An Analytic Longwave Radiation Formula for Liquid Layer Clouds

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ABSTRACT

Many Global Energy and Water Cycle Experiment (GEWEX) Cloud System Study (GCSS) intercomparisons of boundary layer clouds have used a convenient but idealized longwave radiation formula for clouds in their large-eddy simulations (LESs). Under what conditions is this formula justified? Can it be extended to midlevel layer clouds? This note first derives the GCSS formula using an alternative method to effective emissivity. A key simplifying assumption is that the cloud is isothermal in the vertical (and horizontal). However, this assumption does not turn out to be overly restrictive in practice. Then the GCSS formula is compared with a detailed numerical code, BugsRad. Sensitivity studies are performed in which cloud properties, cloud altitude, and thermodynamic profiles are modified. Here, the focus is primarily on midlevel, altostratocumulus layers. The results here show that the GCSS formula can be successfully extended to liquid (ice free), midlevel clouds. The GCSS formula produces remarkably accurate radiative profiles if the parameters are adjusted on a case-by-case basis. However, the formula needs to be calibrated using a more general radiative transfer code.

1. Introduction: Why is an analytic radiative transfer formula useful for cloud simulations?

Large-eddy simulation (LES) is a useful technique for modeling thin cloud layers. Overcast cloud layers often contain turbulence that is driven by longwave radiative transfer. LES models often use analytic within-cloud longwave radiative transfer formulas for three reasons. First, an analytic formula is easy to implement in a model. This aids model intercomparisons in particular, because all participants can easily implement the same radiative formula. Second, LESs usually last 6 h or less, a period too short to heat or cool clear air significantly, thereby vitiating the advantage of accurate multiband radiative calculations for gaseous absorption. Third, an analytic longwave formula is computationally inexpensive. This is advantageous because considerable expense is associated with both LES and numerical radiation calculations. LES is expensive because it requires a small grid size (often tens of meters) and numerous grid columns (often 100 in each horizontal direction). Numerical radiation calculations are expensive in part because longwave radiation can be exchanged over many kilometers in the vertical, from far below cloud base to far above cloud top.

An analytic longwave formula has been used with success in stratocumulus (Sc) intercomparisons by the Global Energy and Water Cycle Experiment (GEWEX) Cloud System Study (GCSS; Bretherton et al. 1999a,b; Duynkerke et al. 1999, 2004; Stevens et al. 2001, 2005). This formula for radiative flux is a simple exponential of liquid water path (LWP). Although this formula can be seen as a special case of the effective emissivity model (Cox 1976; Stephens 1978, 1984; MacVean 1993), the formula’s derivation is apparently
not widely known in the LES intercomparison community because none of the above intercomparisons reference a derivation. One goal of this paper is to provide an alternative derivation to the effective emissivity approach, thereby exposing the assumptions under which it is valid. Another goal is to test the formula’s applicability to thin, overcast, liquid, midlevel clouds. Although these clouds are structurally similar to boundary layer stratocumulus, they reside farther above ground and hence have larger cloud-base heating rates. The GCSS formula has already been applied to a midlevel cloud by Larson et al. (2006).

The outline of this note is as follows. In section 2, we derive the GCSS radiative transfer formula. In section 3, we compare this analytic formula with a sophisticated two-stream numerical radiative transfer model, BugsRad. As a test case, we use an altostratus cloud [i.e., overcast altostratus; see Larson et al. (2006)] that was observed by Fleishauer et al. (2002). We perform various sensitivity studies to test the generality of the analytic formula. Additionally, we test the formula on a boundary layer stratocumulus case. In section 4, we present conclusions.

2. Derivation of the radiative transfer approximation

In this section, we derive the longwave formula that has been used in GCSS intercomparisons. We follow the methodology of Goody (1995) rather than the effective emissivity approach of Cox (1976) and Stephens (1978, 1984). To increase computational speed, the formula retains only a single wavenumber band; that is, the formula is a gray model. We use the method of moments to average over the angular distribution of radiation, leaving a two-stream model with upward and downward streams.

The modeled cloud is idealized. Its geometry is a horizontally infinite uniform slab of finite thickness. We assume that everywhere within cloud there is a constant asymmetry parameter, single-scattering albedo, mass extinction cross section, and temperature. The within-cloud radiation is assumed to be forced by constant downwelling radiation from above and constant upwelling radiation from below.

Given these assumptions, the governing equation for the net upward longwave radiative flux $F$ can be written as (Goody 1995, p. 118)

$$ \frac{d^2F}{d\tau^2} = \alpha^2 F \quad \alpha^2 = 3(1 - \omega)(1 - \omega g). \tag{1} $$

Here, $F$ is the upwelling flux minus the downwelling flux (W m$^{-2}$), $\omega$ is the single-scattering albedo, and $g$ is the asymmetry factor. The single-scattering albedo $\omega$ is the probability that a droplet scatters rather than absorbs, where $\omega = 1$ represents complete scattering and $\omega = 0$ represents complete absorption. The asymmetry factor $g$ indicates the degree of forward or backward scattering, with $g = 1$ for complete forward scattering and $g = 0$ for isotropic scattering. The optical depth $\tau$ ranges from zero at the top of the cloud to a positive number at the cloud base. It is related to the LWP from cloud top to altitude $z$, $\text{LWP}(z)$, by

$$ \tau(z) = \frac{e}{m} \text{LWP}(z). \tag{2} $$

Here, $e$ is the extinction cross section with units of area, and $m$ is the mass of each droplet. We define $\text{LWP}(z)$ as the integral

$$ \text{LWP}(z) = \int_{z'}^{z} \rho(z') r_c(z') dz', \tag{3} $$

where $\rho$ is the density of dry air and $r_c$ is the cloud water mixing ratio.

To solve Eq. (1), boundary conditions are needed at cloud top and base. Just above cloud top, we set the downwelling radiance to $B_t = (\sigma/\pi) T^4$. Here, $\sigma$ is the Stefan–Boltzmann constant and $T$ represents an effective radiative temperature of air above the cloud (not the cloud-top temperature itself). Then we derive the following “mixed” boundary condition (see Goody 1995, 114–115):

$$ \left. \frac{dF}{d\tau} \right|_{\tau=0} = 4\pi(1 - \omega) \left[ \frac{F_{1=0}}{2\pi} - (B - B_t) \right]. \tag{4} $$

Here, $B = (\sigma/\pi) T^4$ is the emitted radiance from the cloud, assumed to have an effective temperature $T$. Also, $F_{1=0}$ is the net flux at cloud top (upwelling minus downwelling), which is unknown because only the downwelling component is specified. Likewise, at cloud base, we set the upwelling radiance to $B_b$ as

$$ \left. \frac{dF}{d\tau} \right|_{\tau=\tau_b} = 4\pi(1 - \omega) \left[ (B_b - B) - \frac{F_{1=\tau_b}}{2\pi} \right]. \tag{5} $$

By inspection, we see that Eq. (1) has a solution of the form

$$ F = Le^{\alpha \tau} + Me^{-\alpha \tau}. \tag{6} $$

The constants $L$ and $M$ are found by substituting Eq. (6) in the boundary conditions [Eqs. (4) and (5)]. We find

$$ L = \gamma[(B - B_t)c_1 e^{-\alpha \tau_b} + (B_b - B)c_2] \quad \text{and} \quad M = \gamma[(B - B_t)c_2 e^{\alpha \tau_b} + (B_b - B)c_1], \tag{7} $$
TABLE 1. Variations of $\kappa$ with effective radius $r_e$ as computed by BugsRad. Also computed is the extinction cross section per mass $e/m$ [see Eq. (2)] and $\alpha = \sqrt{3(1 - \omega)}(1 - \omega)$ [see Eq. (1)]. The calculations use a liquid water content of 0.4 g m$^{-3}$ and a representative longwave radiation wavelength of 13.7 $\mu$m.

<table>
<thead>
<tr>
<th>$r_e (\mu m)$</th>
<th>$e/m$ (m$^2$ kg$^{-1}$)</th>
<th>$\alpha$</th>
<th>$\kappa$ (m$^2$ kg$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0</td>
<td>189</td>
<td>1.13</td>
<td>213</td>
</tr>
<tr>
<td>7.5</td>
<td>152</td>
<td>1.06</td>
<td>160</td>
</tr>
<tr>
<td>10.0</td>
<td>125</td>
<td>1.02</td>
<td>127</td>
</tr>
<tr>
<td>12.5</td>
<td>106</td>
<td>0.99</td>
<td>105</td>
</tr>
<tr>
<td>15.0</td>
<td>91.2</td>
<td>0.98</td>
<td>89.5</td>
</tr>
</tbody>
</table>

where

$$\gamma = \frac{-4\pi(1 - \omega)}{c_1 e^{-\alpha \tau_b} - c_2 e^{\alpha \tau_b}}$$  \hspace{1cm} (8)

and

$$c_1 = \alpha - 2(1 - \omega) \quad c_2 = \alpha + 2(1 - \omega).$$  \hspace{1cm} (9)

Here, $\tau_b$ is the optical depth at cloud base.

In an LES, the cloud field is affected directly not by the radiative flux but rather by the heating rate, defined as

$$\left(\frac{\partial T}{\partial t}\right)_{\text{Rad}} = -\frac{1}{\rho c_p} \frac{\partial F}{\partial z},$$  \hspace{1cm} (10)

where $c_p$ is the specific heat of air at constant pressure. To compute $\partial F/\partial z$, we use the chain rule

$$\frac{\partial F}{\partial z} = \frac{\partial F}{\partial \tau} \frac{\partial \tau}{\partial z} = -\frac{e}{m} \rho r_c \alpha (L e^{\alpha \tau} - M e^{-\alpha \tau}).$$  \hspace{1cm} (11)

In GCSS intercomparisons, the following notation has often been used:

$$F_0 = M \quad F_1 = L e^{\alpha \tau_b} \quad \kappa = \frac{e}{m}.$$  \hspace{1cm} (12)

Substituting Eq. (11) into Eq. (10), letting LWP$_b$ denote LWP at cloud base, and rewriting in the GCSS notation [Eq. (12)], we have finally

![Fig. 1. Radiative and thermodynamic profiles for the DYCOMS-II RF01 boundary layer Sc cloud. (a) The longwave radiative heating rate and (b) net longwave radiative flux (upwelling stream minus downwelling stream). In (a), (b), a solid line denotes the solution obtained by a numerical radiation code, BugsRad, and the dot-dashed line denotes the analytic solution corresponding to Eq. (14). (c)–(e) The profiles of cloud water mixing ratio $r_c$, specific humidity, and temperature, respectively. There is strong cloud-top radiative cooling and minimal cloud-base radiative heating.](image-url)
The $F_0$ term represents cloud-top radiative cooling, and the $F_1$ term represents cloud-base radiative heating. If the cloud is thick enough that the $F_0$ and $F_1$ terms do not overlap (as in Fig. 4 below), then we may interpret $F_0$ as the net radiative flux at cloud top, $F_1$ as the net radiative flux at cloud base, and $\kappa$ as a factor that represents absorptivity. A key to obtaining this simple exponential form is to assume that the cloud layer is isothermal. In this expression, the $\rho$ may be cancelled from the numerator and denominator of the prefactor; they have been written explicitly to preserve the combination $\rho c_p$.

The approximation [Eq. (13)] can be shown to be equivalent to the “effective emissivity” model (Cox 1976; Stephens 1978, 1984) when the within-cloud temperature is constant and the absorption coefficients are assumed equal for upward and downward radiation streams. Stephens (1978) chooses the upward and downward absorption coefficients to be 130 and 158 m$^2$ kg$^{-1}$, respectively, whereas we use $\kappa = 119$ m$^2$ kg$^{-1}$ in the calculations below. The value $\kappa = 130$ m$^2$ kg$^{-1}$ has been used in past intercomparisons (Duynkerke et al. 1999, 2004). To what values of effective radius $r_e$ do these $\kappa$ values correspond? Although there is no unique relationship between $\kappa$ and $r_e$ alone, under typical conditions, $\kappa \approx 1/r_e$ to a very crude approximation. Example calculations using the BugsRad radiative transfer model show that $\kappa = 160$ corresponds roughly to $r_e = 7.5 \mu$m and $\kappa = 105$ to $r_e = 12.5 \mu$m (Table 1). This may be useful for modelers who wish to understand how effective radius affects cooling rates and turbulence levels in layer clouds.

Equation (13) only accounts for radiative heating or cooling within cloud. To represent radiative cooling of...
air above cloud top \((z > z_t)\), Stevens et al. (2005) augment Eq. (13), leading to the combined formula

\[
\frac{\partial T}{\partial t}_{\text{Rad}} = - \frac{1}{\rho c_p} \kappa p_r \left( F_0 e^{-\kappa \text{LWP}(z)} - F_1 e^{-\kappa \text{LWP}(z)} \right) - D(\lambda^3) H(z - z_d)(z - z_d)^{1/3} + z_d(z - z_d)^{-2/3},
\]

The \(F_0\) and \(F_1\) terms are meant to be applied only in cloud. In the last two terms, \(D\) is the large-scale horizontal divergence rate with units of inverse time, \(\lambda = 1 \, \text{K m}^{-1/3}\) is a constant that yields the correct units, \(z_t\) is the altitude of cloud top, \(z_0 = 840 \, \text{m}\) is a constant of the order of the turbulent layer thickness, and \(H\) is the Heaviside step function, which restricts the clear-air cooling to above-cloud areas \([H(z - z_d) = 0 \text{ for } z < z_d \text{ and } H(z - z_d) = 1 \text{ for } z > z_d]\). Because subsidence is known imprecisely, we treat \(D\) as a tuning parameter. We neglect heating and cooling below cloud base. One drawback of this formula is that the \((z - z_d)^{-2/3}\) factor becomes arbitrarily large as \(z\) approaches \(z_d\) from above. This renders the results sensitive to grid spacing.

3. When is the GCSS radiative approximation accurate?

We now compare the simple approximation [Eq. (14)] with calculations by a sophisticated numerical radiative transfer model, BugsRad (Stephens et al. 2001, 2004). BugsRad is a two-stream model that computes hydrometeor scattering and absorption, molecular scattering, and gaseous absorption. Gaseous absorption is computed using the correlated-\(k\) distribution method (Fu and Liou 1992). Cloud droplet optical properties, including extinction cross section \(e\), are computed using anomalous diffraction theory (e.g., Ackerman and Stephens 1987). Cloud droplet size distributions are treated as modified gamma distributions with a dispersion of 2.0 (Stephens et al. 1990). Cloud droplets are assumed to have a fixed effective radius in the vertical
and horizontal, which we set to 10 μm unless stated otherwise. At scales of tens of meters, three-dimensional radiative effects, which are ignored by BugsRad, may be significant. However, three-dimensional effects are likely to be less important for longwave than for shortwave radiation. For our cases, we used a fine within-cloud vertical grid spacing of about 8 m.

In all of our comparisons with BugsRad, the analytic formula [Eq. (14)] will use $g = 0.83$ and $\omega = 0.694$. Additionally, we keep constant the standard-case value of $\kappa (119 \text{ m}^2 \text{ kg}^{-1})$ in all cases except those in which we change the effective radius.

To make contact with prior work on the analytic formula [Eq. (14)], we first compare it with BugsRad calculations for a boundary layer stratocumulus cloud (Stevens et al. 2005) that was observed during Research Flight 01 (RF01) of the second Dynamics and Chemistry of Marine Stratocumulus (DYCOMS-II) field experiment (see Fig. 1). The input fields are the initial profiles (Stevens et al. 2005) from the GCSS LES intercomparison of this case. Stevens et al. (2005) computed radiative transfer through this cloud using the numerical model of Fu and Liou (1993) and the analytic formula [Eq. (14)] with parameter values $F_0 = 70 \text{ W m}^{-2}$, $F_1 = 22 \text{ W m}^{-2}$, and $\kappa = 85 \text{ m}^2 \text{ kg}^{-1}$. Our calculated radiative profiles resemble theirs and our best-fit values to BugsRad are similar: $F_0 = 62 \text{ W m}^{-2}$, $F_1 = 17.7 \text{ W m}^{-2}$, and $\kappa = 100 \text{ m}^2 \text{ kg}^{-1}$. However, when we use $\kappa = 119 \text{ m}^2 \text{ kg}^{-1}$, as in Fig. 1, the fit is still adequate. The best-fit value of $\kappa$ increases somewhat with increasing resolution because the peak in cloud-top cooling is narrow.

Figure 1a shows the heating rate, as calculated by both BugsRad (solid line) and Eq. (14) (dot-dashed line). The two calculations agree well. At cloud top there is strong cooling, and at cloud base there is slight heating. The cloud top cools because it radiates strongly upward and receives little compensating downwelling radiation from above. The cloud base heats because it emits less radiant energy than it receives from the warmer ocean surface below. The cooling minimum produced by BugsRad just above cloud top is due to a thin humid layer there (above the top of the plot). Figure 1b shows that analytic and BugsRad calculations also agree well for net radiative flux (upwelling minus...
Fig. 5. As in the standard case (Fig. 2), except that the cloud water mixing ratio $r_c$ is divided by 5. The analytic solution fits the numerical solution well within cloud.

downwelling). Figures 1c–e display the input profiles of cloud water mixing ratio, water vapor specific humidity, and temperature, respectively.

Next we examine an overcast altostratocumulus (ASC) layer that was observed on 11 November 1999 during the Complex Layered Cloud Experiment 5 (CLEX-5; Fleishauer et al. 2002). This cloud resided roughly 5.6 km above mean sea level and decayed with time. Its cloud-top cloud water mixing ratio evolved from an initial value of $r_c \approx 0.43$ g kg$^{-1}$ to zero (Larson et al. 2001; Fleishauer et al. 2002). Because the analytic radiative Eq. (14) is intended for use in LES models, we input horizontally averaged profiles generated by LES into the radiative scheme. Above and below the LES model domain, we merge a nearby radiosonde profile.

Our standard ASc case is displayed in Fig. 2. The input $r_c$ profile has already decayed, leaving $r_c \approx 0.2$ g kg$^{-1}$ at cloud top. We use the best-fit values $F_0 = 96.2$ W m$^{-2}$, $F_i = 61.2$ W m$^{-2}$, and $\kappa = 119$ m$^2$ kg$^{-1}$. Compared with the aforementioned stratocumulus layer, our ASc cloud has smaller $r_c$ (and higher altitude) and therefore smaller cloud-top cooling. However, the ASc cloud-base heating is substantially larger. This is because the temperature difference between cloud base and ground is much larger for ASc than for stratocumulus.

We now perform sensitivity studies in order to test the range of validity of the analytic formula. To isolate effects, we change only one profile (e.g., cloud water) at a time. In some cases, this procedure leads to incompatible combinations of profiles (e.g., the presence of liquid in subsaturated air). However, this is acceptable because our purpose is to isolate the influence of individual parameters on a radiation formula, not to model the time evolution of cloud fields.

A key assumption of our analytic formula [Eq. (14)] is that the cloud is isothermal. How much can the temperature vary between cloud top and cloud base before the analytic formula breaks down? To test this, we increase the cloud-base temperature until it is 20 K warmer than cloud top (see Fig. 3). Even with this enormous, unrealistically large variation in temperature, the analytic formula is still moderately accurate. However, the analytic flux profile does not vary enough from...
cloud top to base, and adjusting the parameters does not improve the overall shape.

Next we increase the cloud water mixing ratio $r_c$ (see Fig. 4). The analytic formula, with little change in $F_0$ and $F_1$ from the standard case, still matches BugsRad. Cloud-top cooling is increased (to $\sim 13$ K h$^{-1}$) because $r_c$ has increased. If we decrease $r_c$ (Fig. 5) from the standard profile, then our analytic formula again matches BugsRad, with a somewhat larger change in $F_0$ and $F_1$. Because $r_c$ is small, the cloud-top cooling becomes small ($\sim 1$ K h$^{-1}$). In addition, the cloud-base heating vanishes, despite the large temperature difference between ground and cloud base. This is because $r_c$ is especially low near the cloud base, leading to little absorption there. The above-cloud cooling profile in Eq. (14) no longer matches BugsRad well.

Next we vary cloud altitude. We first decrease cloud-top pressure to 300 mb (Fig. 6) and later increase cloud-top pressure to 800 mb, moving the cloud near the planetary boundary layer (Fig. 7). In either case, the cloud-top cooling deviates little from the standard case, but the cloud-base heating changes considerably. When we increase cloud altitude, the temperature difference between ground and cloud base increases, leading to more cloud-base heating and necessitating a larger value of $F_1$ (102 W m$^{-2}$; Fig. 6). When we decrease cloud altitude, the cloud-base heating nearly vanishes and consequently $F_1$ decreases to 14.9 W m$^{-2}$ (Fig. 7).

Next we dry the air above cloud by multiplying the vapor mixing ratio there by 0.4 (not shown). This increases cloud-top cooling and requires a slight increase in $F_0$ to 111 W m$^{-2}$. If we multiply the vapor mixing ratio below cloud by 0.4, then cloud-base heating and $F_1 = 69.9$ W m$^{-2}$ increase slightly (not shown). In both cases, we obtain a good fit.

Finally, we vary the droplet effective radius, defined as $r_e = r^3/r^2$, while holding $r_c$ fixed. If we decrease $r_c$ to 5 $\mu$m (not shown), then cloud-top cooling and cloud-base heating increase slightly, and the heating and cooling rates increase as when $r_c$ is increased. This is because the total cross-sectional area of the droplets has increased, leading to greater cloud optical thickness. In this case, $\kappa (=172$ m$^2$ kg$^{-1}$) must be increased substantially to match BugsRad. If we increase $r_c$ to

![Cloud top to base](image_url)
15 μm (not shown), then the opposite effects occur. In both cases, a good fit is obtained.

The best-fit values of our parameters $F_0$, $F_1$, and $D$ are summarized in Table 2. This illustrates the variation in parameter values over a wide range of conditions. The heating rate errors for our analytical formula are listed in Table 3. In all cases, the root-mean-square heating error is less than 0.33 K h$^{-1}$.

4. Summary of results and conclusions

Our first contribution is to provide an alternative derivation of an analytic within-cloud radiation formula

<table>
<thead>
<tr>
<th>Case</th>
<th>$\kappa$ (m$^2$ kg$^{-1}$)</th>
<th>$F_0$ (W m$^{-2}$)</th>
<th>$F_1$ (W m$^{-2}$)</th>
<th>$D$ (μ s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DYCOMS-II RF01 Sc</td>
<td>119</td>
<td>62</td>
<td>17.7</td>
<td>3.75</td>
</tr>
<tr>
<td>Standard</td>
<td>119</td>
<td>96.2</td>
<td>61.2</td>
<td>1.93</td>
</tr>
<tr>
<td>$\Delta T = 20$ K</td>
<td>119</td>
<td>109</td>
<td>118</td>
<td>1.03</td>
</tr>
<tr>
<td>Quadrupled cloud water</td>
<td>119</td>
<td>94</td>
<td>59.6</td>
<td>2.05</td>
</tr>
<tr>
<td>0.2× cloud water</td>
<td>119</td>
<td>104</td>
<td>59.2</td>
<td>0.58</td>
</tr>
<tr>
<td>300-mb cloud top</td>
<td>119</td>
<td>97.8</td>
<td>102</td>
<td>4.27</td>
</tr>
<tr>
<td>800-mb cloud top</td>
<td>119</td>
<td>96.9</td>
<td>14.9</td>
<td>1.85</td>
</tr>
<tr>
<td>0.4× above-cloud vapor</td>
<td>119</td>
<td>111</td>
<td>60.7</td>
<td>1.93</td>
</tr>
<tr>
<td>0.4× below-cloud vapor</td>
<td>119</td>
<td>96.2</td>
<td>69.9</td>
<td>1.93</td>
</tr>
<tr>
<td>5-μm effective radius</td>
<td>172</td>
<td>96.3</td>
<td>61.3</td>
<td>2.00</td>
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<tr>
<td>15-μm effective radius</td>
<td>89.9</td>
<td>96.7</td>
<td>61.7</td>
<td>1.93</td>
</tr>
</tbody>
</table>
TABLE 3. Vertically averaged RMS errors in the longwave radiative heating rate, given values of $\kappa$. Column 1 lists the case name. Column 2 lists the best-fit value of $\kappa$ for each case, which is not necessarily the value of $\kappa$ (usually 119 m$^2$ kg$^{-1}$) used in the figures. Column 3 lists the range of $\kappa$ values that yield RMS heating rate errors of $<0.5$ K h$^{-1}$. Column 4 lists the RMS heating rate error when the best-fit value of $\kappa$ is chosen. Column 5 lists the RMS heating rate error when $\kappa$ is set to the plotted values listed in Table 2 (usually 119 m$^2$ kg$^{-1}$).

<table>
<thead>
<tr>
<th>Case</th>
<th>Best-fit $\kappa$ (m$^2$ kg$^{-1}$)</th>
<th>Range of $\kappa$ that yields RMS error $&lt;0.5$ K h$^{-1}$ (m$^2$ kg$^{-1}$)</th>
<th>RMS error when $\kappa =$ best-fit value (K h$^{-1}$)</th>
<th>RMS error when $\kappa =$ plotted value (K h$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DYCOMS-II RF01 Sc</td>
<td>100</td>
<td>71--122</td>
<td>0.238</td>
<td>0.323</td>
</tr>
<tr>
<td>Standard</td>
<td>119</td>
<td>80--170</td>
<td>0.078</td>
<td>0.078</td>
</tr>
<tr>
<td>$\Delta T = 20$ K</td>
<td>101</td>
<td>74--131</td>
<td>0.129</td>
<td>0.329</td>
</tr>
<tr>
<td>Quadrupled cloud water</td>
<td>124</td>
<td>93--160</td>
<td>0.11</td>
<td>0.127</td>
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<tr>
<td>0.2× cloud water</td>
<td>118</td>
<td>0--278</td>
<td>0.031</td>
<td>0.031</td>
</tr>
<tr>
<td>300-mb cloud top</td>
<td>126</td>
<td>95--168</td>
<td>0.114</td>
<td>0.142</td>
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<tr>
<td>800-mb cloud top</td>
<td>110</td>
<td>68--166</td>
<td>0.123</td>
<td>0.151</td>
</tr>
<tr>
<td>0.4× above-cloud vapor</td>
<td>122</td>
<td>87--168</td>
<td>0.087</td>
<td>0.095</td>
</tr>
<tr>
<td>0.4× below-cloud vapor</td>
<td>119</td>
<td>82--168</td>
<td>0.085</td>
<td>0.085</td>
</tr>
<tr>
<td>S-μm effective radius</td>
<td>172</td>
<td>120--240</td>
<td>0.146</td>
<td>0.146</td>
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<tr>
<td>15-μm effective radius</td>
<td>90</td>
<td>55--131</td>
<td>0.068</td>
<td>0.068</td>
</tr>
</tbody>
</table>

[Eq. (13)] that has been widely used in GCSS and other intercomparisons of boundary layer clouds (e.g., Bretherton et al. 1999a,b; Duynerker et al. 1999, 2004; Stevens et al. 2001, 2005). The formula is applicable to liquid-only layer clouds. A key assumption is that the cloud layer has constant temperature in the vertical, leading to a simple exponential formula. In practice, however, even large temperature differentials (20 K) do not lead to unacceptable errors (see Fig. 3).

The analytic formula [Eq. (14)] contains four main adjustable parameters: $F_0$ controls cloud-top cooling, $F_1$ controls cloud-base heating, $\kappa$ controls cloud absorptivity, and $D$ controls above-cloud cooling. When these are optimized for individual cases, the formula can yield remarkably accurate fluxes and heating rates. However, these parameter values must be obtained on a case-by-case basis by comparison, for instance, with a sophisticated numerical code such as BugsRad. Although for most cases we have been able to set $\kappa = 119$ m$^2$ kg$^{-1}$, $F_0$ and $F_1$ must change as the cloud-base and -top temperatures change.

On the other hand, if the cloud water in an LES were to change between grid columns or during runtime (e.g., Stevens et al. 2005; Larson et al. 2006), the best-fit parameter values would not change markedly (see Figs. 4 and 5). This allows us to use one set of parameter values for each grid column and time step throughout an LES. Then, to account for the effect of horizontal liquid fluctuations on radiative fluxes, an LES can simulate the horizontal fluctuations in liquid and apply the analytic radiative formula to each column.

What is the value of an analytic formula if we must first run a numerical radiation code to calibrate it? The first benefit is that an analytic formula reduces computational cost during runtime. The second benefit is that the numerical radiation code need only be run in stand-alone mode, thereby avoiding the effort required to implement the radiation code in an LES model. Recall that an LES of layer clouds simulates only a thin slab of the atmosphere. If a numerical radiation code is implemented in the LES, then either radiative fluxes must be specified at upper and lower boundaries, which are usually obtained from a separate stand-alone radiation calculation, or else the radiative fluxes must be computed during runtime far above and/or below the LES boundaries, thereby requiring more computational time.

Finally, we have shown that the GCSS radiation formula, which has been applied with success to boundary layer stratocumulus, can also be applied to midlevel clouds if the parameter values are adjusted. In particular, $F_1$ must be increased in order to account for increased cloud-base heating.

We hope that documenting these tests of the GCSS radiation formula provides a useful service to the community. The authors freely provide the BugsRad code to those who desire it to customize the GCSS formula for their cloud cases.

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REFERENCES


