Markov Chain Models for Hydrological Drought Characteristics

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ABSTRACT

Estimation of drought characteristics such as probabilities and return periods of droughts of various lengths is of major importance in drought forecast and management and in solving water resources problems related to water quality and navigation. This study aims at applying first- and second-order Markov chain models to dry and wet periods of annual streamflow series to reproduce the stochastic structure of hydrological droughts. Statistical evaluation of drought duration and intensity is usually carried out using runs analysis. First-order Markov chain model (MC1) for dry and wet periods is not adequate when autocorrelation of the original hydrological series is high. A second-order Markov chain model (MC2) is proposed to estimate the probabilities and return periods of droughts. Results of these models are compared with those of a simulation study assuming a lag-1 autoregressive [AR(1)] process widely used to model annual streamflows. Probability distribution and return periods of droughts of various lengths are estimated and compared with the results of MC1 and MC2 models using efficacy evaluation statistics. It is found that the MC2 model in general gives results that are in better agreement with simulation results as compared with the MC1 model. Skewness is found to have little effect on return periods except when autocorrelation is very high. MC1 and MC2 models are applied to droughts observed in some annual streamflow series, with the result that the MC2 model has a relatively good agreement considering the limited duration of the records.

1. Introduction

Hydrological droughts are defined as below-average flows in rivers. Droughts are random phenomena that seriously affect the economy, environment, and living standards of human populations. To manage the droughts and take the necessary precautions, it is needed to estimate how often a drought of a certain length is expected to occur. This estimation is also required in solving various water quality and river navigation problems.

Because the available hydrological records are not of adequate length, it is necessary to study the streamflow time series using appropriate models and statistical techniques. Runs analysis (Yevjevich 1972)—the study of the statistical properties of various characteristics of runs defined as a sequence of observations below (above) a threshold value preceded and succeeded by observations above (below) the threshold—has been widely used for investigating the stochastic structure of hydrologic time series $X_t$. When an annual streamflow series is concerned, a year $t$ will be classified as wet or dry with respect to the threshold level $x_0$ for which $P(X \leq x_0) = p$. A year $t$ with the flow $X_t$ less than $x_0$ is to be called a dry year ($D$), and a year $t$ with the flow $X_t$ above $x_0$ is to be called a wet year ($W$). A run consists of a sequence of events of the same kind, preceded and succeeded by one or more events of a different kind. When droughts are concerned a negative run (drought) is a series of dry years between wet years ($\ldots WDD \ldots DW \ldots$). The distributions and parameters of run lengths (duration of a run) and run sums (total water deficit with respect to the threshold level along a run) have been analyzed by the theory of runs (Yevjevich 1972; Sen 1976; Dracup et al. 1980).

The frequency of a dry period (negative run) of a certain length $L = \ell$ can be characterized by its return period (recurrence interval) $T(\ell)$ (Loaiciga and Marino 1991; Fernandez and Salas 1999; Bayazit 2001; Shiau and Shen 2001). The return period $T(\ell)$ is defined as the expected number of years between any two successive occurrences of dry periods of a certain length $L = \ell$.

In this paper, a first-order Markov chain that has been used to model successive wet and dry periods is reviewed, and expressions for the probabilities and return periods...
of droughts are given. Second-order Markov chains are proposed for modeling when autocorrelation of the original process is high. Equations for the probabilities and return periods of droughts of various lengths are derived. Simple expressions can be obtained for the case of median threshold level. Results for the probabilities and return periods of droughts of a simulation study where large samples of a lag-1 autoregressive [AR(1)] model are generated are compared with those of first- and second-order Markov chain models. Applications are carried out on observed annual streamflows at some stations.

2. Literature review

Extensive literature is available related to modeling of droughts; for a recent review see Mishra and Singh (2011). Here, only a review of studies on the probability distribution and return period of lengths of dry periods will be presented.

On the basis of earlier studies by Feller (1968) and Schwager (1983), Fernandez and Salas (1999) presented a method for estimating the return period of droughts when underlying hydrological series (annual streamflows) are autocorrelated. They assumed that the binary process consisting of dry years (D: \( X_t < x_0 \)) and wet years (W: \( X_t \geq x_0 \)) follows a simple (first order) Markov chain with two states (dry and wet). They considered, however, negative runs that are not necessarily preceded and followed by wet years. They provided graphs for the return period \( T \) as a function of run length \( \ell \), probability \( p \), and lag-1 autocorrelation coefficient \( r_1 \) of annual flows. It should be remarked that these graphs are for dry years of length \( \ell \), whether or not limited at both ends by a wet year, and therefore are not applicable to the runs defined as in Yevjevich (1972).

Bayazit (2001) extended this analysis to the case where a run consists of \( \ell \) events \( D(X_t \leq x_0) \) preceded and succeeded by one or more events \( W(X_t \geq x_0) \).

Cancelliere and Salas (2004) derived the probability mass function of drought length assuming a periodic simple Markov chain. They pointed out that the results are particularly useful for drought analysis because the limited hydrological records that are generally available do not allow observing many drought events of long duration.

Bayazit and Onoz (2005) analyzed the probability distribution and return periods of joint droughts of a number of sites assuming that streamflows are cross-correlated first-order Markov processes. They derived a geometric distribution for joint run lengths with a parameter that depends on the threshold probability, lag-1 autocorrelation coefficients of at-site flows, and joint probability of flows at all sites remaining below the threshold.

Studies mentioned above are based on a simple Markov chain model. Chung and Salas (2000) and Cancelliere and Salas (2010) showed that when a lag-1 autoregressive AR(1) process is clipped by a constant threshold level \( x_0 \), the resulting binary process of dry and wet years does not follow a simple Markov chain, and therefore the widely used geometric distribution to model drought length can only be considered as an approximation. They argued that discrete autoregressive moving average (DARMA) models (with \( D = 0, W = 1 \)) represent a better approximation for modeling the sequence of dry and wet years, especially for the lag-1 autocorrelation coefficient \( r_1 = 0.5 - 0.7 \). Drought lengths obtained by this model have a better agreement with simulation results than the drought length probability mass function obtained for a simple (first order, lag-1) Markov chain, especially as \( r_1 \) increases. Chung and Salas (2000) used two DARMA models: DAR(1) and DARMA(1, 1). Return period estimates based on the DARMA(1, 1) model are shorter than estimates based on the DAR(1) model for \( L > 4 \). Historical estimates of the return period follow closely those obtained for DARMA(1, 1) for highly dependent flows, whereas the DAR(1) model is appropriate for weakly dependent flows. No simple expressions could be obtained for the return period of droughts because of the difficulties related to solving complex integrals.

Studies having Markov chains of an order greater than 1 are mainly limited to modeling daily precipitation series. Lowry and Guthrie (1968) applied the Markov chain models of an order greater than 1 to wet and dry periods of daily precipitation data, arriving at the general conclusion that first-order models are sufficient, although orders greater than 1 may be more appropriate in some areas. They suggested that models employing three or four states could be investigated in the future. Chin (1977) compared Markov chains of various orders and selected the one that minimizes the Akaike information criterion, and concluded that the common practice of assuming that the Markov order is always 1 is unjustified—a third-order model being required in one case.

Roldán and Woolhiser (1982) compared a first-order Markov chain and alternating renewal process for the probability distribution of wet and dry periods of daily precipitation series, concluding that the Markov chain model is in general superior. Lall et al. (1996) developed a wet–dry spell model for daily precipitation, and estimated the probability densities of wet and dry spell lengths using kernel estimators.

Schoof and Pryor (2008) examined the choice of Markov model order for daily precipitation occurrence on a monthly basis for a large number of stations in the United States. It is concluded that models chosen on the basis of the Bayesian information criteria that
Similarly, the mean length of the wet period, $E(W)$, can be found as

$$E(W) = 1/[1 - P(W|W)].$$

(2)

Similarly, the mean length of the wet period $L_W$ is

$$E(L_W) = 1/[1 - P(W|W)].$$

(3)

Return period of the drought, $T$ (expected interarrival time between any two successive droughts), equals the sum of $E(L)$ and $E(L_W)$:

$$T = 1/[1 - P(D|D)] + 1/[1 - P(W|W)].$$

(4)

where

$$1 - P(W|W) = [1 - P(D|D)]p/(1 - p).$$

(5)

and therefore

$$T = 1/[1 - P(D|D)] = 1/[P(W|D)p].$$

(6)

Considering a sufficiently long sample of size $N$, the expected value of the total number of runs of any length is

$$N_D = N/T,$$

(7)

and the expected number of runs of length $\ell$ is

$$N_L = f_L(\ell)N_D = Nf_L(\ell)/T.$$  

(8)

Return period of drought of a given length $\ell$ can be found from Eq. (8) as (Bayazit 2001)

$$T(\ell) = N/N_L = T/f_L(\ell).$$

(9)

Shiau and Shen (2001) and Gonzalez and Valdes (2003) derived the same equation by a different approach.

Substituting Eqs. (1) and (6) into Eq. (9), an expression for the return period of a drought of a certain length is found:

$$T(\ell) = 1/[1 - P(W|D)]^2 P(D|D)^{\ell - 1} p.$$  

(10)

Cancelliere and Salas (2010) showed that the simple Markov chain model, MC1, is not adequate when the series $X_t$ exhibits a significant autocorrelation. For large values of $\rho_1$, the difference between the two-step transition probabilities of the AR(1) process and of the first-order Markov chain becomes rather significant. It can be expected that a second-order Markov chain model (MC2) for modeling the sequence of wet and dry years can give a better approximation.

4. Second-order Markov chain model

Let it be assumed that the binary process of successive dry and wet years follows an MC2 such that the state of $X_t$ depends not only on the state of $X_{t-1}$ but also on the state of $X_{t-2}$. In this case, the probability function of drought length, $f_L(\ell)$, is given by

$$f_L(\ell) = \begin{cases} P(W|WD) & \ell = 1(a) \\ P(D|WD)P^{\ell-2}(D|DD)P(W|DD) & \ell = 2(b). \end{cases}$$

(11)

Conditional probabilities in Eq. (11) are two-step probabilities; for example, $P(W|WD)$ is the conditional
probability of \(X_t\) being in state \(W\) given that \(X_{t-2}\) is in state \(W\) and \(X_{t-1}\) is in state \(D\).

Mean length of dry period can be found as follows:

\[
E(L) = P(W|WD) + \sum_{\ell=2}^{\infty} P(D|WD)P^{\ell-2}(D|DD)P(W|DD)\ell
\]

\[
= P(W|WD) + P(D|WD)P(W|DD)\left[\sum_{\ell=1}^{\infty} P^{\ell-1}(D|DD)\ell + \sum_{\ell=1}^{\infty} P^{\ell-1}(D|DD)\right]
\]

\[
= P(W|WD) + P(D|WD)P(W|DD)[P^{-2}(W|DD) + P^{-1}(W|DD)] = 1 + [P(D|WD)/P(W|DD)]. \quad (12)
\]

Similarly, the expression for the mean length of wet period is

\[
E(L_w) = 1 + [P(W|DW)/P(D|WW)]. \quad (13)
\]

Return period of the drought of any length is the sum of \(E(L)\) and \(E(L_w)\):

\[
T = E(L) + E(L_w) = 2 + [P(D|WD)/P(W|DD)] + [P(W|DW)/P(D|WW)]. \quad (14)
\]

\[
T(\ell) = \begin{cases} 
2 \left[1 + \frac{P(D|WD)}{P(W|DD)}\right]/P(W|WD) & \ell = 1\text{(a)} \\
2 \left[1 + \frac{P(D|WD)}{P(W|DD)}\right]/[P(D|WD)P^{\ell-2}(D|DD)P(W|DD)] & \ell \geq 2\text{(b)}
\end{cases}
\]

Conditional probabilities in the above equations can be computed by bivariate and trivariate normal probability functions, assuming that \(X_t\) is normally distributed:

\[
P(W|WD) = P(WD)/P(WD),
\]

\[
P(D|WD) = 1 - P(W|WD),
\]

\[
P(D|DD) = P(DDD)/P(DD),
\]

\[
P(W|DD) = 1 - P(D|DD),
\]

\[
P(W|DW) = P(DWW)/P(DW),
\]

\[
P(D|WW) = P(WWD)/P(WW),
\]

where \(P(WD), P(DD), P(DW), \) and \(P(WW)\) are bivariate normal probabilities, and \(P(WD), P(DD), P(DW), \) and \(P(WW)\) are bivariate normal probabilities.

These can be computed by numerical integration of the multivariate normal probability distribution function (pdf) or using the tables given by Owen (1962).

At the median level \((p = 0.5)\), it is possible to compute the probabilities in Eq. (17) by simple formulas:

\[
P(W|WD) = P(DW) = f_2(-\rho_1),
\]

\[
P(D|DD) = P(DD) = f_2(\rho_1),
\]

\[
P(WDW) = f_3(-\rho_1, \rho_1^2, -\rho_1),
\]

\[
P(DDD) = f_3(\rho_1, \rho_1^2, \rho_1),
\]

\[
P(DWW) = f_3(-\rho_1, -\rho_1^2, \rho_1), \quad \text{and}
\]

\[
P(WWD) = f_3(\rho_1, -\rho_1^2, -\rho_1), \quad (18)
\]

where \(f_2\) and \(f_3\) are bivariate and trivariate normal probabilities of all the variables remaining below the median level \((p = 0.5)\), and can be computed by the following expressions (Abramowitz and Stegun 1965; Owen 1962):

\[
f_2(\rho_1) = \frac{1}{4} + \frac{\sin^{-1}\rho_1}{2\pi} \quad \text{and} \quad (19)
\]

\[
f_3(\rho_A, \rho_B, \rho_C) = \frac{1}{2} - \frac{\cos^{-1}\rho_A + \cos^{-1}\rho_B + \cos^{-1}\rho_C}{4\pi} \quad (20)
\]
5. Simulation study

Three-million-year-long samples of the autoregressive lag-1 AR(1) model, widely used for modeling annual streamflow series with significant autocorrelation, are generated for each value of \( r_1 = 0.1(0.1)0.9 \). Probability mass function and return period of droughts of various lengths are determined for various threshold levels corresponding to \( r = 0.1(0.1)0.5 \). These are compared with the results of first-order (MC1) and (for \( p = 0.5 \) only) second-order (MC2) Markov chains. It should be remarked that the following results are valid for dry periods of AR(1) populations.

Performances of the MC1 and MC2 models are evaluated on the basis of the results for the coefficient of efficiency (CE; Nash and Sutcliffe 1970) and mean relative error (ME):

\[
CE = 1 - \frac{\sum_{i=1}^{m} (X_i - Y_i)^2}{\sum_{i=1}^{m} (Y_i - \bar{Y})^2} \quad \text{and} \quad (21)
\]

\[
ME = \frac{\sum_{i=1}^{m} (X_i - Y_i)}{m \bar{Y}}, \quad (22)
\]

where \( m \) is the number of class intervals, \( X_i \) is the value predicted by the model (MC1 or MC2), and \( Y_i \) is the simulated or observed value of probability or return period of droughts of a given length.

a. Probability distribution of drought lengths

At the median threshold level \( (p = 0.5) \), probability mass function of drought length of both first- and

<table>
<thead>
<tr>
<th>( r_1 )</th>
<th>MC1</th>
<th>MC2</th>
<th>MC1</th>
<th>MC2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.998</td>
<td>1.000</td>
<td>0.1</td>
<td>0.0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.983</td>
<td>0.999</td>
<td>0.5</td>
<td>0.1</td>
</tr>
<tr>
<td>0.6</td>
<td>0.957</td>
<td>0.997</td>
<td>1.1</td>
<td>0.3</td>
</tr>
<tr>
<td>0.9</td>
<td>0.560</td>
<td>0.930</td>
<td>-0.9</td>
<td>-2.6</td>
</tr>
</tbody>
</table>

FIG. 1. Probabilities of droughts of various lengths obtained from MC1 and MC2 models compared with simulation results for \( p = 0.5 \): (a) \( r_1 = 0.3 \), (b) \( r_1 = 0.5 \), (c) \( r_1 = 0.6 \), and (d) \( r_1 = 0.9 \).
second-order Markov chain models are in very good agreement with simulation results for values of \( r_1 \) up to 0.3 (Fig. 1a). For \( r_1 > 0.4 \) and 0.5, the MC2 model gives results that agree with those of simulation, whereas the MC1 model predicts lower probabilities for \( L = 1 \) and higher probabilities for \( L \geq 2 \) (Fig. 1b).

For \( r_1 = 0.6 \), MC2 results agree with simulation results except for \( L = 2 \), where smaller probabilities are estimated. MC1 behaves as for \( r_1 = 0.4 - 0.5 \) (Fig. 1c). For \( r_1 \geq 0.7 \), both Markov chain models deviate from simulation results—differences increasing rapidly with \( r_1 \) (Fig. 1d); MC2 results are closer to those of simulation except for \( L = 2 \).

Results for the statistics CE and ME are shown in Table 1. CE values of the MC2 model are much larger and ME values are generally smaller than those of the MC1 model, especially for higher values of \( r_1 \), indicating better agreement with the simulation results.

The behavior of MC1 and MC2 models described above can be explained as follows. For \( L = 1 \), Eq. (11a) of the MC2 model is exact because \( P(W_1 \ldots W_D) = P(W|WD) \), states of the binary process preceding the \( W \) state in the beginning of a run of length one not being relevant. Therefore, the MC2 model in this case gives results that are in perfect agreement with simulation results for all \( r_1 \) and \( p \) values. The MC1 model gives smaller probabilities because in Eq. (1) \( P(W|WD) \) is replaced by \( P(W|D) \), which is smaller than \( P(W|WD) \), the difference increasing with \( r_1 \).

For \( L = 2 \), the MC2 model gives smaller probabilities than the simulation frequency histogram because \( P(W|DD) < P(W|WDD) \) [see Eq. (11b)]—the difference increasing with \( r_1 \). MC1 has better agreement in this case—\( P(W|D) \) being larger than \( P(W|DD) \) [see Eq. (1)].

For \( L > 2 \), results of the MC2 model agree better with simulation results because \( P(D|DD) > P(D|D) \) [see Eqs. (1) and (11b)], compensating for the differences in other probabilities. For \( L = 3 \) and 4 and \( r_1 \leq 0.6 \), agreement is very good. For longer droughts and higher autocorrelation, probabilities of the MC2 model are somewhat larger than the simulation frequency histogram, but much better than those of the MC1 model, which gives higher probabilities because lag-2 probabilities are closer to the exact values than lag-1 probabilities.

### Table 2. Statistics of efficacy for the drought return periods predicted by the MC1 and MC2 models as compared to the simulation results.

<table>
<thead>
<tr>
<th>( r_1 )</th>
<th>CE MC1</th>
<th>CE MC2</th>
<th>ME (%) MC1</th>
<th>ME (%) MC2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.994</td>
<td>1.000</td>
<td>4.9</td>
<td>-0.2</td>
</tr>
<tr>
<td>0.5</td>
<td>0.975</td>
<td>0.999</td>
<td>6.2</td>
<td>0.4</td>
</tr>
<tr>
<td>0.6</td>
<td>0.976</td>
<td>0.998</td>
<td>2.1</td>
<td>-1.0</td>
</tr>
<tr>
<td>0.9</td>
<td>0.345</td>
<td>0.706</td>
<td>-36.1</td>
<td>-18.6</td>
</tr>
</tbody>
</table>
b. Return period of droughts

Return periods of simulated droughts are estimated as the mean interarrival time for droughts of a certain length, L.

Results for the return period of droughts of various lengths are in general similar to the results of their probabilities because return periods increase as the probabilities decrease.

The MC2 model gives results for the return period of droughts of length one (L = 1) that are in almost perfect agreement with simulation results because \( P(W|\ldots WD) = P(W|WD) \) in this case [see Eq. (16a)], as explained in relation to the probabilities of runs of length one. Although the conditional probabilities in Eq. (15) for the mean return period \( T \) are not exact when computed by the MC2 model, differences from the simulation results are less than 2%. The MC1 model gives larger return periods in this case because \( P(D|\ldots DD) \), \( P(D|D) \) in Eq. (10) (Fig. 2a).

For \( L = 2 \), on the other hand, the MC1 model agrees better with simulation results than the MC2 model (Fig. 2b). This is because \( P(D|WD) < P(D|D) \) and \( P(W|DD) < P(W|D) \) in Eq. (16b), resulting in larger return periods for MC2. Again, differences increase with \( \rho_1 \).

For \( L > 2 \), results of MC2 have better agreement with simulation results because \( P(D|DD) > P(D|D) \), compensating for the effects of other two-step probabilities. For \( L = 3 \) and 4, agreement is almost perfect (Fig. 2c). For longer droughts, return periods of the MC2 model deviate from simulation results for \( \rho_1 > 0.6 \) (Fig. 2d), but are always closer to them than MC1 results, which gives much smaller \( T(L) \) values.

To summarize, the MC2 model leads to drought return periods that agree very well with the results of simulation. The agreement is almost perfect for\( L = 1 \), and much better than those of MC1 except in the case of \( L = 2 \). When the lag-1 autocorrelation coefficient \( \rho_1 \) is less than 0.7, the MC2 model can be used for estimating the return period for all values of drought length. The MC1 model, on the other hand, leads to much larger return periods for \( L = 1 \), and much smaller return periods for \( L \geq 3 \). Only for \( L = 2 \), the MC1 model agrees better with simulation results.

CE and ME values for drought return periods given in Table 2 confirm these results. The MC2 model has much larger CE values and much smaller ME, especially for higher \( \rho_1 \). CE decreases and ME increases rapidly with \( \rho_1 \), indicating a weaker agreement for higher autocorrelations.

For truncation levels other than median (\( p < 0.5 \)), results for probabilities and return periods described above for the MC1 model hold true. As an example, for \( L = 5 \) and \( p = 0.3 \), the MC1 model agrees with simulation for \( \rho_1 \leq 0.6 \), but estimates smaller return periods when \( \rho_1 \) is larger than 0.6 (Fig. 3).

### Table 3. Characteristics of annual streamflow series used in the study.

<table>
<thead>
<tr>
<th>Station</th>
<th>Period of record</th>
<th>Mean (m³ s⁻¹)</th>
<th>Coefficient of variation</th>
<th>Coefficient of skewness</th>
<th>Lag-1 autocorrelation</th>
</tr>
</thead>
<tbody>
<tr>
<td>St. Lawrence River near Ogdensburg, New York</td>
<td>1861–1957 (n = 97)</td>
<td>6819</td>
<td>0.09</td>
<td>-0.29</td>
<td>0.71</td>
</tr>
<tr>
<td>Göta River near Stötp-Vannersburg, Sweden</td>
<td>1808–1957 (n = 150)</td>
<td>536</td>
<td>0.18</td>
<td>-0.06</td>
<td>0.46</td>
</tr>
<tr>
<td>Çine River, station No. 701, Turkey</td>
<td>1938–2005 (n = 68)</td>
<td>6.5</td>
<td>0.49</td>
<td>0.07</td>
<td>0.59</td>
</tr>
<tr>
<td>Porsuk River, station No. 1203, Turkey</td>
<td>1936–2005 (n = 70)</td>
<td>7.9</td>
<td>0.51</td>
<td>0.79</td>
<td>0.54</td>
</tr>
</tbody>
</table>
c. Effect of skewness on return period

Assumption of normal distribution for the original variable (annual streamflow) is usually made in studies related to droughts. Sharma (2000) investigated gamma and lognormal distributed variables and found that the skewness has an effect on the extreme drought parameters such as the longest duration and the greatest severity.

To determine the effect of skewness on the return period of droughts, approximately gamma-distributed

![Time series of observed annual streamflows: (a) St. Lawrence, (b) Göta, (c) Çine, and (d) Porsuk Rivers.](image)

![Frequency histograms of observed flows: (a) St. Lawrence, (b) Göta, (c) Çine, and (d) Porsuk Rivers.](image)
variates are generated by a method described by McMahon and Mein (1986), and are used to generate skewed autocorrelated samples by the AR(1) model. Adjusted skew coefficient \( \gamma_p \) is related to the desired skewness \( \gamma \) and lag-1 autocorrelation coefficient \( \rho_1 \) by the expression

\[
\gamma_p = \frac{1 - \rho_1^3}{(1 - \rho_1^2)^{1.5}} \gamma. \tag{23}
\]

Skewed variates \( \eta \) are generated as

\[
\eta = \frac{2}{\gamma_p} \left( 1 + \frac{\gamma_p^2}{6} - \frac{\gamma_p^3}{36} \right) = \frac{2}{\gamma_p}, \tag{24}
\]

where \( \varepsilon \) is the standard normal variate.

Three-million-year-long series are generated with skew coefficient \( \gamma = 0.25(0.25)1.00 \) and lag-1 autocorrelation coefficient \( \rho_1 = 0(0.1)0.9 \), and return periods of runs of various lengths are determined at threshold levels corresponding to \( p = 0.1(0.1)0.5 \).

Results show that return periods in general increase as skewness increases. This effect is significant only when \( \rho_1 \) is high. For \( \gamma = 1.00 \) and \( \rho_1 = 0.9 \), return periods increase 50% for \( L = 1 \) and 8% for \( L = 10 \). For \( \gamma = 1.00 \) and \( \rho_1 = 0.8 \), the increase is 20% for \( L = 1 \) and 6% for \( L = 10 \). For \( \rho_1 < 0.7 \), return periods do not change significantly with skewness (Fig. 4).

When the skewness is relatively small its effect is negligible. For example, results for \( \gamma = 0.75 \) (not shown in Fig. 4) indicate that the effect is less than 25%, and for \( \gamma = 0.50 \) (also not shown) it is less than 10% for all values of \( \rho_1 \) and \( L \).

6. Applications

Markov chain models MC1 and MC2 have been applied to droughts at the median threshold level (\( p = 0.5 \)), observed in some annual streamflow series (Table 3). These series are selected as long records (\( n \geq 70 \)) with high serial correlation (\( \rho_1 > 0.45 \)). Figure 5 shows the annual streamflow records used in the study.
Frequency histograms of historical flows are shown in Fig. 6, and their correlograms in Fig. 7, together with the correlogram of the AR(1) model with the observed value of $r_1$ and its 90% confidence interval (CI).

It is seen that annual flows in all the rivers except the Porsuk River are nearly symmetrically distributed, whereas Porsuk River flows are positively skewed ($\gamma \approx 0.8$). Normal distribution assumption can be made in three cases. Although Porsuk flows have a rather high skew, the effect of skewness is still small because $r_1$ is less than 0.7. All the historical series can be assumed to follow the AR(1) model, with the possible exception of Göta River flows. Correlograms of the observed streamflows are in general inside the 90% confidence region. One auto-correlation coefficient of St. Lawrence River flows and Porsuk River flows, and two coefficients of Göta River flows, out of an estimated 10 coefficients in each case, lie outside the confidence region.

Frequency histograms and return periods of droughts of various lengths are estimated from the records and shown in Figs. 8 and 9, respectively. At some stations very long droughts ($L = 12–14$) with very large theoretical return periods are not plotted. In three stations, one such drought is observed that cannot be fitted by any model.

Frequency histograms and return periods computed by the MC1 and MC2 models are also plotted in Figs. 8 and 9.

Agreement between the probabilities and return periods of droughts estimated from the observations and those computed by the Markov chain models is reasonably good, considering the limited length of records. Göta River flows with a long record ($n = 150$) and relatively low serial correlation ($r_1 = 0.46$) have droughts that have probabilities and return periods that are in very good agreement with the predictions of both the MC1 and MC2 models. The MC2 model has a better fit with the frequency histograms of the St. Lawrence, Çine, and Porsuk River flows than the MC1 model. Return periods estimated from the records agree reasonably well with the return periods computed by the models for shorter droughts ($L \leq 4$).

The MC2 model has a slightly better fit than the MC1 model for St. Lawrence and Çine River flows. For Porsuk River flows, both the MC1 and MC2 models agree equally well with the observed return periods.

Statistics of efficacy estimated for the observed probabilities are given in Table 4. The CE value of the MC2 model is larger for the St. Lawrence, Çine, and Porsuk Rivers than for the MC1 model, while it is smaller for
Göta River, which has, however, the best agreement with both models. The MC1 and MC2 models have a rather poor fit in the case of the Porsuk River. ME values of both models are high except in the case of Göta River, which has the longest record. Rather poor performance of the models in some cases, especially with respect to the prediction of return periods, can be attributed to the small number of droughts observed during the period of record less than 100 years (all the rivers except Göta). The Porsuk River with a short record and high skew has the poorest agreement. Statistics of efficacy for the prediction of return periods of droughts have the best value in the case of the Göta River: CE is 0.753 for the MC1 model and 0.773 for the MC2 model; ME is 21% in both cases.

7. Conclusions

Successive wet and dry periods of a time series of streamflows are usually considered to follow a first-order Markov chain model. This approach is not adequate when the series has a high serial correlation. In this case, a second-order Markov chain is shown to be the better model. Expressions for the probability mass function and return period of dry periods of various lengths are derived assuming that the binary process of dry and wet periods constitute a second-order Markov chain.

A simulation study has shown that in general the MC2 model has a better agreement than the MC1 model in the case of a lag-1 autoregressive process. The agreement is perfect for droughts of length one, and much better than that of the MC1 model for droughts of length three or higher. Results for the statistics of the coefficient of efficiency CE and mean relative error ME confirm these findings. Skewness of streamflows is found to have a small effect on the return period only when serial correlation is very strong.

<table>
<thead>
<tr>
<th>River</th>
<th>CE (MC1)</th>
<th>CE (MC2)</th>
<th>ME (%) (MC1)</th>
<th>ME (%) (MC2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>St. Lawrence River</td>
<td>0.451</td>
<td>0.715</td>
<td>−35.2</td>
<td>−33.2</td>
</tr>
<tr>
<td>Göta River</td>
<td>0.939</td>
<td>0.883</td>
<td>−2.9</td>
<td>−3.3</td>
</tr>
<tr>
<td>Çine River</td>
<td>0.487</td>
<td>0.692</td>
<td>−18.8</td>
<td>−19.3</td>
</tr>
<tr>
<td>Porsuk River</td>
<td>0.345</td>
<td>0.450</td>
<td>−18.2</td>
<td>−18.6</td>
</tr>
</tbody>
</table>
Droughts observed at some stations with high autocorrelation are compared with the estimates of the MC1 and MC2 models. There is a relatively good agreement, especially for the MC2 model, considering that the observations have a limited period of record, except for some observed very long droughts.

Results of this study will be useful in predicting the frequencies and return periods of droughts of a given length in cases where the period of record is too short for these statistics to be estimated directly from the observations.

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REFERENCES


