A Direct Method for Assimilating Sea Surface Height Data into Ocean Models with Adjustments to the Deep Circulation

K. Haines*

Center for Meteorology and Physical Oceanography, Massachusetts Institute of Technology, Cambridge, Massachusetts

(Manuscript received 23 April 1990, in final form 31 December 1990)

ABSTRACT

There is as yet no consensus on the best method for inserting surface data into ocean models so that appropriate information is transmitted to the deeper layers in a rapid and efficient manner. First we consider the correlation of sea surface pressure (which is related to sea surface height as measured by altimeters) and the surface and subsurface currents within the framework of a four-layer quasi-geostrophic model. We begin by suggesting that the homogenization of potential vorticity, $q$, in the unforced layers discussed by Rhines and Young, ensures that the potential vorticity anomalies in the deep ocean are weak. It follows that instantaneous $q$ information from these deep layers may not need to be accurately known in order to reconstruct the deep eddy currents, i.e., climatological $q$ information may be sufficient. Taking the fields from the model we use the instantaneous surface streamfunction, $\psi_1$, data along with various approximate subsurface $q$ climatologies and attempt to reconstruct the lower layer flow fields, including the eddies, to the best of our ability. We judge the success of the results both visually and by using the global measure of rms errors. This process is remarkably successful showing that much of the information on the deep eddy currents is contained in the surface $\psi_1$ field. We also try to use instantaneous surface $q$ information to reconstruct the deep flow but with much less success. This is because the barotropic mode streamfunction $\psi_b$ is not well constrained by $q$, information alone whereas $\psi_1$, information does constrain $\psi_b$ to some extent.

A new method of directly inserting data within the time integration scheme of the model is then suggested in which the $q$ fields in the subsurface model layers are left unchanged by the assimilation procedure, and the method is tested with a twin experiment (i.e., a control ocean is defined by the same model ocean at a different time in its evolution). In the assimilation run the top layer $q$ field is changed so as to make the consistent $\psi_1$ field coincide with the control ocean values. The appropriate $q$, is not the same as the control ocean $q$, as it must compensate for the incorrect $q$ fields below. The assimilation run is found to converge rapidly to the control ocean even with total surface data coverage every 40 days. The $q$ fields in the deeper layers (particularly at the bottom) converge as the model evolves in between the data assimilation times. The method works very well because of the $q$ homogenization in the unforced model layers; however, we argue that it would also be suitable under the less stringent constraint that the eddy $q$ fields be uncorrelated in the vertical. Finally we discuss the two commonly used methods of nudging and direct insertion in the light of these results and consider whether this new method can be extended into a primitive equation framework.

1. Introduction

The problem of inserting data into numerical models has a long history within the meteorological community but for oceanographers the problem is relatively new because it is only in recent years that significant quantities of data from the oceans are becoming available. Reviews of the rapidly growing interest in this field can be found in Hurlburt (1984) and Ghil and Malanotte-Rizzoli (1991). Even so, the only data source which offers a realistic possibility for global coverage of the oceans at synoptic scales (i.e., ~50 km where most intense waves and eddies develop) is from satellites. Pioneering instruments such as the altimeter aboard Seasat showed that it was possible to obtain measurements of sea surface height at an appropriate accuracy of a few centimeters and other satellites such as Geosat and the Topex/Poseidon experiment to be launched in 1991 and ERS-1 will considerably improve on these observations.

The question that then becomes important is whether such a surface restricted data source can provide enough information to effectively constrain a model of the ocean in which subsurface currents, temperature and salinity anomalies may all be of interest. Fortunately there are several pieces of evidence which suggest that surface data may contain considerable information about the deep flows.

* Present affiliation: Dept. of Meteorology, Edinburgh University, Edinburgh, U.K.

Corresponding author address: Dr. Keith Haines, Department of Meteorology, James Clerk Maxwell Bldg., The Kings Buildings, Mayfield Road, Edinburgh EH9 3JZ, United Kingdom.

© 1991 American Meteorological Society
Observations of Gulf Stream rings, which extend below the main thermocline level, tend to show an equivalent barotropic current structure when they show any structure at all, e.g., Richman et al. (1977), and analysis of the vertical structure over the entire North Atlantic suggests that only perhaps two or three vertical modes are required to describe the distributions of many hydrodynamic and tracer quantities, Fukumori and Wunsch (1990). Modeling efforts with reduced gravity equations also show remarkable success in capturing certain aspects of the ocean circulation (see Hurlburt 1986; Thompson 1986) indicating that high vertical resolution may not be important for many features of interest. Finally the relative success of "test bed" data assimilation schemes based on twin experiments have shown that surface data can effectively constrain the deep currents at least within simple numerical models, e.g., Holland and Malanotte-Rizzoli (1989). Despite this evidence there is as yet no consensus on the best method for inserting surface data into ocean models so that appropriate information is transmitted to the deeper layers in a rapid and efficient manner and without causing transient disturbances.

In section 2 of this paper we consider a method for reconstructing the deep eddy currents using only surface streamfunction, \( \psi \), information along with successive approximations to the potential vorticity climatologies of the deeper layers of the model. The success is assessed both visually and by using the global measure of an rms error. We conclude that a large fraction of the eddy currents in the intermediate layers are strongly correlated with the surface flow and can thus be reconstructed from surface information. Even the bottom currents are positionally related to the surface flow when the eddies are strong, i.e., in the Gulf Stream region. A dynamical explanation for these current relationships is offered in terms of the potential vorticity structure of the circulation. In section 3 we suggest a new method of inserting the surface data as the model is integrated forward in time which makes use of the results and interpretations of section 2 and which allows the rapid transfer of information to the lower layers. We test this new method using a twin experiment simulation (i.e., the true ocean is defined by the same model ocean at a different time in its evolution). In section 4 we discuss some other methods of assimilating data into ocean models, in particular the nudging method and alternative methods of direct insertion, in the light of our results. We also discuss whether our method could be extended to a primitive equation model and its usefulness for assimilating real ocean data.

2. Projecting the deep layer flow

In this section the objective is to demonstrate how much detailed information about the deep eddy currents may be obtained from surface observations. The degree of success here in reconstructing the deep ocean currents will illustrate the dynamical constraints that couple the deep ocean with the surface and will provide a basis for devising and analyzing a new scheme for assimilating altimetry data to be described in section 3.

To find out how well we can do in projecting the lower layer currents from surface information we have used a four-layer quasi-geostrophic box model on a \( \beta \) plane as our surrogate ocean. The model is described in Brugge et al. (1987) and it has been used for several studies of ocean circulation dynamics, e.g., Marshall et al. (1988). The evolution equations can be written as follows:

\[
\frac{\partial q_i}{\partial t} + J(\psi_i, q_i) = \delta_{i,1} F - \delta_{i,4} \epsilon \nabla^2 \psi_i - \kappa \nabla^2 \psi_i, \quad i = 1 \rightarrow 4
\]

where \( \delta_{i,j} \) is the Kronecker delta and \( J \) is the Jacobian operator. Other quantities are described below. In brief the model represents a square basin of 2263 km by 2263 km in extent and has 193 by 193 horizontal gridpoints (grid spacing = 11.79 km). An Arakawa scheme is used to calculate the Jacobian advection so as to conserve potential vorticity and potential enstrophy within the basin. The four layers have thicknesses and relative densities given in Table 1. The center of the basin is taken to be at 40°N and all model parameters have the nondimensional scalings given in the second column of Table 1. The model was spun up from rest with a sinusoidal windstress, \( F \), (see Fig. 1) applied as a body force to the upper layer with a maximum amplitude = \( 1.58 \times 10^{-9} \) s\(^{-2}\). The spin up period lasted for a total of 6260 days and the fields at this time will be taken as our reference or control ocean at time = 0 for the purposes of the experiments described here. A \( \kappa \nabla^2 \psi_i \) friction term is applied to each layer for numerical stability with \( \kappa = 2 \times 10^4 \) m\(^2\) s\(^{-1}\) and an Ekman friction, \( \epsilon \nabla^2 \psi_i \), is applied to the lower layer with \( \epsilon = 2.07 \times 10^{-7} \) s\(^{-1}\). Subscripts will refer to the model layers, 1 being the top layer and 4 the bottom, except where otherwise stated.

An unstable jet develops at the zero wind-stress curl line between the two counter rotating gyres and eventually a statistically steady state is reached in which the eddies breaking off of this midocean jet provide a means of transferring potential vorticity between the two upper layer gyres to balance the opposite signed vorticity input of the wind stress. The four instantaneous \( \psi \) and \( q \) fields at time 0 are shown in Fig. 2. Two related features of these fields are most important to note. The \( q_{2,3} \) fields contain large "pools" in which the values of potential vorticity are almost uniform (see also the time mean \( q_{2,3} \) fields in Fig. 6 indicating the climatological nature of this feature). This is the result of potential vorticity mixing and weak diffusion as discussed in Rhines and Young (1982). It has previously been noted to occur in quasi-geostrophic models; see Holland et
Table 1. The left-hand column shows the physical parameters of the model with $q_{1,4,4,4}$ being the fractional density change between the layers. The right-hand column shows the nondimensional scales for various quantities in the model. All numbers appearing on maps and graphs are in these units unless otherwise stated.

<table>
<thead>
<tr>
<th>Model parameters</th>
<th>Nondimensional scales</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basin size: $L = 2263$ km</td>
<td>Length: $L = 2263$ km</td>
</tr>
<tr>
<td>Coriolis parameter: $f_0 = 0.93 \times 10^{-4}$ s$^{-1}$</td>
<td>Time: $T = (\beta L)^{-1} = 7$ h</td>
</tr>
<tr>
<td>$\beta = 1.76 \times 10^{-11}$ m$^{-1}$ s$^{-1}$</td>
<td>Potential vorticity: $Q = \beta L = 3.98 \times 10^{-5}$ s$^{-1}$</td>
</tr>
<tr>
<td>Thickness</td>
<td>Velocity: $U = 0.002$ m s$^{-1}$</td>
</tr>
<tr>
<td>Layer 1: 300 m</td>
<td>Streamfunction: $\Psi = UL = 4530$ m$^2$ s$^{-1}$</td>
</tr>
<tr>
<td>Layer 2: 300 m</td>
<td></td>
</tr>
<tr>
<td>Layer 3: 700 m</td>
<td></td>
</tr>
<tr>
<td>Layer 4: 3700 m</td>
<td></td>
</tr>
<tr>
<td>$\sigma_{1,2} = \sigma_{3,4} = 3 \times 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td>$\sigma_{2,3} = \sigma_{3,4} = 8 \times 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td>$\sigma_{3,4} = \sigma_{4,3} = 8 \times 10^{-4}$</td>
<td></td>
</tr>
</tbody>
</table>

al. (1984) and data from the ocean plotted on isopycnal surfaces also suggest large areas of almost uniform potential vorticity (Kefler 1985). In consequence the transient eddy currents seen in the $\psi$ fields are closely equivalent barotropic with near vertical phase lines down into the third layer. This is consistent with the potential vorticity budget in a statistically steady state. The equivalent barotropic eddies transport fluid across the Gulf Stream at all levels but in the lower unventilated layers this does not imply a N–S flux of potential vorticity because the fields there have already become uniform.

This simple structure in the potential vorticity of the lower layers suggests that the details of these fields may be relatively unimportant compared with the active top layer when we consider the $q$ “sources” for the eddy currents and thus we might replace them with surrogate $q$ fields, which capture only the essential features. Therefore, although we are unlikely to have instantaneous data from the deep ocean, this may not be so important for obtaining the deep eddy currents provided we have at least a rough climatological knowledge of the deep ocean $q$ fields. Just how rough these lower layer $q$ estimates can be will now be investigated. From looking at Figs. 2f,g,h the simplest suggestion is that the $q_2$ and $q_3$ fields could be replaced entirely over the whole ocean with the value of planetary vorticity in midbasin. This was the average value of potential vorticity at these layers at the beginning of the spinup and it will not have changed. Note that in the unventilated thermocline regions of the ocean the value of the uniform potential vorticity may be set by remote processes of deep-water formation. For a discussion of the broader implications for the gyre structure, see Hall (1990). We could also initially replace the $q_4$ field by the uniform planetary vorticity gradient, which it resembles over most of the basin.

Now we must choose how to introduce the eddy information. We could retain the instantaneous upper layer $q_1$ field along with the $q_{2,3,4}$ climatologies and invert for all of the $\psi$ fields as we would do on each timestep when integrating the model forward using the potential vorticity equation. This will be tried at the end of this section and will provide an interesting and enlightening comparison. Alternatively, and in the spirit of using satellite altimetry, we could retain the instantaneous upper layer $\psi_1$ field and discard the $q_1$ field. Either the $q_1$ or $\psi_1$ fields combined with our $q_2$, $q_3$, $q_4$ “rough climatologies,” contain enough information for the complete inversion if the boundary conditions on $\psi$ are assumed known. The boundary condition question is discussed more fully in the next section on data assimilation. We will compare both inversion methods below and discuss some of the important differences.

The problem of inverting four $q$ fields to obtain the four $\psi$ fields within this model is solved on every timestep and provides no new conceptual issues. However, the inversion of one streamfunction field $\psi_1$ and three potential vorticity fields $q_2$, $q_3$, $q_4$ is a little more interesting, and so we will discuss the inversion method below. Consider the definition of $q_{1,2} = 2 \rightarrow 4$:

$$q_2 = \nabla^2 \psi_2 + \beta y - \gamma_{2,1}(\psi_2 - \psi_1) - \gamma_{2,3}(\psi_2 - \psi_3) \tag{2a}$$

$$q_3 = \nabla^2 \psi_3 + \beta y - \gamma_{3,2}(\psi_3 - \psi_2) - \gamma_{3,4}(\psi_3 - \psi_4) \tag{2b}$$

$$q_4 = \nabla^2 \psi_4 + \beta y - \gamma_{4,3}(\psi_4 - \psi_3) \tag{2c}$$
where $\gamma_{j,j+1}$ is the inverse of the interfacial Rossby radius between layers $j$ and $j \pm 1$, given by

$$\gamma_{j,j+1}^{-2} = \frac{f_0^2}{gH_j \sigma_{j,j+1}}$$

with $H_j$ the thickness of layer $j$, $\sigma_{j,j+1} = \Delta \rho_{j,j+1}/\rho_0$ the fractional density changes between the layers, and $g = 9.81 \text{ m s}^{-1}$. With a little shuffling we can rewrite the problem with the unknowns ($\psi_2$, $\psi_3$ and $\psi_4$) on the left and the known forcings on the right:

$$\nabla^2 \psi_2 - \gamma_{2,3}^2 (\psi_2 - \psi_3) - \gamma_{2,4}^2 \psi_2 = q_2 - \beta \gamma_{2,1} \psi_1,$$

(3a)

$$\nabla^2 \psi_3 - \gamma_{3,3}^2 (\psi_3 - \psi_2) - \gamma_{3,4}^2 (\psi_3 - \psi_4) = q_3 - \beta \gamma_{3,1} \psi_1,$$

(3b)

$$\nabla^2 \psi_4 - \gamma_{4,3}^2 (\psi_4 - \psi_3) = q_4 - \beta \gamma_{4,1} \psi_1.$$

(3c)
The known $\psi$, field now appears on the right as a forcing in Eq. (3a). This problem looks very much like the problem of inverting for $\psi$ from $q$ in a three-layer model. The method is straightforward. First, we project onto vertical modes such that each modal equation has the form:

$$\nabla^2 \psi_m - \gamma_m^2 \psi_m = F_m, \quad m = 1, 2, 3. \quad (4)$$

Note that these modes are not the same as the modes in the original four-layer model but represent the appropriate modes of a "3½" layer model based on the lower three layers of the four-layer model.¹ The $\psi_m$ can be obtained from a Helmholtz solver and then corrected to obtain fields that satisfy appropriate boundary conditions. In this section we will assume that we know the mean layer thicknesses and hence $\psi_2, \psi_3$.¹

¹ These modes of both the 4 layer and 3½ layer models, are not to be interpreted as the best vertical structure representations of the actual dynamical variations within the model. Instead they merely indicate an orthogonal vertical basis set which allows the inversion of the elliptic operators.
where $\bar{\psi}^{xy}$ is a horizontal integral over the basin. The modes are then reprojected to obtain the three lower layer $\psi$ fields. For completion we note that the $q_1$ field can now be computed as follows:

$$q_1 = \nabla^2 \psi_1 + \beta y - \gamma_{1,2}(\psi_1 - \psi_2)$$

(5)

using the $\psi_2$ field obtained above. Note that this $q_1$ field will be different from the true $q_1$ (given that we will be using incorrect $q_2$, $q_3$ and $q_4$ data). In fact, we shall see below that the differences may be of crucial importance for obtaining the best reconstruction of the lower layer flow.

We are now in a position to carry out some simple inversions in order to test our ability to reconstruct the lower layer $\psi$ fields using only surface eddy information. First of all we will define an integral measure to assess the similarity of two fields. The simplest such measure will be used namely a root mean square (rms) difference. If we have two horizontally varying fields $G_1$ and $G_2$ then the rms difference is defined as

$$F_{\text{rms}} = \left[ \frac{\sum_{i,j=1}^{193} (F1(i,j) - F2(i,j))^2}{(193 \times 193)} \right]^{1/2}.$$  

(6)

In order to make sensible comparisons we need baseline values for the mean rms differences between the streamfunction and potential vorticity fields in the various layers for any two realizations of the model which are well separated in time. To obtain these values the model was integrated forward from the fields shown in Fig. 2 and the rms differences between the fields at time $t$ and the initial conditions of Fig. 2 were calculated. Figure 3 shows the evolution of these rms difference fields as a function of time. The first two columns of Table 2 show the mean $q$ and $\psi$ differences over the last 350 days of the integration. The rest of the table summarizes our success in reproducing the lower layer $\psi$ fields for the various inversions outlined below. Using the simple estimated $q$ fields mentioned above, namely, $q_2 = q_3 = \beta y_0$, $q_4 = \beta y$, where $y_0$ is the midocean $y$ coordinate, and retaining the $\psi_1$ field shown in Fig. 2a, we invert to obtain the three lower layer $\psi$ fields shown in Fig. 4. Visual comparison with the "true" fields, Figs. 2b,c,d, shows that layers 2 and 3 have been captured quite well in the region of the midocean jet although the intensity of the eddies tends to be too large. The $\psi_4$ field is not particularly good however with large recirculation regions in layers 3 and 4. We can see that when comparing rms errors with the decorrelated field values our reconstructed $\psi$ rms errors, Table 2 (column 3), are 85% better for layer 2 although layer 3 is unimproved and the error is nearly three times worse in layer 4. Our failure with layer 4 suggests that the assumption of $q_4 = \beta y$ as a climatology is probably rather poor. We may suspect that layer 4 also contained the most important eddy $q$ contributions of any of the deeper layers so that discarding this information will inhibit the accurate reconstruction of the $\psi_2$ field.

Still we might feel we can at least make better guesses for the lower layer $q$-field climatologies. We can allow for $q$ homogenization over only parts of layers 2 and 3 and assume uniform gradients of $q$ to the north and south reaching their appropriate planetary vorticity values at the northern and southern boundaries. We could also assume some homogenization in the lowest $q_4$ layer, which might be confined to only the western half of the basin (as appears to be the case in Fig. 2h).
Table 2. Rms error values for the various flow reconstructions. Columns 1 and 2 in the upper row give the mean "decorrelated" \( q \) and \( \psi \) field differences for any two realizations of the model fields well separated in time. Columns 3, 4, and 5, in the upper row and 6, in the lower row show the attempts to reconstruct the fields with various climatological approximations to \( q \) from the subsurface layers. Columns 7 and 8, lower row, show the errors in the assimilation runs before and after the first assimilation insertion step.

<table>
<thead>
<tr>
<th>Layer</th>
<th>Average rms ( q ) difference (from Fig. 3)</th>
<th>Average rms ( \psi ) difference (from Fig. 3)</th>
<th>Rms ( \psi ) error ( q_2 = q_3 = \beta y ), ( q_4 = \beta y )</th>
<th>Rms ( \psi ) error. Best analytic ( q_{2,3,4} ) climatology</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.786</td>
<td>2.958</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0.0683</td>
<td>2.459</td>
<td>0.385</td>
<td>0.204</td>
</tr>
<tr>
<td>3</td>
<td>0.0776</td>
<td>1.399</td>
<td>1.408</td>
<td>0.786</td>
</tr>
<tr>
<td>4</td>
<td>0.0351</td>
<td>0.779</td>
<td>2.236</td>
<td>1.233</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Rms ( \psi ) error. Model ( q_{2,3,4} ) climatology</th>
<th>Rms ( \psi ) error. Using instantaneous ( q_i ) and model ( q_{2,3,4} ) climatology</th>
<th>Rms ( \psi ) error before first assimilation insertion</th>
<th>Rms ( \psi ) error after first assimilation insertion</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>2.791</td>
<td>3.043</td>
</tr>
<tr>
<td>2</td>
<td>0.093</td>
<td>2.795</td>
<td>2.537</td>
</tr>
<tr>
<td>3</td>
<td>0.331</td>
<td>2.818</td>
<td>1.460</td>
</tr>
<tr>
<td>4</td>
<td>0.373</td>
<td>2.871</td>
<td>0.805</td>
</tr>
</tbody>
</table>

This is as far as we bothered to go and our best-guess analytic \( q \) fields are shown in Fig. 5 along with the derived \( \psi_2 \), \( \psi_3 \) and \( \psi_4 \) fields. There is now a considerable reduction in eddy intensity in layer 3 and a more accurate eddy structure away from the jet reflecting a reduction in the recirculation intensity. Layer 4 is still showing an anomalous recirculation however. Table 2, column 4, gives the \( \psi_{\text{rms}} \) deviations for these fields showing a 92% and 44% reduction in the \( \psi_2 \), \( \psi_3 \) rms errors over the respective decorrelated values. However, layer 4 has again worsened, suggesting that a still more accurate \( q_4 \) climatology is required. As a final improvement therefore we can use the true climatological values obtained from an average over 700 days of integration time of the model Eq. (1). These climatological \( \psi \) and \( q \) fields for all of the layers are shown in Fig. 6. Figure 7 shows the instantaneous \( \psi_{2,3,4} \) fields reconstructed with the \( q_2 \), \( q_3 \), \( q_4 \) 700-day climatology model climatologies (again using \( \psi_1 \) instantaneous, for the inversion). The \( \psi_{\text{rms}} \) errors are in Table 2, column 5, showing that the climatological fields give the best results of all and the lower layer \( \psi \) fields in Fig. 7 are now very realistic in appearance, although eddy intensities are still considerably in error. We are now doing better than the decorrelated values in all the layers with improvements of 96%, 76% and 26% in the \( \psi_2 \), \( \psi_3 \) and \( \psi_4 \) fields, respectively. This is probably the best that we can hope for in the absence of instantaneous eddy \( q \) information particularly from layer 4. It seems to be telling us the approximate percentage of eddy current information from the lower layers which is correlated with the surface (although see section 3 for the possibility of eddy \( q \) correlations in the vertical which have been explicitly neglected here).

All of these inversions were carried out using the upper layer \( \psi_1 \) field. Within the context of this model they demonstrate that instantaneous information is not necessary from the deep layers in order to reconstruct with some success the deep eddy circulation from a set of surface observations. However, it is interesting to compare the results of using the \( q_1 \) field as our "upper-layer information" instead. If \( q_1 \) is retained (Fig. 2e), and the 700-day climatological \( q_2 \), \( q_3 \) and \( q_4 \) values are used for the inversion, then the four \( \psi \) fields shown in Fig. 8 are obtained. If these are compared with the fields in Fig. 7 from the best \( \psi_1 \) inversion and with the truth fields in Fig. 2, some remarkable differences are apparent. Although both inversions get many features of the flow correctly positioned (at least in layers 2 and 3), the \( q_1 \) inversion considerably overestimates the strength of the eddies in some regions and underestimates them in others. What is more, the major errors in Fig. 8 appear to vary on a large scale with \( \psi_1 \) being broadly too low in the eastern half of the basin and too high in the west. This is very apparent in Fig. 8d with a large low pressure region over the whole of the eastern basin. These problems are reflected in the rms errors for this inversion shown in column 6 of Table 2. We have the apparently surprising result that an incorrect \( q_1 \) field derived through Eqs. (3), (4), (5) is actually considerably better at generating the lower-layer flow fields than the true \( q_1 \) field.

What appears to be happening is that the \( q_1 \) field calculated from Eqs. (3), (4), (5), which has the effect of generating the correct \( \psi_1 \) field, must be compensating for the effects of having incorrect \( q \) fields in the lower layers. This compensation must occur exactly when considering the generation of the \( \psi_1 \) field. However, having adjusted \( q_1 \) to get \( \psi_1 \) correct, we have also managed to make useful corrections to the lower-layer \( \psi \) fields so that they too are considerably more accurate that those obtained from using the correct \( q_1 \) field. Therefore, a modification is required to the statement with which we began the section when it was suggested that the similarities in the lower-layer \( q \) fields between any two realizations of the model implied that the instantaneous variations in the lower-layer \( q \) fields may be unimportant in generating the deep eddy currents. It now appears that the time-varying component of the lower-layer \( q \) fields may indeed generate important...
eddy currents but that the eddies, at least down into layer 3, can also be effectively generated by substituting an appropriate eddy $q_1$ field at the surface. Furthermore, we manage to do this quite effectively for all of the lower layers simultaneously if the $q_1$ field is chosen so as to get the $\psi_1$ field exactly correct as results from solving Eqs. (3), (4), (5).

The mechanism is further revealed by noting that the rms errors obtained from the true $q_1$ inversion are very similar in all of the layers. This is indicating that the errors in the lower layer $q$ fields are mainly producing errors in the barotropic mode $\psi_0$ field. In fact, the wrong barotropic mode accounts for nearly all of the differences between the $q_1$ inversion (Fig. 8) and the $\psi_1$ inversion (Fig. 7). This can be expected for two reasons. First of all, we note that the lowest layer in particular has a very strong projection onto the barotropic mode due to its thickness so that errors in the $q_1$ field (if they are not judiciously canceled by errors in the $q_0$ field) will produce a large error in the baro-
Fig. 5. The left-hand panels show the best-fit analytic $Q_{2,14}$ fields, which were used as “rough climatologies.” The right-hand panels show the $Q_{2,14}$ fields reconstructed using these climatologies and the $\psi_1$ data in Fig. 2a. The fields are considerably improved over those in Fig. 4. Contour intervals are as in Fig. 2.
tropic mode $q_B$ projection. Then, because of the infinite Rossby radius of the barotropic mode (finite but basin scale in the real ocean) the local errors in $q_B$ will spread out to produce large-scale $\psi_B$ errors over the entire ocean basin. The extreme sensitivity of the barotropic mode streamfunction to alterations or errors in the potential vorticity fields is an important factor to be considered in any assimilation scheme. It will be shown later that after an assimilation step has been performed the majority of the remaining $\psi$ errors reside in the gravest vertical mode structures of the flow [see Eq. (13)].

In the following section we will also consider relaxing the condition that $q_1$ alone be adjusted although only marginal improvement will be found with this modification if only $\psi_1$ information is available. We conclude
then that the upper-layer $\psi_1$ information is actually much more useful for inferring the lower-layer $\psi$ fields than the $q_1$ information because by allowing for a slightly incorrect $q_1$ field we can compensate to some extent for not knowing the correct $q$ fields in the lower layers. This compensation is particularly important for keeping the errors in the barotropic mode small so that we generate quite accurate $\psi$ fields throughout the ocean. This will also be demonstrated and discussed in more detail later in the data assimilation section.

These results will allow us to project sea surface height data, $\psi_1$, into the deep ocean and to directly insert this information about the lower layers in a data assimilation experiment. However, the method of guessing lower-layer $q$ fields as used above, or even of enforcing a lower-layer $q$ climatology seems to be a little arbitrary and restricting. Fortunately, this can be avoided in an assimilation experiment, and we can even
hope to regenerate the correct lower-layer $q$ anomalies in time as outlined in the next section.

3. A method for data assimilation

The method used to regenerate the lower-layer eddy $\psi$ fields in the above section suggests a very simple variation, which should allow a rapid and efficient assimilation of the deep-ocean currents when surface streamfunction (sea surface height) is available. It is proposed that the surface streamfunction data, $\psi_1$, should be assimilated in such a way as not to change the lower-layer $q$ fields. Since we expect the intermediate layers of our model to contain large regions of homogenized potential vorticity and the bottom layer $q_b$ field may have only weak (although possibly still important) perturbations about climatology, this assimilation method should work reasonably well. However, we point out that from a statistical standpoint this method would also be the best we can do if there are no vertical correlations in the eddy potential vorticity fields. It is very clear from Fig. 2 that $\psi$ is highly correlated in the vertical compared with the corre-

Fig. 7. The best reconstructed $q_{2, 3}$ fields obtained using the lower layer climatological $q_{2, 3}$ fields of Fig. 6g,h along with $\psi_1$, Fig. 2a. Most eddy features are captured except some of those in layer 4 away from the unstable jet region. Contour intervals as in Fig. 2.
Fig. 8. Streamfunction fields obtained by inversion of $q_1$ from Fig. 2c and $q_{2,3,4}$ climatologies from Fig. 6f.g.h. (a)-(d) Layers 1–4, respectively. Most features are captured in layers 1 and 2 but the lower layer flows are swamped by large scale anomalous gyres resulting from errors in the barotropic mode. Contour intervals as in Fig. 2.

Corresponding $q$ fields. This point will be discussed further at the end of the paper. The development of the framework for achieving this assimilation scheme provides illuminating comparisons with previous schemes and is set out below.

a. Assimilation dynamics

The problem is most easily defined in the following notation. Consider the model fields before the assimilation to be given by $\psi_i, q_i$ ($i = 1 \rightarrow 4$) although in general an $N$-layer model could be used. The "observed" field is given by $\psi_{\text{obs}}$. Thus, we have the initial known error field in the upper layer,

$$\Delta \psi_1 = \psi_1 - \psi_{\text{obs}},$$

where $\psi_1$ is the model upper-layer streamfunction field. This quantity can be determined whenever a new set of satellite observations becomes available, although...
perhaps with incomplete spatial coverage. However, we have a four-layer model, and therefore another three fields are needed to solve for the entire flow. We suggest here that the best assumption to make is that

\[ \Delta q_j = 0, \quad j = 2 \rightarrow 4. \]

Note that this is very different from the direct insertion methods of Berry and Marshall (1989) and Hurlburt (1986), who effectively used

\[ \Delta \psi_j = 0, \quad j = 2 \rightarrow 4, \]

corresponding to no change in the lower-layer pressure fields, a choice which clearly prevents any immediate transfer of current information in the vertical. We will aim to solve for the \( \Delta \psi_j, j = 2 \rightarrow 4 \) and use these values (along with \( \Delta \psi_1 \)) to update the lower layer \( \psi \) fields. If we take Eq. (3) from section 2 as our starting point and consider what happens if we are solving for the field changes \( \Delta \psi \), then the equations become

\[ \nabla^2 \Delta \psi_2 - 2 \frac{1}{3} (\Delta \psi_2 - \Delta \psi_3) - \frac{1}{2} \Delta \psi_2 = -\frac{1}{2} \Delta \psi_1, \quad (7a) \]

\[ \nabla^2 \Delta \psi_3 - 2 \frac{1}{3} (\Delta \psi_3 - \Delta \psi_2) - \frac{1}{2} (\Delta \psi_3 - \Delta \psi_4) = 0 \quad (7b) \]

\[ \nabla^2 \Delta \psi_4 - 2 \frac{1}{3} (\Delta \psi_4 - \Delta \psi_3) = 0. \quad (7c) \]

We can solve these equations provided we have additional information about boundary conditions on \( \Delta \psi_1 \). We have chosen to ensure that the mean temperatures before and after the assimilation are conserved, or equivalently the mean thicknesses of the model layers. We apply the constraints:

\[ \Delta (\psi_i - \psi_{i+1}^{xy}) = 0, \quad i = 1, 2, 3, \quad (8) \]

which translate into

\[ \Delta \psi_i^{xy} = \Delta \psi_2^{xy} = \Delta \psi_3^{xy} = \Delta \psi_4^{xy} \quad (9) \]

and since \( \Delta \psi_1^{xy} \) can be calculated we have our constraints on the other fields. The last equality, \( \Delta \psi_2^{xy} = \Delta \psi_1^{xy} \), ensures that \( \Delta \psi_2^{xy} = 0 \), and therefore the assimilation does not change the total potential vorticity in the model. The equations are all homogeneous except for the layer 2 equation which feels the “topography” of the top layer \( \Delta \psi_1 \) as a forcing on the right-hand side. However, the equations are coupled. To solve them we project onto the three independent vertical modes \( m \) as for Eq. (4) in the previous section. On making the appropriate vertical projection, the equations take the form:

\[ \nabla^2 \Delta \psi_m + \gamma m^2 \Delta \psi_m = \Delta F_m, \quad m = 1, 2, 3 \]

where each equation is now a forced Helmholtz equation that can be solved for the \( \Delta \psi_m, \Delta \psi_m = 0 \) on the horizontal boundaries. The area-averaged values appropriate for each \( \Delta \psi_m \) are then calculated from Eq. (9) and the fields obtained from the Helmholtz solver are corrected with multiples of the homogeneous solutions. These \( \Delta \psi_m \) can then be reprojected onto the layers to give the \( \Delta \psi_{2,3,4} \) which satisfy condition (8).

Allowances can be made for error estimates in model and data fields; for example, on the right-hand side of Eq. (7a), \( \Delta \psi_1 \rightarrow \omega \Delta \psi_1 \) where \( 0 < \omega < 1 \) can allow for smaller changes to the model fields. The size of \( \omega \) reflects our confidence in the new data compared to the model values.

We now have all the information we need to update the \( \psi \) fields of the model. The update could be summarized as follows:

\[ \psi_{i,\text{new}} = \psi_{i,\text{old}} - \Delta \psi_i, \quad i = 1 \rightarrow 4 \quad (10) \]

with \( \psi_{1,\text{new}} \) now becoming \( \psi_{i,\text{obs}} \) of course. If this method is diluted to allow for errors as described above, then the factor \( \omega \) would appear in front of the error correction fields in Eq. (10). To make adjustments for incomplete data coverage in the upper layer, two possible methods could be used. Either the \( \Delta \psi_1 \) values are projected horizontally by some statistical method so that they are assumed to be known entirely over the upper layer. Otherwise, and in imitation of, the vertical projection method used here, we could allow changes in \( \Delta q_1 \) only where we have observations of the upper-layer \( \psi_1 \) field. These possibilities will be discussed further in section 4. While the \( \psi \) field adjustments could be implemented directly in the model, there is an even easier method based on simply altering the \( q_1 \) field (remembering that the \( q_{1,3,4} \) fields are to remain unchanged). Having obtained the \( \Delta \psi \) fields as above we could then compute

\[ \Delta q_1 = \nabla^2 \Delta \psi_1 - \gamma \Delta \psi_2 \quad (11) \]

Now the assimilation can be done in a single step by changing the \( q_1 \) field; thus,

\[ q_{1,\text{new}} = q_{1,\text{old}} - \Delta q_1. \quad (12) \]

The new \( \psi \) fields will then be obtained by inversion as a normal part of the model integration cycle. This was the method that we used in the experiments described below.

We point out again that the new \( q_1 \) field obtained in the above process is not the same as the true \( q_1 \) field. It is also worth noting that the success of a single assimilation step may be judged differently according to the measure used. We will show below that even a single assimilation step with complete \( \psi_1 \) data will considerably improve the second and third layer \( \psi \) fields in a four-layer quasi-geostrophic model. Alternatively if we measure the rms potential vorticity error, the lower-layer fields will not be changed at all by the assimilation process. However, we show that even the lower-layer \( q \) fields converge well after applying this assimilation technique intermittently in combination with model integration over an extended time period.

b. A twin experiment

To demonstrate the effectiveness of the above technique a twin experiment was performed. A control
ocean was defined by running the model over a period of 18 months from the initial fields shown in Fig. 2. The $\psi$ and $q$ fields for all levels are saved every 2.26 days during this period so that the $\psi_i$ field can be used as the observed data, $\psi^{obs}$, and the other fields can be used for verification of the convergence of the data assimilation run. The assimilation was begun from the fields obtained at the end of the control ocean integration, i.e., 18 months after the Fig. 2 fields. These initial conditions are the same as those corresponding to day 559 on the rms deviation plot in Fig. 3 from which the initial rms errors can be found. The final two columns of Table 2 give the rms $\psi$ deviations before and after the first data insertion (using the $\psi_i$ field at the beginning of the control ocean run shown in Fig. 2a). Although the lower-layer $q$ fields are unaltered, a significant improvement occurs immediately in the $\psi_2$ and $\psi_3$ fields, although the $\psi_4$ field has not improved very much according to this rms measure.

The model was run on and the data insertion technique was used approximately every 20 days (actually 18.85 days) assuming that the entire $\psi_i$ field is known each time. More frequent data insertion every 4.5 days was tried but the results were found to be insignificantly better than using the 20-day insertion frequency. As will be discussed later, the limiting timescale for the convergence may depend upon the time taken to alter the potential vorticity in the lower layers, particularly layer 4. Although the surface current and potential vorticity structure will strongly influence the deep-layer flows through vortex stretching, the fields cannot converge until the deep potential vorticity fields have been redistributed correctly. Therefore, if assimilation is too frequent the lower-layer fields cannot be improved at a faster rate. Figures 9a–d show the evolution of the rms errors for the $\psi$ fields throughout the 18 months of the assimilation run. Figures 9e–h show the evolution of the rms errors of the $q$ fields during this time. Crosses on the $y$ axis denote the initial rms differences before the first assimilation at day 0. Notice that only the $q_1$ field improves stepwise at each assimilation time. By measure of the $\psi$ rms errors the assimilation run converges rapidly to the true ocean in all layers, and there is no significant improvement after about 450 days, i.e., after 23 insertion times although the $q_2$ and $q_3$ errors still appear to be slowly decreasing at this time. The final deviation from the truth for each of the fields $\psi_i$ are given in Table 3 as percentages of the decorrelated values (before any assimilation). These results compare very favorably with previous work, which have used perfect data with other direct insertion or nudging techniques (for example, Holland and Malanotte-Rizzoli 1989; Berry and Marshall 1989).

To further test the method we repeated the above assimilation experiment but with data insertion every 40 days (37.7 days) instead of every 20 days and continuing for 23 months. The model still converges well although rather more slowly. The evolution of the $\psi^{rms}$, $q^{rms}$ errors in all the layers are shown in Fig. 10. Figure 11 shows the $\psi$ fields for the true ocean on the left and the assimilation ocean on the right corresponding to day 695 of the 40 day insertion run. Figure 12 shows the same comparison for the $q$ fields. The $\psi$ fields have converged very closely but perhaps the most stringent test comes from a $q$-field comparison. Although no immediate changes are made to $q_{2,3,4}$ at each assimilation time, the fields in all layers are showing remarkable similarities after these 700 days of integration and data insertion. Virtually every small-scale structure of the $q$ fields is reproduced, often with the correct intensities.

Some points about the convergence are worth discussing in more detail. If the $\psi$ rms error plots are compared we can see that in the upper three layers the improvements occur in a stepwise fashion at the assimilation times with the model mostly diverging from the control ocean between the assimilation times. This contrasts with the bottom layer whose rms error remains nearly the same or even gets worse in a stepwise fashion at the assimilation times but shows a net convergence with the control ocean in between assimilation times. One way of interpreting this result is to say that the projection method works well for transmitting current information downward as far as the third layer, but not into the bottom layer. However, as the model evolves the dynamical coupling between the layers, mediated by vortex stretching, causes the bottom layer currents to converge with the "control" ocean over a period of time. It is interesting to consider what happens to the potential vorticity fields as this process occurs. The success of the projection method in reproducing the eddy currents down into the third layer is explained by the fact that there are only very weak eddy potential vorticity anomalies in both layers 2 and 3 due to homogenization. However, this is not true of layer 4 where significant potential vorticity anomalies, which are essentially errors, will remain in the model after the vertical projection. These local potential vorticity errors in layer 4 strongly distort the bottom currents and ensure that the $q_{4}^{rms}$ error is not immediately improved. (These bottom potential vorticity errors will of course distort the currents in layers 2 and 3 also but less strongly and the currents in these layers remain dominated by the surface potential vorticity field.)

As the model begins to evolve the current fields in layer 4 will alter. It is useful to consider separately the two influences on these bottom-layer currents. First, there will be the influence of the flow (or more succinctly, the $q_i$ field) from the layers above, which is transmitted downwards through vortex stretching. These upper layers have been greatly improved through the assimilation procedure and we may be confident that this downward influence will continue to provide an improved component to the flow in layer 4. Second, there is the continuing influence of the incorrect potential vorticity field in layer 4 itself. This will distort the currents in the bottom layer as stated previously; however, there is no reason why this distortion should get worse with time. Indeed we expect the errors in $q_4$
to decay due to both the damping effects of Ekman friction (as in the bottom layer of the Berry and Marshall (1989) study) and also perhaps to the correcting influence of the currents generated from above. The bottom layer currents cannot, after all, converge until the local potential vorticity field has become redistributed correctly.

Another interesting illustration of this convergence mechanism is provided by the following observation. If the $\psi_4$ field is examined, particularly just after the first assimilation step is carried out, it is found that visually the field appears to be more similar to the true ocean $\psi_4$ than before the assimilation; however, the intensity of the eddies is considerably too large. This explains why the $\psi_4$ rms error does not improve at the insertion time but afterwards there are rapid improvements as the $q_4$ field is redistributed and the errors reduced. A correlation coefficient for the bottom-layer fields, before and after the first assimilation was calculated as follows:
FIG. 9. (Continued)

\[ C = \frac{193 \sum_{i=1}^{193} \psi^a \psi^b}{\|\psi^a\| \|\psi^b\|} \]

where \(\psi^a\) and \(\psi^b\) are the two fields to be compared and \(\|\psi\|\) is an \(L_2\) norm. Before the first assimilation, correlating \(\psi^4_{\text{model}}\) with \(\psi^4_{\text{control}}\) one obtains only a weak correlation, \(C = 0.09\). However after the assimilation, \(C = 0.49\). This is despite the fact that the \(\psi^4_{\text{rms}}\) has not improved, and it demonstrates that the bottom layer eddies have undergone positional improvements due to the component of the flow generated from the surface potential vorticity field. However, the \(q_4\) errors are still present and the implication is that the \(q_4\) eddy field is the second most important \(q\) field for generating the eddy currents. The convergence mechanism is consistent with this proposal as the \(\psi^4_{\text{rms}}\) error decreases on the same timescale as the \(\psi^4_{\text{rms}}\) errors.

The overintense \(\psi_4\) field after assimilation can be explained by considering the effects of Ekman friction...
Table 3. Summary of the success of the assimilation runs showing the remaining errors in the various fields as a percentage of the initial errors before any assimilation was performed.

<table>
<thead>
<tr>
<th>Layer</th>
<th>Percent rms $\psi$ error residual after 550 days</th>
<th>Percent rms $q$ error residual after 550 days</th>
<th>Percent rms $\psi$ error residual after 680 days</th>
<th>Percent rms $q$ error residual after 680 days</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.2</td>
<td>3.5</td>
<td>2.4</td>
<td>4.2</td>
</tr>
<tr>
<td>2</td>
<td>1.6</td>
<td>32.6</td>
<td>2.6</td>
<td>50.4</td>
</tr>
<tr>
<td>3</td>
<td>4.8</td>
<td>39.6</td>
<td>5.1</td>
<td>35.0</td>
</tr>
<tr>
<td>4</td>
<td>5.2</td>
<td>10.5</td>
<td>8.5</td>
<td>13.1</td>
</tr>
</tbody>
</table>

in layer 4. Eddy $q$ anomalies in the top layer will tend to produce relative vorticity anomalies of the same sign all the way down into layer 4 (although they cannot change $q$ in the lower layers of course). In layer 4 however, the Ekman friction will act to reduce the relative vorticity induced from layer 1 by generating a local $q_4$ anomaly of the opposite sign. This process will be going on all the time as the ocean model evolves and it is this process that prevents much $q$ homogenization from occurring in layer 4. When data is assimilated by changing $q_1$, particularly on the first insertion, the new circulation anomalies penetrate to layer 4 but the $q_4$ anomalies are not now in the right place to reduce the relative vorticity intensity in layer 4. Therefore, the $\psi_4$ eddies are often close to their correct positions but anomalously strong. A similar effect was noted in a two-layer model of the vortex pairs discussed in Haines (1989).

Further consequences of this can be tested by some simple analysis. Earlier in this section, Eqs. (7) were derived from Eqs. (3) of section 2 by assuming that we were solving for the field changes required for assimilation. Using Eqs. (3) again as a base we will now show how to solve for the field errors, $\delta \psi_{2,3,4}$, which are present after the assimilation inversion. In addition we will make explicit our belief that the most important $q$ errors are in layer 4, therefore $\delta q_{2,3} = 0$. We then obtain

$$
\nabla^2 \delta \psi_2 - \gamma_{2,3}^2 (\delta \psi_2 - \delta \psi_3) - \gamma_{2,1}^2 \delta \psi_2 = 0, \\
\nabla^2 \delta \psi_3 - \gamma_{3,2}^3 (\delta \psi_3 - \delta \psi_2) - \gamma_{3,4}^3 (\delta \psi_3 - \delta \psi_4) = 0, \\
\nabla^2 \delta \psi_4 - \gamma_{4,3}^4 (\delta \psi_4 - \delta \psi_3) = \delta q_4. \\
$$

If we wish to solve (13), we project onto the three orthogonal vertical modes, which are the same as those in Eq. (4). However, we are not going to solve the set but we merely note two important factors about the modal equations. 1) The error field $\delta q_4$, on the right-hand side of Eqs. (13) has its largest projection onto the equivalent barotropic mode. This is one reason for expecting a large streamfunction response in this mode. 2) The equivalent barotropic mode has much the largest Rossby radius (a factor of 4.88 times larger than that of the first baroclinic mode, although recall that these are not the true modes of the four-layer model). This means that the streamfunction errors will have a wide horizontal distribution around the $\delta q_4$ errors. These two factors ensure that it is the equivalent barotropic mode associated with Eqs. (13) that completely dominates the large-scale streamfunction error fields.

We merely need to know the relative contributions of this mode when projected back onto the layers in order to predict the ratios of the final errors $\delta \psi_{2,3,4}$ after an assimilation. For the parameters used in these runs we have:

$$
6.39 \delta \psi_2 = 1.76 \delta \psi_3 = 1.06 \delta \psi_4.
$$

The rms errors in column 8 of Table 2 are seen to have very nearly these same ratios. This is also consistent with the fact that the $\psi_2$, $\psi_3$, $\psi_4$ and $q_1$, $q_4$ errors all appear to be decreasing on approximately the same timescale, whereas the $q_2$ and $q_3$ errors appear to decay rather more slowly (in fact at approximately the same rate in both the 20-day and 40-day assimilation experiments). A possible reason for the slower improvement in $q_2$, $q_3$ may be that it is not actually advection which is the dominant redistribution mechanism for $q$ in these layers over most of the basin since these $q$ fields are almost homogeneous with many small scale perturbations. Instead, the lateral diffusion timescale could be playing an important role, and this might explain the same slower convergence rate for the two experiments. We note, however, that the largest grid point $q_4$ errors are confined to the edge of the basin where the assumptions used to derive relation (13) are probably invalid.

Finally a further experiment was performed in order to see if any improvement could be obtained by relaxing the constraint that only $\psi_1$ be adjusted. Instead, we allow some $q_4$ variations such that $\Delta q_4 = \alpha \Delta \psi_1$ thus allowing for the anticorrelation discussed above. This can be implemented by substituting a $\Delta q_4$ on the right-hand side of Eq. (7c) and then using the above linear relation with $\Delta \psi_1$ and substituting from Eq. (11). A slightly different set of modes are needed to solve for the $\Delta \psi$ fields, but otherwise we proceed as before. In the model both $q_1$ and $q_4$ are then adjusted by the appropriate amounts. We will not present any specific
results here but we note that some very small improvements are obtained using a value of $\alpha = -0.02$. The results deteriorate for more negative $\alpha$ or for positive $\alpha$. This anticorrelation is again due to the penetration of the $q_1$-induced circulation combined with the strong Ekman friction in layer 4 noted above. If the relative vorticity could be entirely wiped out in layer 4, then vortex stretching alone would require an approximate anticorrelation between $q_1$ and $q_2$ (assuming $q_2$ and $q_3$ are unaltered due to uniformity) with a value of $\alpha = -0.081$. This would remove all projection onto the barotropic mode, which is dominated by layer 4. However, the Ekman friction cannot wipe out all the relative vorticity in layer 4. The anticorrelation with $\alpha = -0.02$ presumably indicates a playoff in timescales between the generation of $q_1$ anomalies through instability and the generation of a $q_4$ component through friction. The improvements found with the above method were not significant enough to merit application in the model although in other situations an adjustment of this type may be more useful.

Many further experiments could be done by extending the method for incomplete data coverage in the upper layer or with the inclusion of artificial noise and hence an error analysis. Studies could also be made with degraded models that would assess the ability of assimilation to reproduce the correct climatologies as well as the eddy fields. However, the purpose of this paper is to introduce this new method of projecting in the vertical, and therefore we will not go on to investigate these other aspects here. We can conclude this section on an optimistic note with the results indicating that satellite altimeter coverage of the ocean even on a 40-day timescale would be sufficient to constrain an ocean model at least of quasi-geostrophic complexity and to reproduce deep eddy currents to good accuracy. The extent to which this will be true of other more complicated models needs to be investigated [a study with a shallow water model, Haines et al. (1991) has been submitted] although these results suggest that even longer periods than 40 days between repeat times may still be useful. The experiments discussed above compare most favorably with equivalent assimilation methods which have been previously used and we will devote the rest of this paper to a discussion of why this is so and under what conditions it will remain true.

4. Discussion

Some discussion of comparable methods for assimilating altimetry data will help to put this work in perspective. As an alternative to inserting data at an instant of time, the nudging method has been found to be very successful both in quasi-geostrophic, Holland and Malanotte-Rizzi (1989), Verron and Holland (1989) and primitive equation, Malanotte-Rizzi et al. (1989), experiments. An artificial forcing term applied to the right-hand side of the dynamical equations is used to relax the model fields towards the observed fields over some period of time. The most conspicuous success has been achieved in the quasi-geostrophic models when the forcing term is applied to the right-hand side of the top layer potential vorticity equation. This method is similar to the method used here in that only the top layer potential vorticity is being changed directly by the forcing and the lower layer potential vorticity fields then change due to advection (recalling lagrangian $q$ conservation) and Ekman friction as the model evolves. However, it is becoming clear that the faster the data can be inserted into the model, the better the model response. Holland and Malanotte-Rizzi (1989) showed that if the nudging term is applied with a Gaussian weighting around the observation time then, provided the forcing is strong enough for long enough, the model responds better for the shorter forcing periods. This seems to be related to the damping effect of the nudging on the high frequency modes (Capotondi and Malanotte-Rizzi, private communication) which distorts the model if the nudging is applied continuously—implying that it is better to let the model run freely between assimilation times.

A simple thought experiment is useful for suggesting how nudging could, in principle, produce the same results as the direct insertion method discussed here, even in a more generalized framework. 1) We consider that the observed field essentially gives us the current information at the surface (a geostrophic or cyclostrophic assumption is needed to make this true for altimeter data). 2) We assume that nudging is applied in such a way that the forcing term only appears in the top layer potential vorticity equation, and 3) the forcing is strong enough to make the upper layer currents adjust to the observed values very rapidly. The combination of assumptions 2 and 3 mean that the only mechanism that can have time to change the lower layer currents will be gravity waves, which do not alter the potential vorticity fields to any significant degree. This assumes then that the nudging is hard enough to alter the top layer fields on something like the geostrophic adjustment timescale. Of course, in practice it may be difficult to implement such a rapid nudging numerically, and therefore the direct insertion method may be a sensible alternative that does not distort the model physics. A study of this problem has now been completed in the framework of a three-layer shallow water model (Haines et al. 1991).

Previous work in the area of direct insertion has been divided along two lines. Either no attempt has been made to project the current information in the vertical, as in Hurlburt (1986) or Berry and Marshall (1989); or else a statistical means has been used to project current information based on long time period EOF analyses of the numerical model itself, see De Mey and Robinson (1987), with supporting comparisons made with vertical mode observations such as Richman et al. (1977) and Mercier and Colin de Verdiere (1985). It was mentioned in section 3 that the first method,
which avoids any vertical current projection, is equivalent to making a $\Delta \psi = 0$ assumption instead of a $\Delta q = 0$ assumption in the lower layers. In our quasi-geostrophic model this would imply a $\Delta q_2 = \gamma_{2,1} \Delta \psi_1$ change (with $\Delta q_{3,4} = 0$) which would effectively cancel out the $\Delta q_1$ change in the upper layer and prevent the circulation changes from penetrating to the layers below.

The second method, which uses a statistical basis for the vertical current projection, has its merits and should work well if care is taken in its application; however it has little to say on the dynamics that explain the vertical correlations. It may indeed work better than the method suggested here if there were strong evidence for the vertical correlation of potential vorticity anomalies in the ocean. However, apart from the compensating effect of Ekman friction discussed in section 3 and perhaps deep convection processes which are very locally confined, there seems to be little likelihood of such vertical correlations over large regions of the
ocean. The method discussed here of not changing the lower layer $q$ is much easier to apply and gives more insight into the mechanisms of model convergence as illustrated by considering the relationships between rms errors in the different layers.

Several important points must still be considered. The emphasis of this paper has been on the effects of vertical current projections where good evidence is available for the noncorrelation of $q$ anomalies in the vertical (at least within the models and with a supporting mechanism based on Rhines and Young's theory). The method of dealing with a limited data coverage of $\Delta \psi_1$ observations at the upper surface was not really addressed. Now $q_1$ anomalies certainly are correlated in the horizontal so it is not immediately clear how to change the $q_1$ field in order to generate the correct $\psi_1$ field within certain limited areas. A minimal change was one suggestion made in section 3 where $q_1$ is only allowed to change at the points (or along the tracks) where $\psi_1$ observations are available. The in-
version to obtain the full 3-D $\Delta \psi$ field with this assumption can then be performed. In this case the nudging method may be easier to apply because the forcing term that alters the $q_1$ field can easily be localized along satellite tracks; however it is not clear that this is the best approach given the expectation that $q_1$ anomalies are horizontally correlated. If $q_1$ is allowed to change at more points than those for which we have observations of $\psi_1$, then there is of course no unique solution. We could assume a correlation scale for $q_1$ and describe an analytic relation between $q_1$ changes at the observation points and in the surrounding region or apply an optimal interpolation scheme on the $q_1$ fields, although it seems that there may be little to gain from this over assuming such a correlation scale for $\psi_1$ instead. If a $\psi_1$ correlation scale or optimal horizontal
projection is used, then the problem becomes greatly simplified because we can assume $\Delta \Psi_1$ (after the horizontal interpolation) is known everywhere again and then invert for the lower-layer $\Delta \Psi$ as described here. The most interesting conceptual problem with this procedure is in understanding the horizontal correlations in the barotropic mode which may have anomaly or error correlations over the whole ocean. Work is underway to consider this problem in more detail.

Perhaps the other major question is the applicability of this method to primitive equation models, which many people advocate as being able to capture important non–quasi-geostrophic effects such as a surface thermocline. Although many primitive equation models, especially those with a level formulation, do not guarantee a potential vorticity conservation law numerically, the dynamics of the lower layers should provide for approximate Lagrangian conservation, and therefore potential vorticity mixing and homogenization may still take place in the absence of forcing. The
equations for the circulation changes, equivalent to Eqs. (7) here, can be obtained and solved (with some balance assumption imposed), thus providing the required changes in the lower-layer velocity and density fields. Such a direct insertion scheme has recently been implemented for a three-layer shallow water model and the results will appear later. All of the evidence so far seems to support the notion that \( q \) anomalies should not be correlated in the vertical within the unventilated thermocline regions of primitive equation models under a wide range of oceanic conditions.

To summarize, a direct insertion data assimilation method is advocated, that involves changing the top layer potential vorticity so as to obtain the streamfunction/sea surface height values which have been observed. The alteration of \( q_t \) will imply a change of circulation in all deeper layers of the ocean. These current changes reduce the \( \psi^{\text{rms}} \) errors in all but the bottom
layer. The deep $q$ fields are left unchanged by this process and it is the remaining errors, in the $q_4$ field particularly, which prevent immediate improvements to the bottom currents. However, the influence on the bottom currents due to vortex stretching from the improved layers above allows advection and Ekman friction to reproduce more correct $q_4$ (and hence $\psi_4$) fields as the model evolves over a period of time. Many practical details still remain to be investigated. However, these results show that even a coverage of the ocean on a 40-day timescale is sufficient to reconstruct the "true" ocean to considerable accuracy. The method naturally operates in a forecast mode so that no assumption of future observations is necessary at any stage. In particular, the analysis in terms of potential vorticity errors and the explanation of the noncorrelation of $q$ in the vertical due to Rhines and Young homogenization provides a powerful framework for the
interpretation of results, which has been lacking in previous assimilation studies.

Acknowledgments. I thank P. Malanotte-Rizzoli, J. C. Marshall and C. Wunsch for various useful comments and also A. Capotondi for lively discussions in the early stages of this study. This work was supported by the NSF under Grant OCE 8614369 and by NASA Project UPN 161-25-63-03. Acknowledgment is made to the National Center of Atmospheric Research, which is sponsored by the NSF, for the computing time used in this research.

REFERENCES


