 \lambda_1 C_3) \frac{[(60 \lambda_3 + 20 \lambda_2) \lambda_4 + 15 \lambda_1] C_3 + 12 \lambda_4}{450 \lambda_1 C_3^2},$$

(C22)

$$s'' = (12 \lambda_3^2 - 9 \lambda_1 \lambda_3 - 4 \lambda_2^2 + 3 \lambda_1 \lambda_2) \lambda_4 \times \frac{[(60 \lambda_3 + 20 \lambda_2) \lambda_4 + 15 \lambda_1] C_3 + 12 \lambda_4}{1350 \lambda_1 C_3^2}.$$  

(C23)


