Self-Advection of Density Perturbations on a Sloping Continental Shelf

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ABSTRACT

Bottom water movement on the continental shelf is modeled by the nonlinear interaction between longshore bottom geostrophic flow and the density field. Bottom geostrophic velocity, subject to linear steady momentum equations with linear bottom friction, can be generated by along-isobath density variations over a sloping bottom. At the same time, the density field is slowly advected by the velocity field. Away from boundary layers, the interplay is governed by Burgers' equation, which shows the formation and self-propulsion of strong density gradients along an isobath. The direction of propagation of a dense water blob is to have shallow water on the right-(left-) hand side facing downstream in the Northern (Southern) Hemisphere. The propagation of a light water blob is opposite to that of a dense water blob.

The problem is further investigated by solving the governing equations numerically. Under forcing by localized surface cooling, the flow in the mid-shelf region shows the characteristics of the solution of Burgers' equation. A coastal buoyancy source generates a shore-hugging plume, slowly moving along the coast in the direction of Kelvin wave propagation. The flow associated with coastal dense water discharge has different characteristics: the dense water moves away from the coast initially. The accumulation of dense water on the mid-shelf then invokes the same self-advection process as found for surface cooling.

The theory sheds light on bottom water movements in the Adriatic Sea, over the Antarctic continental shelf, and in the Middle Atlantic Bight. It also describes the dispersion of river water and dense water outflow on the shelf. The model results agree qualitatively with the observed distribution of bottom water and give correct order-of-magnitude estimates for the propagation speed of density perturbations.

1. Introduction

Dense water is produced at the sea surface in several places in the World Ocean, where the atmospheric influence is strong. This process is usually associated with strong vertical mixing. If the water depth is shallow, such as is the case on continental shelves and in shallow seas, dense water may reach the bottom. In the Weddell Sea of the Antarctic continental shelf, cold and saline water is formed on account of the freezing of surface water (Gill, 1973). In the Middle Atlantic Bight, winter water is produced by surface heat loss and is observed as a pool of cold water below the seasonal thermocline in spring and summer (Bigelow, 1933). Both evaporation and cooling are responsible for the formation of dense water in the Adriatic Sea in winter (Hendershott and Rizzioli, 1976). In all the cited cases, dense water always reaches the bottom of a shallow sea. Moreover, the dense water thus formed is not stationary. Observations show that the outflow of the Gulf of Maine Intermediate Water reaches the New England shelf in summer (Hopkins and Garfield, 1979). In the Middle Atlantic Bight, the cold winter water inside the cold pool moves southward (Beardsley et al., 1976; Houghton et al., 1982). The Antarctic Bottom Water flows along the Weddell Sea shelf break westward to the tip of the Antarctic Peninsula (Foster and Carmack, 1976). In the Adriatic Sea, a cyclonic gyre is a prominent feature of the winter circulation (Hendershott and Rizzioli, 1976). The bottom water movement over mid-latitude shelves is important to the dispersion of pollutants and the distribution of bottom fauna, both of concern to fisheries.

At present, the dynamics of bottom water movement over shallow seas and continental shelves is not well understood. It has been thought that the bottom water movement in the Middle Atlantic Bight is due to passive advection by a mean flow of unknown origin, perhaps associated with the deep ocean circulation (Sverdrup et al., 1942; CSANADY, 1976; Beardsley and Winant, 1979). However, similar bottom water movements are also observed in semi-enclosed basins like the Adriatic Sea or the Weddell Sea, where any deep ocean influence should be absent. A reasonable alternative hypothesis is that the bottom flow is density driven. Holland (1973) has shown in a numerical deep ocean circulation model that flow...
may be generated by the misalignment of constant density isopleths at the bottom with isobaths. This process is likely to be relatively more important in shallow water.

Numerous attempts have been made to model the circulation on the continental shelf including density effects. The complexity of the problem has necessitated various simplifications. Basically, there are three categories of models in order of increasing sophistication. In the first category, bottom topography is only allowed a passive role. The models of Stommel and Leetmaa (1972) and of Csanady (1976) on the eastern North American shelf, and that of Killworth (1974) on the Antarctic shelves belong to this category. The effect of the density field on the flow is in these models limited to the generation of thermohaline velocity described by the thermal-wind equation. There is no interaction between the flow field and the density field. Models in the second category take into account both topography and stratification but not the effect of density advection by the flow. The models of Pedlosky (1974) and Csanady (1979) belong to this category, as do many “diagnostic” numerical models, e.g., Hsueh and Peng (1978). The major shortcoming of these models is the neglect of the important dynamics of density advection, which causes the density perturbation to propagate slowly away from its source region. Hendershot and Rizzoli (1976) included all the essential dynamic factors in their numerical calculations of the winter circulation in the Adriatic Sea. Their model belongs to a third category, in which stratification, topography and density advection are all considered. In order to understand the dynamics of density-driven flow in detail, it is desirable to analyze further the problem of interaction between the density field and flow over a sloping bottom, and to develop simple analytical models of key phenomena.

In this paper, the dynamics of density-driven flow are investigated both analytically and numerically using simple methods of the third category. Dynamic equations are solved prognostically to understand the behavior of the movement of density perturbation over topography. Solutions can be applied to the near-bottom circulation on an open continental shelf under forcing by surface density flux. The theory can also be used to analyze the effects of river efflux and dense water outflow onto a continental shelf under weakly stratified conditions, when the interaction between a nonuniform density field and topography is important.

The central result of the analysis is a simple model of density self-advection over the mid-shelf region away from any coastal and shelf edge boundary layers. The dynamical argument contains two steps: the generation of bottom geostrophic flow and the subsequent advection of the density perturbations by this flow. The first process can be understood from a kinematic consideration of the flow field over a sloping bottom, also taking into account the dynamical constraint of vorticity conservation. Suppose that an isolated column of dense water is produced in a flat bottom ocean in the Northern Hemisphere, e.g., by atmospheric cooling. After a period of initial geostrophic adjustment, the steady velocity field is shear flow satisfying the thermal-wind relation with reference to the bottom as shown in Fig. 1a. The density perturbation is of no further significance in this case. On the other hand, if the density perturbation is over a sloping bottom, it is clear that total cross-isobath transport by the thermohaline flow will be larger in deeper water (Fig. 1b). A steady state similar to that shown in Fig. 1a cannot exist. The divergence in the thermohaline flow can only be compensated for by a depth-independent flow component, which is the bottom geostrophic flow in this paper. Because the bottom geostrophic flow component extends to the bottom, it stretches vortex lines, unless the bottom geostrophic velocity is along isobaths as shown by the arrows inside the density perturbation in the top view of Fig. 1b. It is clear that with a steeper bottom slope and a stronger density perturbation, the divergence of the thermohaline velocity is larger. Therefore, the magnitude of the compensating bottom geostrophic flow is also proportional to the bottom slope and the magnitude of the density perturbation. The advection of the density field by the flow field forms the second part of the problem. A nonlinear interaction between

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Fig. 1. Schematic diagram showing the effect of bottom topography on the generation of bottom flow.
the two results from the dependence of advection velocity on density itself, and can be described by an equation similar in form to the one used by Burgers (1948) to model turbulence, which also describes shock formation and propagation. The known solutions of this equation provide simple analytical models of density self-advection.

2. Mathematical formulation

We consider density-driven steady (i.e. frictionally controlled) flow on a shelf with a long and straight coast, where the bottom depth depends on the offshore coordinate only. The Coriolis parameter \( f \) is supposed constant. The coordinate system is shown in Fig. 2, where \( x \) is offshore, \( y \) longshore and \( z \) upward. The pressure at a depth \( z \) is

\[
p = p_s + g \int_0^z \rho dz,
\]

where \( p_s \) is the pressure at \( z = 0 \) caused by the sea level elevation, the atmospheric pressure being supposed zero, \( g \) the gravitational constant, and \( \rho \) the deviation of density from a reference value \( \rho_0 \). The geostrophic velocity at the bottom \( [z = -h(x)] \) can be defined as

\[
v_b = \frac{1}{\rho_0 f} \left[ \frac{\partial p}{\partial x} \right]_{x=-h(x)} = \frac{1}{\rho_0 f} \frac{\partial p_s}{\partial x} + \frac{g}{\rho_0 f} \int_{-h(x)}^0 \frac{\partial \rho}{\partial x} dz,
\]

\[
u = \frac{1}{\rho_0 f} \left[ \frac{\partial p}{\partial y} \right]_{x=-h(x)} = -\frac{1}{\rho_0 f} \frac{\partial p_s}{\partial y} - \frac{g}{\rho_0 f} \int_{-h(x)}^0 \frac{\partial \rho}{\partial y} dz.
\]

Cross-differentiation of (2) and (3) gives the divergence of the bottom geostrophic velocity, which can be expressed as

\[
\frac{\partial u_b}{\partial x} + \frac{\partial v_b}{\partial y} = -\frac{gs}{\rho_0 f} \frac{\partial \rho_b}{\partial y},
\]

where \( s = dh/dx \) is the bottom slope and \( \rho_b = \rho(z = -h) \). Eq. (4) shows that along-isobath density variations generate convergence or divergence in the bottom geostrophic flow.

A second relationship between the bottom geostrophic velocity components may be deduced from the depth-integrated momentum equations. To exhibit the effects of density forcing in isolation, the wind stress is taken to be zero. As in the analogous wind-forced steady flow problem (Csanyi, 1978), the cross-isobath momentum balance is reasonably supposed geostrophic. The along-isobath component of the bottom stress is parameterized by a linear drag law, with the stress proportional to the (geostrophic) velocity at the bottom and a proportionality constant close to \( r = 0.05 \) cm s\(^{-1}\) (Scott and Csanyi, 1976; Winant and Beardsley, 1979). Let \( U \) and \( V \) represent the horizontal transport in the \( x \) and \( y \) directions, respectively. The depth-integrated, linear, steady flow equations are then

\[
-fV = -\int_{-h}^0 \frac{1}{\rho_0} \frac{\partial p}{\partial x} dz,
\]

\[
fU = -\int_{-h}^0 \frac{1}{\rho_0} \frac{\partial p}{\partial y} dz - rv_b,
\]

\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0.
\]

![Fig. 2. The bottom topography and the coordinate system.](image-url)
Eq. (7) can be used to eliminate $U$ and $V$ in (5) and (6), the resulting equation being

$$r \frac{\partial v_y}{\partial x} - f s u_b = 0,$$

(8)

where (3) has been used to eliminate the pressure term. This is a vorticity tendency equation expressing balance between bottom stress curl and vortex stretching due to cross-isobath flow. In the absence of density variations it yields a parabolic equation for the pressure distribution over a sloping shelf (Csanady 1978, 1981). The present formulation is valid for arbitrary stratification.

Eqs. (4) and (8) can be solved to find the bottom geostrophic velocity from a known density distribution at the bottom without regard to the stratification in the water column. Such a calculation is “diagnostic”, that is to say, it ignores how a given bottom density distribution is generated or maintained, which is the essence of the problem of density-driven flow.

3. The density field

In order to understand the interplay of the velocity and density fields it is necessary to consider the advection and diffusion of density. For simplicity, it will be supposed that density perturbation is smoothed out by turbulence as any other scalar property, so that

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z}$$

$$= K_H \left( \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} \right) + K_V \frac{\partial^2 \rho}{\partial z^2},$$

(9)

where $K_H$ and $K_V$ are horizontal and vertical eddy diffusivities. The time-dependent term has been retained in this equation in order to allow consideration of changes on a slow time scale, brought about primarily by advection.

At first sight it seems inconsistent to neglect both time-dependent and advective terms in the momentum equations (5)–(7), while retaining them in the diffusion equation (9). For a suitably shallow water column, however, the approximations leading to (5)–(7) are readily justified, mainly because the cross-isobath Coriolis force and the along-isobath bottom friction force are relatively large. Except in singular locations, no large flux divergence terms occur in the density equation, nor is there, of course, an analog of the Coriolis force, so that the balance in Eq. (9) is primarily between local and advective change, i.e. between the terms on the left-hand side. This point is readily established formally by scale analysis (see further comments below).

The nonlinearity introduced into the problem by density advection is difficult to cope with and forces one to resort to further simplification. Having in mind density perturbations caused by localized surface cooling and overturning we will confine attention to a vertically well-mixed layer, subject to horizontal density gradients, $\rho = \rho_b(x, y)$. In such a layer the “thermohaline” velocity varies linearly with distance above the sea floor, i.e.,

$$u_c = \frac{g}{\rho_b \partial f} \int_h^z \frac{\partial \rho}{\partial y} dz = \frac{g}{\rho_b \partial f} (z + h)$$

$$v_c = -\frac{g}{\rho_b \partial f} \int_h^z \frac{\partial \rho}{\partial x} dz = -\frac{g}{\rho_b \partial f} (z + h)$$

(10)

The thermohaline velocity was defined by integrating the thermal wind equation with reference to the bottom. The total velocity in the interior is the sum of the bottom geostrophic velocity, the thermohaline velocity, and a frictional velocity, the latter significant in a bottom boundary layer. Eqs. (10) show that the thermohaline velocity does not contribute to density advection:

$$u_c \frac{\partial \rho}{\partial x} + v_c \frac{\partial \rho}{\partial y} = 0.$$

(10a)

On depth-integration, the frictional velocities yield the Ekman transport, $-fv_b$ in the cross-isobath direction. The depth-integrated density equation is therefore

$$h \frac{\partial \rho}{\partial t} + \left( h u_b - \frac{r}{f} v_b \right) \frac{\partial \rho}{\partial x} + h v_b \frac{\partial \rho}{\partial y}$$

$$= h K_H \left( \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} \right) + Q,$$

(11)

where $Q = K_V \partial \rho / \partial z|_0$ is the surface density flux, parameterizing cooling or salt release. Eqs. (4), (8) and (11), for the three variables $u_b$, $v_b$, and $\rho = \rho_b$ form a closed set and may be solved for given initial and boundary conditions. Unlike the commonly used formulation of the density equation in terms of a stream-function, e.g., that by Hendershott and Rizzoli (1976), only the bottom geostrophic velocity is present in (11). Because thermohaline transport usually dominates the total transport, Eq. (14), which does not contain the thermohaline component, is potentially more useful in numerical and analytical model calculations.

4. Scale relationships

We will nondimensionalize the above equations by the scale of forcing, $L_0$, the greatest bottom depth, $H_0$, and the mean bottom slope, $S_0 = H_0 / L_0$. With a density perturbation scale $\Delta \rho$, Eq. (4) gives a velocity scale $U_0 = g(\Delta \rho / \rho_b) S_0 / f$. The time scale is the advective one, $T_0 = L_0 / U_0$, and $Q_0$ is the magnitude of the surface density flux. In nondimensional form, Eqs. (4), (8) and (11) become
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial \rho}{\partial y}, \quad (12)
\]
\[
\epsilon \frac{\partial v}{\partial x} - su = 0, \quad (13)
\]
\[
\frac{\partial \rho}{\partial t} + \left( u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} \right) = \gamma \left( \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} \right) + \alpha \frac{Q}{h}, \quad (14)
\]

where \( \epsilon = r(fS_0L_0) \), \( \gamma = K_H/(U_0L_0) \) and \( \alpha = Q_0 \rho_0/(\Delta \rho^2 g S_0^2) \). The subscript \( b \) was dropped from the non-dimensional bottom geostrophic velocity.

Typical values of the various scaling parameters are \( L_0 = 100 \text{ km}, S_0 = 10^{-3}, r = 5 \times 10^{-4} \text{ m s}^{-1}, f = 10^{-4} \text{ s}^{-1}, K_H = 100 \text{ m}^2 \text{ s}^{-1} \) and \( \Delta \rho = 0.24 \text{ kg m}^{-3} \). These give a velocity scale \( U_0 \) of 0.024 m s\(^{-1}\) and an advective time scale \( L_0/U_0 = 4 \times 10^6 \text{ s} \) or about 50 days. The importance of the forcing is measured by the parameter \( \alpha \). A negligible \( \alpha \) corresponds to the case of free propagation ("self-advection" is a more descriptive term) of the density perturbation field. Given the above, typical parameters, the density flux \( Q \) has to be about \( 6 \times 10^{-6} \text{ kg m}^{-2} \text{ s}^{-1} \) to yield \( \alpha = 1 \). This is about the typical winter heat loss in the Mid-Atlantic Bight.

The friction and diffusion parameters \( \epsilon \) and \( \gamma \) are both typically 0.05 or smaller. The Rossby number \( U_0/fL_0 \) is \( 2 \times 10^{-3} \), while the ratio of advective to frictional terms in the momentum equation, \( H_0 U_0/\rho R_0 \), is 0.05. The linearization of the momentum equations is therefore justified, the neglected contribution to Eq. (13) being of order \( H_0 U_0 / \rho R_0 \) times the retained frictional term \( \epsilon \partial v / \partial x \), itself small.

5. Interior solutions

Given the choice of the scaling parameters, all the dependent variables and their various derivatives in (12)–(14) should be order 1 quantities, except where the boundary conditions cannot be satisfied without boundary layers. Outside such boundary layers, the terms multiplied by \( \epsilon \) and \( \gamma \) in these equations may be neglected, leaving the following relationships describing an "interior" or mid-shelf region (outside the forcing region):

\[
u = 0, \quad (15)
\]
\[
\frac{\partial v}{\partial y} = -\frac{\partial \rho}{\partial y}, \quad (16)
\]
\[
\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial y} = 0. \quad (17)
\]

Flow is thus along isobaths and only boundary conditions in \( y \) are needed. Atmospheric effects are supposed to have produced some initial density anomaly in a spatially limited region:

\[
\rho = F(y), \quad t = 0, \quad (18)
\]
\[
\rho = 0, \quad v = 0, \quad y = \pm \infty. \quad (19)
\]

Eq. (16) is readily integrated with condition (19) to yield

\[
v = -\frac{s}{\rho} \frac{\partial \rho}{\partial y}. \quad (20)
\]

Substitution into Eq. (17) then gives

\[
\frac{\partial \rho}{\partial t} - s \frac{\partial \rho}{\partial y} = 0. \quad (21)
\]

This is a first-order, nonlinear equation of the same form as arises in the theory of shock wave propagation (e.g., Whitham, 1974). Anticipating a boundary layer region within a density front analogous to a shock, we augment Eq. (21) with the \( y \)-diffusion term from the full diffusion equation (14):

\[
\frac{\partial \rho}{\partial t} - s \frac{\partial \rho}{\partial y} = \gamma \frac{\partial^2 \rho}{\partial y^2}. \quad (22)
\]

This equation is reduced to a canonical form by the transformation

\[
\eta = \frac{y}{s}, \quad (23)
\]

the transformed equation being

\[
\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial \eta} = \lambda \frac{\partial^2 \rho}{\partial \eta^2}, \quad (24)
\]

where \( \lambda = \gamma/s^2 \). This is Burgers' equation. Illuminating results follow from its known analytical solutions. The characteristics of the flow depend on the sign of the initial density anomaly \( F(y) \), \( F(y) > 0 \) corresponds to an excess of density, and \( F(y) < 0 \), a deficiency of density. For a positive delta function perturbation

\[
F(y) = A \delta(\eta), \quad A > 0, \quad (25)
\]

the solution of (24) is given by

\[
\rho(\eta, t) = \frac{(\lambda t)^{1/2}}{\pi^{1/2} + (e^\mu - 1) \int_0^{(4A\lambda t)^{-1/2}} \exp(-t^2) dt}, \quad (26)
\]

where \( \mu = A/(2 \lambda) \) (Whitham, 1974). Two parameter ranges of \( \mu \) in (26) are of particular interest. For \( \mu \ll 1 \), the denominator in (26) is \( \pi^{1/2} + O(\mu) \) and (26) becomes

\[
\rho(\eta, t) = A(4 \pi \mu t)^{1/2} \exp(-\eta^2/(4 \lambda t)). \quad (27)
\]

This is the solution of the heat conduction equation for an initial delta function distribution of temperature. The density perturbation diffuses away from the source. The other parameter range \( \mu \gg 1 \) is of greater interest. We write \( \sigma = \eta/(2A\lambda)^{1/2} \) and expand the integral in (26) asymptotically for \( \eta > 0 \):
\[
\int_{\mu^{1/2}}^{\infty} \frac{\exp(-t^2)dt}{\sigma^2} = \frac{e^{-\mu \sigma^2}}{2\sigma \mu^{1/2}} \times \left( 1 - \frac{1}{2\mu \sigma^2} + \cdots \right), \quad \mu \gg 1.
\]

Eq. (26) can therefore be simplified to
\[
\rho \approx \left( \frac{\lambda}{\tau} \right)^{1/2} \frac{\exp(\mu - \mu \sigma^2)}{\pi^{1/2} + \exp[(\mu - \mu \sigma^2)/(2\sigma \mu^{1/2})]}.
\]

To the leading order, this is
\[
\rho \approx \frac{(2A/t)\sigma [1 + 2\sigma(\pi \mu)^{1/2} \exp(\mu \sigma^2 - \mu)]}{\sigma > 1, \quad \mu \gg 1.}
\]

In this limit, the term \(2\sigma(\pi \mu)^{1/2} \exp(\mu \sigma^2 - \mu)\) is much greater than 1 for \(\sigma > 1\), and is much smaller than 1 for \(\sigma < 1\). Eq. (28) can be further approximated by
\[
\rho(\eta, t) \approx \begin{cases} \eta/t, & 0 < \eta < (2At)^{1/2} \\ 0, & \text{otherwise}. \end{cases}
\]

This solution has a saw-tooth shape in a diagram of \(\rho(2At)^{-1/2}\) against \(\eta(2At)^{-1/2}\). A density front is formed at \(\eta = (2At)^{1/2}\) and the density value jumps from zero to \((2At)^{1/2}\) across the front. It is clear from (29) that the front moves in the \(+\eta (-y)\) direction with speed \((A/2t)^{1/2}\) and that the amplitude of density perturbation at the face decreases as \(t^{-1/2}\) independently of \(\gamma\). However, the total density perturbation along an isobath is conserved, because the area under the saw-tooth in (29) is independent of time. The evolution of the density structure of a dense water blob described by (29) is sketched schematically in Fig. 3. The solution for an initial density deficiency is the same as (29) if \(\eta\) is defined as \(+y/s\). Therefore, the direction of propagation is reversed. That solution is also shown in Fig. 3.

The propagation speed of the front is of interest because it is readily compared with observation. It is easily shown (Whitham, 1974) that in the \(y\) coordinate, the propagation speed \(c\) is the average of the particle velocities across the front, i.e.,
\[
c = -s \left[ \rho(a^+) + \rho(a^-) \right]/2 = [v(a^+) + v(a^-)]/2
\]
\[
\approx -s \left( \frac{A}{2t} \right)^{1/2},
\]
where \(a\) is the location of the front at time \(t\). Suppose that at \(t = 0\), the density perturbation along an isobath is concentrated at \(y = 0\) with a total density excess of 0.24 kg m\(^{-3}\) \(\times\) 100 km, i.e., \(A = 2.4 \times 10^4\) kg m\(^{-2}\). With \(K_\mu = 100\) m\(^2\) s\(^{-1}\) and \(s = 10^{-3}\), \(\mu\) is 12. Eq. (29) shows that after 47 days, the amplitude of the saw-tooth-like density wave will be 0.34 kg m\(^{-3}\), have a propagation speed of 0.017 m s\(^{-1}\), and will be stretched out over some 140 km. After six months, these values will become 0.17 kg m\(^{-3}\), 0.009 m s\(^{-1}\) and 277 km, respectively.

Some limitations of the interior solutions should be noted. The propagation speed of the density front depends both on the amplitude of the perturbation and the bottom slope. Where these vary in the \(x\)-direction, differential advection will eventually cause the formation of sharp cross-isobath density gradients. The terms in Eqs. (13) and (14), multiplied by the small parameter \(\epsilon\), and those containing \(x\)-gradients of along-isobath velocity and of density then also become important, and the interior solution is

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**Fig. 3.** Solutions of Burgers' equation at \(t > 0\) showing the evolution of a positive (solid line) and a negative (dashed line) density perturbation. The initial condition is a delta function at \(t = 0\) and \(y = 0\).
no longer valid. At the shelf break, where the slope varies rapidly across the isobaths, a boundary layer region should form especially fast.

Although in the above derivation vertical homogeneity in the water column has been assumed, the formulation is more general. Notice that the equations of motion (15) and (16) are valid independently of stratification. The limitation comes in only through the density equation (17). As long as the near-bottom density balance is dominated by the time rate of change and advection, Eq. (17), and thus (22), should be a good approximation to the processes near the bottom. For example, in the Middle Atlantic Bight in spring and early summer, the balance is still dominated by the time rate of change and advection. Therefore, (22) can be used to describe the bottom water movement in the cold pool.

6. Boundary layers

If the order $\varepsilon$ term is retained in Eq. (13) and the cross-isobath velocity is eliminated from (12) and (13), the result is

$$\varepsilon \frac{\partial}{\partial x} \left( \frac{1}{s} \frac{\partial v}{\partial x} \right) + \frac{\partial v}{\partial y} = -s \frac{\partial p}{\partial y}. \quad (31)$$

According to the along-isobath momentum equation (6) with $U = 0$, the coastal boundary condition of zero normal transport implies

$$uh - \varepsilon v + \frac{h^2}{2} \frac{\partial p}{\partial y} = 0, \quad x = 0, \quad (32)$$

where (1) and (3) have been used to eliminate the pressure terms.

Adopting the most realistic depth distribution, $h \to 0$ at the coast, Eq. (32) becomes simply $v = 0$. To match this to the interior solution, a coastal boundary layer of a width of order $\varepsilon^{1/2}$ is required. Problems of this kind have been discussed by Birchfield (1972), Pedlosky (1974) and Csanady (1978, 1979, 1981). Eq. (31) is a one-dimensional heat conduction equation, with a source term (analogous to interior heat generation) on the right. A positive gradient $\partial p/\partial y$ generates positive $v$ along the parabolic characteristics of Eq. (31) emanating from the source region. Similarly, boundary conditions imposed along a given sector of the coast make their influence felt along such characteristics.

While the analytical solutions quoted above offer useful guidance, they do not come to grips with the complex interplay of geostrophic flow and density advection represented by Eqs. (14) and (31). Below, some numerical solutions for an idealized shelf are presented, to simulate the effects of surface cooling and river discharge. The latter is modeled as diffusive flux at the coast, in the manner of Hendershot and Rizzoli (1976), by imposing the boundary condition

$$\gamma h \frac{\partial p}{\partial x} = R, \quad x = 0, \quad -1 \leq y \leq 0, \quad (33)$$

where $R$ is the nondimensional buoyancy flux per unit length of the coastline (in units of $U_0 H_0 \Delta \rho$).

7. Numerical solutions

The bottom topography used in the model calculations was shown in Fig. 2: it is a smoothed version of the continental shelf south of New England. Isobaths are supposed to run parallel to the coast. In the river inflow solution, in order to avoid a singularity at zero depth, a shallow vertical wall $(25 \text{ m deep})$ replaces the coastline. The distorting effect of this assumption may be shown to be negligible (Shaw, 1982). At the shelf break a constant pressure (no onshore bottom flow) boundary condition is imposed, representing the "insulating" effect of the steeper slope beyond. "Backward" boundary conditions, imposed at $y = 0.5$, are zero flow and negligible density perturbation. Conditions imposed at the "forward" boundary, $y = -4.0$, do not affect the domain of interest and have been chosen for technical convenience: they are vanishing cross-isobath velocity and along-isobath density flux.

a. Flow driven by surface cooling

Typical winter heat loss in the Middle Atlantic Bight is $250 \text{ W m}^{-2}$ (Beardsley and Boicourt, 1981). With a thermal expansion coefficient of $10^{-4} \text{C}^{-1}$ for sea water at $5^\circ \text{C}$, this heat loss amounts to a density flux of $6 \times 10^{-6} \text{ kg s}^{-1} \text{ m}^{-2}$ into the water. We will use this value for the scale parameter $Q_0$. The scale for the density perturbation was chosen to be typical, and also, incidentally, to yield $\alpha = 1$ in Eq. (14). Table 1 lists the scale parameters used in the numerical calculations.

The ocean is motionless with zero density perturbations initially. Cooling, which begins at $t = 0$, is supposed uniform between the coast and the shelf break, with a longshore dependence of

$$Q(y) = \begin{cases} \pi/2 \sin(\pi y), & -1 < y < 0, \\ 0, & \text{otherwise}. \end{cases} \quad (34)$$

The cooling lasts for a nondimensional time $T$. Fig. 4 shows the density field at $t = 2$, with $T = 2$ and $\gamma = 0.05$. These values correspond to cooling over a longshore distance of 100 km for three months. The

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**Table 1. Scale parameters for flow driven by surface cooling.**

| $L_0$ | 100 km | $\Delta \rho$ | 0.24 kg m$^{-3}$ |
| $H_0$ | 100 m | $U_0$ | 0.024 m s$^{-1}$ |
| $S_0$ | 0.001 | $T_0$ | $4 \times 10^8$ s (47 days) |
| $K_H$ | 100 m$^2$ s$^{-1}$ | $\gamma$ | 0.04 |
| $Q_0$ | $6 \times 10^{-6}$ kg s$^{-1}$ m$^{-2}$ |
The formation and propagation of a density front over the mid-shelf is demonstrated in Fig. 6, where density perturbation along $x = 1$ isobath is plotted as a function of $y$. At $t = 1$, the nearly symmetric distribution of density perturbation is the result of local response to cooling. The shock-wave-like front begins to form at $t = 2$ with a mean front propagation speed of 0.6, which agrees with the speed given by (30). The front continues to propagate after the forcing period $0 < t < 2$. However, the magnitude of the density perturbation decreases because of diffusion and longshore dispersion. The longshore propagation speed, which depends on the magnitude of the density perturbation, also decreases.

Another way to look at the longshore dispersion of a density perturbation is to show the constant parameter $\mu$, which determines the behavior of Burgers' equation in the last section, is 20 in this case. One expects that a density front will form. The closely-spaced contour lines in the region $-1 < y < -2$ confirm this expectation. At the same time, the density perturbation is advected in the $-y$ direction, again as suggested by the analytical solution. Fig. 5 shows the associated bottom geostrophic velocity in the $y$ direction. A high-velocity zone is located on the mid-shelf between $y = -0.7$ and $-1.7$. The strong flow at the shelf break is somewhat unrealistic because the boundary condition imposed is approximate, and because the assumption of vertical homogeneity breaks down.
Density lines on the $y-t$ plane. Fig. 7 is plotted from the solution with $T = 2$ and $\gamma = 0.1$. The propagation speed of the front is seen to be fairly high during the cooling period ($t \leq 2$) and shortly after the cooling stops. Later, the erosion of the density maximum in the free propagation stage reduces the parameter $\mu$ and with it the propagation speed. The density field eventually behaves like heat diffusion, a condition seen at $t = 4$. The propagation is so slow that the density maximum is almost stationary.

Table 2 is a summary of the numerical results for different parameters and forcing. The maximum density perturbation and the mean propagation speed of the density front at the end of forcing do not vary significantly for different $\gamma$ and the duration in forcing (cases 1, 2 and 4). The effect of a larger $\gamma$ is to have a faster decay at the end of forcing (cases 1 and 2). Increasing the longshore range of cooling from $\Delta y = 1$ to 2 has a stronger effect on the density and velocity fields than the duration of forcing (cases 3 and 4): when cooling takes place along an isobath over a longer range, the maximum density perturbation is larger and the propagation is faster.

b. Flow driven by buoyancy flux at the coast

Eqs. (11) and (31) have been solved for light river water and dense water discharge at the coast. The forcing comes from the coastal boundary condition (33). A value of 2500 m$^3$ s$^{-1}$ has been used as the scale for the river stream flux at the coast. This falls between St. Lawrence River's 10 000 m$^3$ s$^{-1}$ (e.g., Sutcliffe et al., 1976) and the Hudson estuary's 1000 m$^3$ s$^{-1}$ in the Middle Atlantic Bight (e.g., Beardsley and Boicourt, 1981). We assume that this flux spreads

<table>
<thead>
<tr>
<th>Cases</th>
<th>$T$</th>
<th>Range of cooling $\Delta y$</th>
<th>$\gamma$</th>
<th>$\text{Max}_\rho$</th>
<th>Speed of front$^1$</th>
<th>Decay rate$^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0.05</td>
<td>2.0</td>
<td>0.59</td>
<td>0.36</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0.10</td>
<td>1.8</td>
<td>0.51</td>
<td>0.44</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>0.05</td>
<td>2.9</td>
<td>0.71</td>
<td>0.52</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>1</td>
<td>0.05</td>
<td>2.0</td>
<td>0.48</td>
<td>0.48</td>
</tr>
</tbody>
</table>

$^1$ Value at the end of forcing.
over 100 km of the coastline to avoid large density gradients at the coast. Using 25 kg m\(^{-3}\) as the density contrast between the river water and the shelf water, we have a density flux of 0.625 kg m\(^{-1}\) s\(^{-1}\) per unit length of the coastline. The scale parameters used in the numerical calculation are listed in Table 3.

In the numerical computation, a coastal wall is placed at \(x = 0.2\) in the topography shown in Fig. 2. The depth at the coast is 0.25 (25 m) at the coast. The nondimensional strength of the buoyancy source is 0.0625 between \(y = -1\) and \(y = 0\). The dispersion of the density perturbation in the \(-y\) direction is clearly seen at \(t = 10\) (Fig. 8). The density field is confined to a coastal boundary layer. In the region \(y > 0\), the dominant process is density diffusion and the dispersion of density perturbation is small. The mean propagation speed of the density maximum in the numerical solution is about 2.6 km day\(^{-1}\). The bottom geostrophic velocity associated with the density distribution is also shown in Fig. 8. The velocity is alongshore and positive off the source at the coast where \(\frac{\partial \rho}{\partial y}\) is positive (corresponding to the heat generation analogy). This region extends into the “forward” \(-y\) portion of the shelf along parabolic characteristics. In the region \(y < -1\) a coastal boundary layer forms with negative longshore velocity generated by negative \(\frac{\partial \rho}{\partial y}\). The “self-advective” of the density field by the coastal boundary layer is significant, and is able to disperse the light river water in a plume directed toward the “forward” \(-y\) direction.

In the case of dense water discharge from the coast (a “sinking” plume), the characteristics of the flow are quite different from the one with buoyancy discharge. The value \(R\) is now negative between \(y = -1\) and \(y = 0\), and is of the same magnitude as in the previous example. The other parameters are kept the same. Fig. 9 shows the density and the velocity fields at \(t = 10\). At \(y > 0\), the density diffusion again dominates and the density field is not different from a diffusing cloud. Within the source region a plume of negative alongshore velocity originates, engendered by negative \(\frac{\partial \rho}{\partial y}\). In the region \(y < -1\), the density perturbation field is detached from the coast. This is due to the positive along-isobath velocities generated by the positive \(\frac{\partial \rho}{\partial y}\) region. The converging nearshore flow advects the density field in the offshore direction. Once over the mid-shelf, the dense water propagates and disperses much as in the case of surface cooling, discussed earlier.

The different behavior of light and heavy coastal discharge strikingly illustrates the influence of the coastal boundary layer. The alongshore bottom geostrophic velocity field in the boundary layer spreads toward the \(-y\) direction in the Northern Hemisphere in both cases. However, the flow direction is determined by the sign of density perturbation. In the region \(y < -1\), the alongshore velocity is negative near the coast for light water discharge. The flow direction is the same as that of the characteristic curves of (31). Therefore, the density advection is enhanced. On the other hand, the flow direction for \(y < -1\) in the case of dense water discharge is positive, opposite to that of the characteristics. The advection of density perturbation thus works against the frictionally induced spreading of the velocity field toward negative \(y\). The escape of dense water to mid-shelf, outside the topographic boundary layer, eventually invokes the self-advective mechanism in the interior region.

In reality, river discharge often forms a light surface layer in the ocean. The interaction of the buoyancy discharge with bottom topography is then greatly reduced. It is only during the winter months or with strong tidal mixing that the density deficiencies reach the bottom. The present model applies only to such cases.

8. Comparison with observations on continental shelves

Although the above models are considerably simplified, they should identify key features of a density-driven flow over a sloping shelf under well-mixed conditions. In this section, we will discuss the observational evidence on bottom water movement under well-mixed conditions in the Adriatic Sea, over the Antarctic continental shelf, and in the Middle Atlantic Bight, as well as on river water dispersion at a coast. Because of the various idealizations in the model and the difficulty in measuring model-predicted quantities such as the propagation speed of the density front, the comparison is mostly qualitative. However, in the Adriatic Sea and in the Middle Atlantic Bight, an order of magnitude comparison is possible.

a. Winter circulation in the Adriatic Sea

Circulation in the Adriatic Sea in the winter of 1965–66 was studied by Hendershott and Rizzoli (1976). A cold air mass outbreak from eastern Europe reached the Adriatic Sea on 6 January 1966, and resulted in intense cooling, which lasted for about 20 days. From the meteorological data, Hendershott and Rizzoli estimated a surface density flux of \(1.5 \times 10^{-5}\).
kg m$^{-2}$ s$^{-1}$ during this period. Their density sections of February 1966 show that the water was vertically homogeneous, and that the density increased by 0.5–1.0 kg m$^{-3}$ from the November value.

The most significant flow features during this cooling period were a cyclonic gyre in the northern Adriatic Sea and the advection of a density field by this gyre. In their Fig. 4, Hendershott and Rizzoli show a “tail” of dense water in the horizontal density distribution in February 1966. This dense water extended southward to Jabuka Pit along the bottom topography on the western side of the basin. Along
the Italian coast, a band of low-salinity water from the Po River was also observed. These features were demonstrated in a numerical model by Hendershott and Rizzoli to be caused by surface cooling and coastal buoyancy flux. The along-isobath dispersion of the density perturbation and the formation of a cyclonic gyre were exhibited in their results.

The theory formulated in this paper sheds further light both on the numerical and the observational results. It is first necessary to estimate the amplitude parameter $\mu$ of Eq. (26). The complex bottom topography and geometry of the basin makes it difficult to assign a value to the slope, but $10^{-3}$ should be realistic. For a density perturbation of 1.0 kg m$^{-3}$ existing over an along-isobath distance of 50 km, $\mu$ is 15. Therefore, the cooling event in the Adriatic Sea in early 1966 should have been dominated by the self-advection process. Comparison can be made between the predicted and observed propagation speeds of the "tail" (or front) of dense water. In the observed hor-
izontal density distribution, the tail reached Jabuka Pit in early February. If we estimate that the dense water covered a distance of 200 km from its main source region in the northern Adriatic Sea, the mean propagation speed would be 6.6 km day\(^{-1}\). A rough estimate of the propagation speed from the solution of Hendershott and Rizzoli (1976) is about 200 km in 40 days, or 5 km day\(^{-1}\). With the above value of \(\mu\) the propagation speed from Eq. (30) here is 4.3 km day\(^{-1}\). The flow predicted by the model is both in the same direction and of the same order of magnitude as the observed one.

b. Bottom water movement over the Antarctic continental shelf

Bottom water movement has also been observed over the Antarctic continental shelf. Gill (1973) has suggested that the dense water in the Weddell Sea is formed by salt release during freezing of surface water. He observed that there was a westward increase of 0.4% in salinity, and that the local horizontal salinity gradient could be large. The dense water flows westward on the continental shelf after sinking to the bottom. This westward movement of bottom water was clearly shown in a map of bottom potential temperature distribution given by Foster and Carmack (1976) [see Fig. 1.6 in Warren’s (1981) review]. Dense water accumulation on the western side of the Ross Sea was observed by Jacobs et al. (1970). Current measurements by Foldvik and Kvinge (1974) indicate a strong westward flow (~7 cm s\(^{-1}\)) at the shelf break in the Weddell Sea. The present theory predicts both the westward propagation of bottom water and the formation of strong salinity fronts. It is reasonable to conclude that the self-advection process is important on the Antarctic shelves.

c. Cold water pool in the Middle Atlantic Bight

In the Middle Atlantic Bight, bottom water movement is known to occur in the cold water pool and the outflow of the Gulf of Maine Intermediate Water during spring and summer (Hopkins and Garfield, 1979). The numerical calculation in Section 7a showed that the parameter \(\mu\) may reach 25 after 47 days for an alongshore cooling range of 100 km. Density fronts are expected to form under these circumstances.

Flow features corresponding to the model results have been reported in several publications. A tongue of cold bottom water extending to the southwest in April 1929 is clearly shown in Bigelow’s (1933) bottom temperature distribution. Recently Houghton et al. (1982) analyzed a more extensive data set in 1979. They basically reproduced Bigelow’s picture (their Fig. 5). Fig. 7 here can be compared directly with a similar plot in the paper by Houghton et al. (their Fig. 6). There is considerable qualitative agreement. The maximum propagation speed estimated by Houghton et al. was 3.2 cm s\(^{-1}\). The theory predicts the correct direction of cold bottom water movement with a speed of 1.7 cm s\(^{-1}\) for an alongshore cooling range of 200 km. Therefore, the mean southward flow may well be partly caused by the self-advection process.

d. Coastal discharge on the eastern North American shelf

The largest river system on the eastern shelf of North America is that of the St. Lawrence River with a stream flux of the order of 10 000 m\(^3\) s\(^{-1}\). The downstream influence of the St. Lawrence River was studied by Sutcliffe et al. (1976). The southward propagation of the river water along the coast was clearly demonstrated by their correlation analysis of temperature and salinity distribution in both surface and subsurface layers. According to their calculation, the river water reaches Boston, which is 2300 km downstream, in 9 months. The mean propagation speed is 7 km day\(^{-1}\) (8.1 cm s\(^{-1}\)). This value is larger than the value 2.6 km day\(^{-1}\), given by the solution for a much smaller stream (flux of 2500 m\(^3\) s\(^{-1}\)). Because it is not possible to estimate the amount of density perturbation which reaches the bottom, we will not pursue a quantitative comparison further.

The outflow of the Gulf of Maine Intermediate Water during cold years onto the shelf south of New England (already mentioned above as the source of the Mid-Atlantic Bight cold pool) is an example of the outflow of dense water from the coast. Hopkins and Garfield (1979) traced the direct outflow of the Gulf of Maine Water through the Great South Channel and the Northeast Channel. In both cases, the outflow water moves southwestward in the mid-shelf region, a fact qualitatively in agreement with the model of the dense water outflow from the coast. Hopkins and Garfield estimated the export of the Gulf of Maine Intermediate Water to be \(1.6 \times 10^5\) m\(^3\) s\(^{-1}\). Using a temperature difference of 2°C between the Intermediate Water and the shelf water, we have an equivalent density flux of \(3.2 \times 10^4\) kg s\(^{-1}\). If this flux is spread over a coastline of 100 km, it comes to about half of that in the earlier numerical example.

9. Summary and conclusion

In this paper, we have investigated the density-driven flow over topography. The interaction between the along-isobath bottom density difference and bottom topography is dynamically important. Away from coastal and shelf-edge boundary layers, a near-bottom dense water blob moves in the direction with
shallow water on the right- (left-) hand side facing downstream in the Northern (Southern) Hemisphere. The direction of propagation of a light water blob is reversed. A strong density front may appear in the forward face of the density perturbation. This process is similar to shock wave formation, described by Burgers’ equation.

A similar kind of self-advective can also be produced by density discharge at the coast when the density perturbation reaches the bottom. The light water from a river outflow always moves along the coast within a coastal boundary layer. For dense water discharge from the coast, the density perturbation is not confined to a boundary layer. The water moves offshore initially. The movement is then dominated by the self-advective process which governs the propagation of a dense water blob in the mid-shelf region.

The model can be used to explain qualitatively bottom water movement observed in the Adriatic Sea, over the Antarctic continental shelf and in the Mid-Atlantic Bight. It also realistically describes the advection and dispersion perturbations from a coastal discharge. It is reasonable to conclude that the self-advective process of density perturbations over topography is an important mechanism of circulation over continental shelves.

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