A "Geostrophic Map"

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ABSTRACT

If a base map is constructed so that its scale is proportional to the sine of the latitude the geostrophic wind speed would be everywhere constant for a given contour spacing. Such a map construction is presented and an analysis is made of the errors introduced due to its being slightly non-conformal at certain latitudes.

A map whose scale is proportional to the sine of the latitude would facilitate geostrophic wind computations, in that a given distance between isobaric contours would then represent the same geostrophic wind speed at all latitudes. Such a mapping will hereafter be referred to as a "geostrophic map."

Unfortunately, it is not possible to design such a map to include all latitudes. A very simple one however, can be constructed which will serve the purpose adequately between the latitudes 30° and 60°.

Consider a map on which the meridians and parallels form a rectangular grid, of coordinates \( x' \) and \( y' \), with the spacing given by:

\[
\begin{align*}
y' - y'_0 &= K(\cos \phi - \cos \phi_0), \quad \text{or} \quad dy' = K \sin \phi d\phi, \quad (1) \\
x' - x'_0 &= K \eta(\lambda - \lambda_0), \quad \text{or} \quad dx' = K \eta d\lambda, \quad (2)
\end{align*}
\]

where:

\[
\begin{align*}
y' - y'_0 &= \text{map distance between any parallel of latitude } \phi, \text{ and a reference parallel, } \phi_0, \\
x' - x'_0 &= \text{map distance between any meridian, } \lambda, \text{ and a reference meridian } \lambda_0, \\
K &= \text{a constant which is determined by the choice of map size*}, \text{ and} \\
\eta &= \text{a parameter whose optimum value is to be chosen.}
\end{align*}
\]

An example of a mapping made according to equations (1) and (2) with \( \eta = 0.47 \) is shown in Figure 1.

The errors introduced by adopting this mapping can best be understood by considering the degree of correspondence between an ideal "geostrophic map" (i.e., one in which the scale is everywhere proportional to \( \sin \phi \)) and the map described by equations (1) and (2).

*Example: If 1 inch is to represent 5° of longitude, then

\[
K = \left(\frac{1}{2}\right) \frac{360}{2\pi} \frac{1}{\eta} = \left(11.45 \frac{\text{inches}}{\eta}\right)
\]

A rectangular \( x, y \) coordinate system for the earth can be defined in a manner similar to equations (1) and (2), where \( x, \) and \( y \) corresponds to \( x' \) and \( y' \), respectively, viz.,

\[
\begin{align*}
y - y_0 &= a(\phi - \phi_0), \quad \text{or} \quad dy = ad\phi, \quad (3) \\
x - x_0 &= a \cos \phi(\lambda - \lambda_0), \quad \text{or} \quad dx = a \cos \phi d\lambda, \quad (4)
\end{align*}
\]

where, \( a = \text{radial distance to the earth's center.} \)

Taking the map-to-earth ratio of corresponding distances, we have:

a. From equations (1) and (3),

\[
\frac{dy'}{dy} = \frac{K}{a} \sin \phi. \quad (5)
\]

It is obvious that the parallels of latitude are spaced on the chart such as to fulfill the stated requirement exactly (i.e. map scale proportional to \( \sin \phi \)) for distances measured in the north-south direction. The east-west component of the geostrophic wind will therefore be constant for a given pressure gradient as measured on the map, at all latitudes, with no error introduced due to the map construction.

b. From equations (2 and 4)

\[
\frac{dx'}{dx} = \frac{K \eta}{a \cos \phi}. \quad (6)
\]

Here it is seen that the spacing of the meridians on the map does not fulfill the required condition exactly. However, \( \eta \) can be so chosen as to make the function \( (\eta/\cos \phi) \) very closely approximate \( \sin \phi \) for a given range of \( \phi \). In Table 1, \( \eta/\cos \phi \) for \( \eta = 0.47 \) is compared with \( \sin \phi \) for various values of \( \phi \). The percentual differences (\( D\% \)) between them are seen to be small within the range of latitudes 30° < \( \phi < 60° \). \( \text{Minus} \ D\% \), of course, represents the percentage errors in the north-south component of the geostrophic wind computed using the proposed geostrophic map.

The differences, \( D\% \), are reflected in a non-conformal mapping of directions. If \( \psi' \) is the
map representation of a direction $\psi$, on the earth's surface, then it can be shown that the directional error $\Delta \psi = \psi - \psi'$ is given by

$$\tan \Delta \psi = \frac{1 - \sin 2\phi}{2\eta} \tan \psi.$$  

(7)

The maximum value of $\Delta \psi$ as a function of $\phi$ is then

$$\Delta \psi_{\text{max}} = \tan^{-1} \left( 1 - \frac{\sin 2\phi}{2\eta} \right) \sqrt{\frac{2\eta}{\sin 2\phi}}.$$  

(8)

In Table I are tabulated values of $\Delta \psi_{\text{max}}$ as a function of $\phi$ according to equation (8) with $\eta = 0.47$. It is seen that within the range of latitude $30^\circ \leq \phi \leq 60^\circ$ the directional errors are negligible and the mapping can be considered practically conformal.

A geostrophic wind scale for use in conjunction with the proposed map can readily be constructed, since it consists merely of a straight line calibrated in appropriate units of speed. The equation for the speed of the geostrophic wind $V_g$ on a constant pressure surface is:

$$V_g = \frac{g}{2\Omega \sin \phi} \frac{\Delta \phi}{\Delta n}.$$  

(9)

where:  
$g$ = acceleration of gravity,  
$\Omega$ = angular speed of the earth,  
$\phi$ = latitude,  
$\Delta \phi$ = height interval of adjacent contours, and  
$\Delta n$ = distance between adjacent contours.

Table I. Comparison of $\sin \phi$ vs $\eta/\cos \phi$

<table>
<thead>
<tr>
<th>Latitude</th>
<th>$\sin \phi$</th>
<th>$\frac{\eta}{\cos \phi}$</th>
<th>$\eta = 0.47$</th>
<th>$D%$</th>
<th>Angular error $\Delta \psi_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25°</td>
<td>.423</td>
<td>.518</td>
<td>+22.4%</td>
<td>-5.9°</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>.500</td>
<td>.543</td>
<td>+ 8.7</td>
<td>-2.3</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>.574</td>
<td>.574</td>
<td>0.00</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>.643</td>
<td>.613</td>
<td>- 4.7</td>
<td>+1.2</td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>.707</td>
<td>.664</td>
<td>- 6.0</td>
<td>1.5</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>.766</td>
<td>.730</td>
<td>- 4.7</td>
<td>+1.2</td>
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</tr>
<tr>
<td>55</td>
<td>.819</td>
<td>.819</td>
<td>0.00</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>.866</td>
<td>.941</td>
<td>+ 8.7</td>
<td>-2.3</td>
<td></td>
</tr>
<tr>
<td>65°</td>
<td>.906</td>
<td>1.111</td>
<td>+22.4%</td>
<td>-5.9°</td>
<td></td>
</tr>
</tbody>
</table>
Subject to the differences discussed above $\Delta n$ can be expressed in terms of its map representation $\Delta n'$ by the relation

$$\Delta n = \frac{a\Delta n'}{K \sin \phi}.$$  \hspace{1cm} (10)

Expressing $\Delta n'$ in terms of the number of radians of longitude $\Delta \lambda$ that an equivalent distance would subtend along a parallel, we have from equation (2):

$$\Delta n' = K\eta \Delta \lambda.$$ \hspace{1cm} (11)

Substituting from equation (10) and (11) in (9) gives

$$V_g = \frac{g\Delta \rho}{2\Delta \eta \Delta \lambda}.$$ \hspace{1cm} (12)

If $V_g$ is expressed in knots, $\Delta \rho$ in feet, $\Delta \lambda$ in degrees of longitude and $\eta$ is taken as 0.47, equation (12) becomes:

$$V_g = .762 \frac{\Delta \rho}{\Delta \lambda^0} \text{ knots.} \hspace{1cm} (13)$$

Using equation (13) a single geostrophic wind scale applicable for all latitudes between 30°–60° can be readily constructed. For example, if the geostrophic wind speed is 50 knots the spacing between 200-ft contours is:

$$\Delta \lambda^0 = \frac{.762 \times 200}{50} = 3.05 \text{ degrees of longitude.}$$

**Acknowledgement**

The author wishes to acknowledge the assistance of Major A. F. Gustafson in the preparation of this article.

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**MINUTES OF THE COUNCIL**

(Continued from the September 1950 Bulletin, p. 266)

**Meeting of the Council, Boston, Mass., August 10–11, 1950**

*Place*: New quarters of the Society, at 3 Joy St., Boston. *Time*: 10, Aug. 10 to 0040, Aug. 11. *Present*: President Yates, Vice-President George, Secretary Brooks, Treasurer Ward, Councilors Harrison, Haurwitz, Holzman, Houghton, Merewether, Petterssen, and Wexler; and, by invitation, Past-Councilor Ferguson, and Executive Secretary Spengler. All votes were unanimous.

34. Voted, to recommend to the membership that the following amendments be made to the Constitution and By-Laws:

Constitution, Art. VI, par. 3: In 2nd sentence change “majority” to “plurality” and for the 3rd sentence, which now reads, “In the event that no candidate for a given office receives a majority of the votes cast, the retiring Council shall choose by ballot between the two candidates who received the largest number of votes.”, substitute the following three sentences: “In the event of a tie between the leading candidates, the retiring Council shall elect one by ballot. Councilors shall be elected to the available vacancies in descending order in accordance with the number of votes received. In case there is a tie vote for the last remaining vacancy counted in this manner the retiring Council shall elect one.”

By-Laws, Art. III, par. 1: At end add: “and a minimum of a two-thirds vote of the Council members present shall be required for the resolution of any question.”

By-Laws, Art. III, par. 2: For the 2nd sentence, which reads, “A minimum of eleven affirmative votes shall be required for the approval of any question.”, substitute the following sentence: “A minimum of a two-thirds vote of the entire Council shall be required for the resolution of any question by mail ballot.”

By-Laws, Art. IV: Change “February 1” to “March 1,” and “October 1” to “October 15.”


36. Voted, in the case of a certain candidate for professional membership challenged by a professional member, to accept the renewed recommendations of COPES and the Committee on Admissions that this candidate meets the requirements of the Constitution for professional membership.

37. Voted, to thank Dr. D. M. Ludlum for his offer of a block subscription for “Weatherwise” at a very reasonable price and of sponsorship of this publication by the Society; but to decline it (since it does not appear that such an arrangement would be any more workable now than it was, on the same basis, except for sponsorship, when abandoned by the Society two years ago), yet assure Dr. Ludlum of the Society’s continued interest in this excellent journal and of its desire to consider an alternative proposition which would assure its unbroken continuation and a maximum circulation.

38. Voted, to appoint a permanent Committee on the Upper Atmosphere consisting of Dr. M. Ference, Chairman, Dr. Richard Craig, Dr. Bernhard Haurwitz, Dr. Victor Regener, and Dr. Harry Wexler.

39. Voted, to refer to the Committee on the Upper Atmosphere for early consideration the proposal by the McGraw-Hill Book Co., that the Society cooperate and possibly participate in the preparation of a technical book on upper atmosphere studies.

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