The Evolution of Internal Wave Undular Bores: Comparisons of a Fully Nonlinear Numerical Model with Weakly Nonlinear Theory

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ABSTRACT

The validity of shallow-water, weakly nonlinear theory for describing the evolution of a single large internal wave depression into an undular bore is explored by comparing theoretical results with results obtained from a fully nonlinear numerical model. Inclusion of second-order nonlinear and dispersive terms significantly improves the agreement. Solutions of the KdV and extended KdV equations, which includes second-order nonlinearity, overpredict the wave amplitudes in the undular bore. Inclusion of all second-order nonlinear and dispersive terms significantly improves the predicted amplitudes; however, the resulting evolution equation breaks down for sufficiently large waves. This can be corrected by modifying the linear terms in the equation to give a modified equation. Solutions of this modified second-order equation are in much better agreement with the model results than are the solutions of the KdV equation and the extended KdV equations.

1. Introduction

Large amplitude, vertically trapped internal waves have been observed in the ocean in many locations worldwide, particularly in coastal regions where strong tidal currents flow over large topographic features (Huthnance 1989). Deep water observations have also been made (e.g., Zheng et al. 1995). While their behavior has been explained with considerable success by weakly nonlinear theories, detailed comparisons between observations and weakly nonlinear theories are difficult to make because it is difficult to follow the space–time evolution of the waves. To further complicate matters, observed waves are often three-dimensional and are affected by many physical processes that are not often modeled in the theories. These include rotational effects, horizontal variation of the background currents, shear, stratification, and topography, as well as interactions with other waves, eddies, etc. Dissipation effects are also often not included or are parameterized in a crude fashion.

Numerical modeling of internal wave evolution in an idealized ocean offers an opportunity to validate weakly nonlinear theories by comparing wave evolution in a fully nonlinear numerical model with the evolution predicted by weakly nonlinear theories. Differences in the wave evolution can then be clearly attributed to shortcomings of the weakly nonlinear theory. Such comparisons are made in this paper. This work complements the comparisons of laboratory experiments with weakly nonlinear theory for two-layer fluids made by Koop and Butler (1981), Segur and Hammack (1982), and Melville and Helfrich (1987). Cummins (1995) has recently made some comparisons between fully nonlinear numerical calculations and weakly nonlinear theory for a two-layer fluid. In contrast, these numerical experiments use a continuously stratified fluid.

Large depressions are formed when tidal currents flow over topographic features. These depressions propagate away and, depending on the stratification, may evolve into undular bores (Lamb 1994). The goal of this study is to understand how accurately weakly nonlinear theory predicts this evolution. To this aim, a numerical model is used to simulate the evolution of a single wide depression into an undular bore in a fluid of constant depth and in the absence of rotation. An example is shown in Fig. 1. The model was initialized with a single rightward propagating, mode-one depression (Fig. 1a). The wave front steepens due to nonlinearities and then dispersive processes result in the formation of an undular bore (Fig. 1b). The linear propagation speed is \( c = 0.842 \text{ m s}^{-1} \) and the background horizontal velocity is \( -0.942 \text{ m s}^{-1} \). Hence, the leading edge of the undular bore has propagated about 87 km through the fluid in 24 hours. The linear propagation speed accounts for about 73 km of this. At the rear of the depression nonlinearities have reduced the wave steepness so that dispersive effects are almost nonexistent.

Weakly nonlinear theories for vertically trapped internal waves in a continuously stratified fluid can be

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categorized according to the relative sizes of the fluid depth $H$, a length scale $h$ associated with the density stratification, and wavelength $L$ (Koop and Butler 1981). For shallow-water theory (Benjamin 1966; Benney 1966), which results in the Korteweg–de Vries (KdV) equation at first order, the length scales formally satisfy $L/H \gg 1$ and $h/H = O(1)$. Deep-water theory has $L/H \ll 1$ and $L/h \gg 1$ and results in the Benjamin–Ono equation (Benjamin 1967; Ono 1975). Finite-depth theory has the balance $L/h \gg 1$ and $h/H \ll 1$. This results in the Joseph equation (Joseph 1977; Kubota et al. 1978).

The background density profile used for the most part in this study is

$$\bar{\rho}(z) = 1027.31 - 3.3955 e^{(z-300)/50} \text{ kg m}^{-3}, \quad (1.1)$$

with $z$, the height above the bottom, between 0 and 300 m. Some model runs were done using a second density profile

$$\bar{\rho}_2(z) = 1027.31 - 10^{-4} \frac{z}{g}$$

$$- 1.696 \left[ 1 + \tanh \left( \frac{z - 200}{40} \right) \right] \text{ kg m}^{-3}, \quad (1.2)$$

where $g = 9.8 \text{ m s}^{-1}$ is the gravitational constant. Both density profiles are shown in Fig. 2 along with the cor-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example_plot.png}
\caption{Fig. 1. Contour plots of the density field from a model run. The waves are propagating to the right. A background flow of 0.942 m s$^{-1}$ to the left is used to keep the wave in the computational domain. The front of the initial depression (upper panel) steepens due to nonlinear effects and then dispersive effects result in the formation of an undular bore at $t = 24$ h (lower panel).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example_plot2.png}
\caption{Fig. 2. Undisturbed density and buoyancy frequency profiles: first density profile (solid), second density profile (dashed).}
\end{figure}
responding buoyancy frequency profiles computed using the Boussinesq approximation $N = (-g \beta' (z)/\rho_0)^{1/2}$ with $\rho_0 = 1000$ kg m$^{-3}$. The second density profile has an upper mixed layer about 50 m thick with the pycnocline 100 m thick centered 100 m below the surface. The two density profiles have the same values at the surface and the bottom with roughly the same maximum density gradient. For the second stratification much longer times are required to form undular bores so emphasis is placed on the first stratification.

The waves shown in Fig. 1b have an amplitude of about 25 m (vertical isopycnal displacement) and a wavelength of $L \approx 500$ m. The fluid depth is $H = 300$ m. Thus, using $h = 50$ m, $L/H \approx 1.6$, $h/H \approx 1/6$, and $L/h \approx 10$. Deep water theory is clearly inappropriate. The formal requirements for finite-depth theory are met, although the requirement that most of the density variation be confined to a thin thermocline is perhaps not satisfied in this case.

The formal requirements for shallow water theory are not satisfied as $L/H$ is not large. Laboratory experiments of Koop and Butler (1981) and Segur and Hammack (1982), however, showed that for a two-layer fluid, shallow-water theory was much more robust than could be formally expected, better in fact than finite-depth theory even for cases when one would expect otherwise. Segur and Hammack showed that this is because the range of validity of the asymptotic expansion in powers of the wave amplitude is much smaller for finite-depth theory than for shallow-water theory. For this reason shallow-water theory is considered in this paper.

In shallow-water weakly nonlinear theory for vertically trapped internal waves each dependent variable (streamfunction and density perturbation) is written as an asymptotic series in powers of two small parameters (see section 3). Each term is separated into the product of a function that depends on the vertical coordinate $z$ and a function that depends on the horizontal coordinate $x$ and on time $t$. The theory relates all of the functions of $x$ and $t$ to a single function $B(x, t)$, which we call the wave profile. The theory yields evolution equations for $B$, that is, the KdV equation and its higher-order extensions.

Our goal in this paper is twofold. First we investigate the existence of the wave profile $B(x, t)$. This is done by verifying that the asymptotic expansion of the density perturbation is consistent with the density perturbation in the nonlinear model results, that is, that the model results can be used to determine $B(x, t)$. The particular test we use is to fix $z$, use the asymptotic expansion to solve for $B(x, t)$, and then see how $B$ depends on the choice of $z$. If the complete asymptotic expansion could be used, then $B$ should be independent of $z$. Use of a finite number of terms means that $B$ will, in fact, depend on $z$. For the truncated asymptotic expansion to be useful this dependence should be small and should decrease as more terms in the series are used.

Our second objective is to investigate the validity of the weakly nonlinear evolution equations by comparing the time evolution of $B$ in the model results with the predictions of weakly nonlinear theory. To do this an initial wave profile $\tilde{B}$ is extracted from the model results at an early time and then evolved forward in time using a theoretical evolution equation. The resulting wave profile is compared to the wave profile extracted from the model results at the same time. Attention is focused on the front of the depression where the undular bore is formed.

The plan of the paper is as follows. In section 2 the nonlinear numerical model is briefly described. In section 3 second-order, shallow-water, weakly nonlinear theory is reviewed. The theoretical asymptotic expansions derived using weakly nonlinear theory involve a number of undetermined constants, the values of which must be chosen somewhat arbitrarily. Some new results showing the dependence of the asymptotic solutions on these constants are presented. Section 4 describes the procedure used to compare the weakly nonlinear theory with the numerical model results and discusses resolution tests. The results, using density (1.1), are presented in section 5 in three parts. First, in section 5a the choice of the values of the undisturbed constants is discussed and the existence of the wave profile $B(x, t)$ is confirmed. Next, in section 5b, the evolution of the wave profile is compared with the predictions of weakly nonlinear theory. Results using the KdV equation and two higher-order extensions are presented, along with results obtained using modified equations. Finally, the sensitivity of the results on the values of the undetermined constants is discussed in section 5c. A summary and discussion is given in section 6, where some comparisons to results obtained using density (1.2) are made.

2. The numerical model

The numerical model used is identical to the one used in Lamb (1994) to investigate internal wave generation by tidal flow across a bank edge. In this case the Coriolis terms are not included since the KdV equation and its extensions do not include rotational effects. The governing equations solved by the model are the inviscid, incompressible Boussinesq equations

\begin{align}
\frac{\partial U}{\partial t} + U \cdot \nabla U &= -\nabla p - \rho g k, \tag{2.1a} \\
\rho_t + U \cdot \nabla \rho &= 0, \tag{2.1b} \\
\nabla \cdot U &= 0. \tag{2.1c}
\end{align}

Here $U(x, z, t) = (u, w)$ is the velocity vector in the vertical plane (in m s$^{-1}$) with $u$ the horizontal velocity in the $x$ direction and $w$ the vertical velocity in the upward $z$ direction; $\nabla$ is the gradient operator ($\partial / \partial x$, $\partial / \partial z$), and $t$ is time. In these equations $\rho$ is the dimen-
sionless density variation, the dimensional density being \( \rho_0(1 + \rho) \) where \( \rho_0 \), taken as 1000 kg m
\(^{-3}\), is the reference density. The pressure \( p \) is likewise a dimensionless pressure, the full dimensional pressure being \( \rho_0 g z + \rho_0 p \cdot g = 9.8 \text{ m s}^{-2} \) is the gravitational constant and \( k \) is the unit vector in the upward direction.

The equations are solved in a domain of constant depth \( H \) with lower boundary at \( z = 0 \) and a rigid lid at \( z = H \). The inviscid boundary conditions include no normal flow at the upper and lower boundaries. The boundary conditions at the left and right consist of a prescribed, vertically uniform horizontal velocity \( u \) at the inflow boundary and a radiation condition at the outflow boundary (Lamb 1994).

3. Weakly nonlinear theory

The weakly nonlinear theory, which leads to the KdV equation and its higher-order extensions, was originally developed by Benney (1966) and extended to second-order by Lee and Beardsley (1974). In terms of a streamfunction \( \Psi(x, z, t) \) the governing equations (2.1) can be written as

\[
\frac{\partial}{\partial t} \nabla^2 \Psi - b_x = J(\Psi, \nabla^2 \Psi)
\]

\[
b + N^2(z) \Psi_x = J(\Psi, b),
\]

(3.1)

where

\[
(u, w) = (\Psi_z, -\Psi_x) \quad \rho = \bar{\rho}(z) + \rho'
\]

\[
b = g \rho' \quad N^2(z) = -g \frac{d\bar{\rho}}{dz}
\]

\[J(A, B) = A_z B_z - A_z B_x.
\]

(3.2)

The density field has been separated into a background state \( \bar{\rho}(z) \) plus a perturbation \( \rho'(x, z, t) \); \( N(z) \) is the buoyancy frequency of the background state. The boundary conditions are \( \Psi = b = 0 \) at \( z = 0 \) and \( H \).

The equations are nondimensionalized using the fluid depth \( H \) as the vertical length scale, a typical horizontal wavelength \( L \) as the horizontal length scale, a velocity scale \( U \), and the horizontal convective time scale \( L/U \). Thus we set

\[
(x, z, t) = \left( L \tilde{x}, H \tilde{z}, \frac{L}{U} \tilde{t} \right)
\]

\[
(\Psi, b) = \epsilon \left( U H \tilde{\Psi}, \frac{U^2}{H} \tilde{b} \right)
\]

\[
N^2 = \frac{U^2}{H^2} \bar{N}^2,
\]

(3.3)

where \( \epsilon \) is a dimensionless parameter measuring the wave amplitude. The dimensionless equations, after dropping the tildes, are

\[
\frac{\partial}{\partial t} \Psi_{zz} - b_z = \epsilon J(\Psi, \Psi_{zz}) - \mu \frac{\partial}{\partial t} \Psi_{xx} + \epsilon \mu J(\Psi, \Psi_{xx})
\]

\[
b + N^2(z) \Psi_x = \epsilon J(\Psi, b),
\]

(3.4)

where

\[
\mu = \frac{H^2}{L^2}.
\]

(3.5)

Assuming that the waves have small amplitude (\( \epsilon \ll 1 \)) and are long (\( \mu \ll 1 \)), the search for separable asymptotic solutions leads to solutions of the form

\[
\Psi \sim A(x, t) \phi(z) + \epsilon A^2(x, t) \phi^{(i)0}(z) + \mu A_{xx}(x, t) \phi^{(i)1}(z) + \epsilon^2 A^3(x, t) \phi^{(i)2}(z)
\]

\[
+ \epsilon \mu \left[ A(x, t) A_{xx}(x, t) - \frac{1}{2} A_x^2(x, t) \right] \phi^{(i)1}(z) \]

\[+ \frac{1}{2} A_x^2(x, t) \phi^{(i)0}(z) + \frac{1}{2} A_x^2(x, t) \phi^{(i)1}(z) + \frac{1}{2} A_x^2(x, t) \phi^{(i)2}(z) + \cdots
\]

(3.6a)

\[
b \sim A(x, t) \frac{N^2(z)}{c} \phi(z) + \epsilon A^2(x, t) D^{(i)0}(z) + \mu A_{xx}(x, t) D^{(i)1}(z) + \epsilon^2 A^3(x, t) D^{(i)2}(z)
\]

\[+ \epsilon \mu \left[ A(x, t) A_{xx}(x, t) - \frac{1}{2} A_x^2(x, t) \right] D^{(i)1}(z) \]

\[+ \frac{1}{2} A_x^2(x, t) D^{(i)0}(z) + \frac{1}{2} A_x^2(x, t) D^{(i)1}(z) + \frac{1}{2} A_x^2(x, t) D^{(i)2}(z) + \cdots
\]

(3.6b)

\[\phi(0) = \phi(1) = 0.
\]

(3.7)

The above equation defines the operator \( \mathcal{L} \). This eigenvalue problem has an infinite set of solutions \( \phi_n(z), c_n \) with \( c_1 > c_2 > c_3 > \cdots > 0 \). These represent the vertical structure and phase speed of mode-\( n \), linear, hydrostatic internal waves. Only mode-one waves propagating to the right are considered, so we set \( \phi = \phi_1 \) and \( c = c_1 \). Here \( \phi \) is uniquely specified by the condition
max \phi = 1. \quad (3.8)

The higher-order vertical structure functions \( \phi^{i,j}_s(z) \) and \( D^{i,j}_s(z) \) come from the \( O(\epsilon^i \mu^j) \) problems, which have the form

\[
\begin{align*}
\mathcal{L}(\phi^{i,j}_s) &= -r_{i,s} \frac{2N^2}{c^2} \phi + S^{i,j}_s \quad (3.9a) \\
\phi^{i,j}_s(0) &= \phi^{i,j}_s(1) = 0 \quad (3.9b) \\
D^{i,j}_s &= \frac{N^2}{c} \phi^{i,j}_s + r_{i,s} \frac{N^2}{c^2} \phi + T^{i,j}_s, \quad (3.9c)
\end{align*}
\]

where the \( S^{i,j}_s \) and \( T^{i,j}_s \) are known functions of solutions of the lower-order problems (see appendix A). The subscript \( s \) is present only for the \( O(\epsilon \mu) \) problem, which naturally splits into two parts denoted by subscripts \( a \) and \( b \). The homogeneous boundary conditions give the integral condition

\[
\int_0^1 \phi \mathcal{L}(\phi^{i,j}_s) dz = 0, \quad (3.10)
\]

which determines the \( r_{i,s} \) to be

\[
r_{i,s} = \frac{I^{i,j}_s}{I}, \quad (3.11)
\]

where

\[
I = 2 \int_0^1 \phi^{1,2}_s(z) dz \quad (3.12a)
\]

\[
I^{i,j}_s = c \int_0^1 \phi S^{i,j}_s dz. \quad (3.12b)
\]

Details are contained in the appendix A.

The general solution to (3.9) can be written as the sum of a particular solution \( \phi^{i,j}_{s,p} \) plus an arbitrary multiple of the solution of the homogeneous equation. Hence

\[
\phi^{i,j}_s = \phi^{i,j}_{s,p} + \alpha^{i,j} \phi, \quad (3.13)
\]

where the particular solution is specified by setting

\[
\frac{d}{dz} \phi^{i,j}_{s,p}(0) = 0. \quad (3.14)
\]

It is important to recognize that the asymptotic theory gives no way of determining the value of \( \alpha^{i,j} \). Thus, each term in the asymptotic series is nonunique, although as long as the series converges, the full asymptotic expression should not depend on the \( \alpha \) values (we drop subscripts and superscripts when referring to the collection of all the \( \alpha^{i,j} \)). Extremely large \( \alpha \) values can be expected to result in the breakdown of the asymptotic expansion (3.6). The authors are unaware of any discussion in the literature on how these values should be chosen. In later sections it is shown that the use of a truncated asymptotic series results in solutions that are somewhat sensitive to the values used. Optimal \( \alpha \) values could vary with time as the undular bore evolves and the relative strengths of the terms in the asymptotic expansion (3.6) varies. This aspect of the problem has not been investigated.

Results will be given in terms of the wave profile

\[
B(x, t) = \frac{A(x, t)}{c}, \quad (3.15)
\]

which, in the small amplitude, long-wave limit, gives the maximum vertical isopycnal displacement. From (3.6b) we have the second-order approximation

\[
\frac{b}{N^2} = B \phi + \epsilon B^2 E^{1,0} + \epsilon^2 B^3 E^{2,0} + \epsilon B_{a \mu} E^{0,1} + \epsilon^2 B_{a \mu}^2 E^{0,2} + \epsilon \mu \left( B_{a \mu} \frac{1}{2} B_x E^{1,1}_a \right) + \epsilon \mu \left( B_{a \mu} B_{a \mu} \frac{1}{2} B_x E^{1,1}_a \right) + \mu^2 B_{a \mu} E^{0,2}. \quad (3.16)
\]

Definitions of the \( E^{i,j}_s \) follow from a direct comparison with (3.6b).

In general, the particular solutions of (3.9) and (3.14) depend on the \( \alpha \) values from the lower-order problems. Letting \( \phi^{i,j}_s \) be the particular solutions obtained when all the \( \alpha \) are zero, the general solutions are given by

\[
\begin{align*}
\phi^{1,0} &= \phi^{1,0}_s + \alpha^{1,0} \phi \\
\phi^{0,1} &= \phi^{0,1}_s + \alpha^{0,1} \phi \\
\phi^{2,0} &= \phi^{2,0}_s + 2 \alpha^{1,0} \phi^{1,0}_s + \alpha^{2,0} \phi \\
\phi^{1,1}_a &= \phi^{1,1}_a + 2 \alpha^{0,1} \phi^{0,1}_s + \alpha^{1,1}_a \phi \\
\phi^{1,1}_b &= \phi^{1,1}_b + 6 \alpha^{1,0} \phi^{1,0}_s + 2 \alpha^{0,1} \phi^{0,1}_s + \alpha^{1,1}_b \phi \\
\phi^{0,2} &= \phi^{0,2}_s + \alpha^{0,1} \phi^{0,1}_s + \alpha^{0,2}_b \phi
\end{align*}
\]

with the \( E^{i,j}_s \) given by

\[
\begin{align*}
E^{1,0} &= E^{1,0}_s + \alpha^{1,0} \phi \\
E^{0,1} &= E^{0,1}_s + \alpha^{0,1} \phi \\
E^{2,0} &= E^{2,0}_s + 2 \alpha^{1,0} \phi^{1,0}_s + \alpha^{2,0} \phi \\
E^{1,1}_a &= E^{1,1}_a + 2 \alpha^{0,1} \phi^{0,1}_s + 2 \alpha^{1,0} \phi^{2,0}_s + \alpha^{1,1}_a \phi \\
E^{1,1}_b &= E^{1,1}_b + 6 \alpha^{1,0} \phi^{1,0}_s + 2 \alpha^{0,1} \phi^{0,1}_s + \alpha^{1,1}_b \phi \\
E^{0,2} &= E^{0,2}_s + \alpha^{0,1} \phi^{0,1}_s + \alpha^{0,2}_b \phi
\end{align*}
\]

where, again, the \( * \) values refer to values when all the \( \alpha^{i,j} \) are equal to zero.

To second order in \( \epsilon \) and \( \mu \), the wave profile \( B \) satisfies the evolution equation
This equation has been derived by a number of authors for both internal waves (Lee and Beardsley 1972) and for surface waves (Marchant and Smyth 1990). This evolution equation will be referred to as the full extended KdV equation (feKdV). Dropping all the second-order terms results in the KdV equation. Adding the second-order nonlinear term to the KdV equation \((e^2 r_{20} c^2 B^2 B_x)\) results in the extended KdV equation (eKdV). This latter equation was used by Lee and Beardsley for their comparisons with experimental results and observations. A forced version of the eKdV equation was used by Melville and Helfrich (1987).

In general, the evolution equations depend on the \(\alpha\) values through the constants \(r_{ij}\), the exception being the KdV equation because condition (3.8) fixes \(r_{10}\) and \(r_{01}\). The values of \(r_{02}\) and \(r_{11a}\) are also fixed; however, \(r_{20}\) and \(r_{11b}\) depend on \(\alpha^{1.0}\) and \(\alpha^{2.0}\) via

\[
r_{20} = r_{20}^* + \frac{2}{3} \alpha^{1.0} r_{10}
\]

\[
r_{11b} = r_{11b}^* + 6\alpha^{1.0} r_{01} - 4\alpha^{2.0} r_{10},
\]

where the asterisk values are those obtained when all \(\alpha\) are zero.

In comparing the wave evolution in the primitive equation numerical model to that predicted by weakly nonlinear theory, initial and final \(B\) profiles are calculated from the density perturbation in the numerical model results using (3.16). This is done by first specifying \(z\) and then solving (3.16) for \(B(x, t)\). The procedure for doing so is described in appendix B. In general the resultant wave profile depends on the values of \(z\). Further, \(B\) depends on each \(\alpha^{i,j}\) since \(E_{i,j}^i(z)\) does. This is an important point because, for example, the nonlinear propagation speeds for the first- and second-order evolution equations, \(c = \epsilon_2 r_{10} c B^2 c B + \epsilon_2 r_{20} c^2 B^2\), respectively, depend on \(B\). The wave evolution predicted by the KdV equation is sensitive to the values of \(\alpha^{1.0}\) because \(B(x, 0)\) depends on \(\alpha^{1.0}\) while \(r_{10}\) does not. The dependence of \(r_{20}\) on \(\alpha^{1.0}\), however, results in the nonlinear propagation speed of the second-order equation being much less sensitive to \(\alpha^{1.0}\). Indeed, let \(B^*\) be the wave profile obtained when all \(\alpha^{i,j}\) are zero and let \(B\) be the wave profile obtained for some nonzero \(\alpha^{i,j}\). Assuming for the moment that the dispersive terms are negligible (as is the case at early times), we have

\[
\frac{b}{N^2} \sim B^* \phi + \epsilon_2 B^* E_{i,j}^{1.0} + \epsilon_2 B^* E_{i,j}^{2.0} + \cdots
\]

\[
= B \phi + \epsilon_2 B^2 E_{i,j}^{1.0} + \epsilon_2 B^2 E_{i,j}^{2.0} + \cdots.
\]

Using (3.18) and assuming \(B\), \(\alpha^{1.0}\) and \(\alpha^{2.0}\) are small, we find that

\[
B = B^* - \epsilon \alpha^{1.0} c B^2
\]

\[
+ \epsilon^2 (2\alpha^{1.0} c^2 - \alpha^{2.0} c^2) B^3 + \cdots.\]  

Substituting this into the second-order nonlinear propagation speed we have

\[
c = \epsilon_2 r_{10} c B - \epsilon_2 r_{20} c^2 B^2
\]

\[
= c - \epsilon_2 r_{10} c B^* - \epsilon_2 r_{20} c^2 B^2 + O(\epsilon^3).\]  

Thus, to \(O(\epsilon^2)\) the nonlinear propagation speed of the eKdV and feKdV equations is independent of small changes in \(\alpha^{1.0}\) and \(\alpha^{2.0}\). The KdV propagation speed is \(c = \epsilon_2 r_{10} c B \approx c = \epsilon_2 r_{10} c B^* + \epsilon_2 2\alpha^{1.0} r_{10} c^2 B^2\), and hence is sensitive, at \(O(\epsilon^2)\), to the value of \(\alpha^{1.0}\) if \(B\) is sufficiently large. This sensitivity arises through the dependence of the initial wave profile on \(\alpha^{1.0}\). It can be expected that other quantities computed from \(B\) (e.g., the horizontal surface velocity) will depend on the \(\alpha\) values and that this dependence would decrease as more terms in the asymptotic expression were included.

4. Model setup

4.1 Model initialization

The model domain is 40 km long and extends from \(x = -30\) km to \(x = 10\) km. It is 300 m deep. Model runs are initialized with a single rightward propagating mode-one depression using second-order nonlinear and first-order dispersive theory. Because the initial depression has wide fronts at each end, the use of the dispersive term does not effect the initialization. Hence, while the initialization formally depends on \(\alpha^{1.0}\), \(\alpha^{2.0}\), and \(\alpha^{0.1}\), it does not in practice depend on \(\alpha^{0.1}\). Because the dispersive term is negligible, higher-order dispersive and the nonlinear dispersive terms were not used.

A reference frame moving with a speed \(U\), close to the wave propagation speed, is used. For small depressions we use \(U = c\). For larger waves, with their larger nonlinear propagation speeds, a slightly larger value of \(U\) is used. The initial velocity fields are obtained from the initial streamfunction

\[
\Psi(x, z, 0) = -Uz + Bc \phi(z) + \epsilon B^2 c^2 \phi^{1.0}(z)
\]

\[
+ \epsilon^2 B^3 c^3 \phi^{2.0}(z) + \mu B_{\alpha \phi} \phi^{0.1}(z),
\]

and the density field is initialized using

\[
\frac{b(x, z, 0)}{N^2(z)} = B \phi(z) + \epsilon B^2 E_{i,j}^{1.0}
\]

\[
+ \epsilon^2 B^3 E_{i,j}^{2.0}(z) + \mu B_{\alpha \phi} E_{i,j}^{0.1}(z).
\]
The initial pressure gradients are given by

\[ P_z(x, z, 0) = c^2 B_x \phi_z(z) + \mu B_{xx} (c^2 \phi_z^2(z) - r_{01} c \phi_z(z)) + \epsilon B_x \left[ -2r_{10} c \phi_z(z) + 2c^3 \phi_z^1 \phi_z(z) - c^3 \left( \phi_z^2(z) + \frac{N^2}{c^2} \phi^2 \right) \right] + \epsilon^2 c^5 B_x^2 B_x \left( 3c^4 \phi_z^2 - 4r_{10} \phi_z^1 - 3r_{01} \phi_z - 3 \phi_z^1, \phi_z^1, \phi_z \right) - 3 \frac{N^2}{c^2} \phi \phi^1 - 2r_{10} \frac{N^2}{c^3} \phi^2 + \frac{(N^2)^2}{c^3} \phi^1 \phi^1, \phi^1, \phi^1, \phi^1, \phi^1, \phi^1 \),

\[ P_z(x, z, 0) = -b(x, z, 0) - \mu c^2 B_x \phi_z(z). \] (4.3)

(The \( \epsilon \) and \( \mu \) are retained here to label the different terms for future discussion. Their values are taken to be 1 and all terms are dimensional.)

For the initial wave profile \( B \) we use

\[ B = B(x, 0) = -a \frac{1}{4} \left[ 1 + \tanh \left( \frac{x - x_i}{d_i} \right) \right] \times \left[ 1 - \tanh \left( \frac{x}{d_i} \right) \right]. \] (4.4)

with the initial depression amplitude \( a > 0 \). The left and right edges of the initial depression are at \( x = x_i \) = -20 km and at \( x = 0 \) respectively. Here \( d_i \), determine the width of the initial edges. The smaller the value of \( d_i \), the earlier the undular bore forms. We use \( d_i = d_r = 100a \).

b. Resolution tests

For the resolution tests a depression amplitude \( a = 20 \) m was used. For purposes of the resolution tests \( B \) profiles were computed for \( z = 200 \) using the first approximation \( B^{(0)} \) given by (B2).

The evolution equation (3.19) is solved using the pseudospectral method of Fornberg and Whitham (1978) suitably modified to include the higher-order terms. This requires evenly spaced horizontal grid points with the number of them a power of 2. One thousand, twenty-four grid points in the horizontal gave accurate solutions of the evolution equations. Model grids had \( I = 1024n \) equally spaced grid cells in the horizontal for integer values of \( n \); \( J \) evenly spaced grid cells were used in the vertical.

Model runs were done using maximum time steps of 2.5, 5.0, and 10.0 s; \( I \) varied between 1024 and 4096, while \( J \) varied between 40 and 120. Memory limitations restricted \( J \) to 80 for \( I = 4096 \). Some results are shown in Fig. 3. These tests verified the second-order convergence of the numerical scheme. In the tests the most noticeable difference was a slight forward shift in the undular bore as the resolutions (temporal and spatial) increased. Larger waves, with the larger fluid ve-
Table 1. scalings using $H = 300$ m, $L = 500$ m, and $U = 0.5$ m s$^{-1}$.

<table>
<thead>
<tr>
<th>Dimensional quantity</th>
<th>Scaling</th>
<th>Approximate value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>$H$</td>
<td>300 m</td>
</tr>
<tr>
<td>$c$</td>
<td>$U$</td>
<td>0.5 m s$^{-1}$</td>
</tr>
<tr>
<td>$\phi^{1,0}$, $\alpha^{1,0}$</td>
<td>$UUH$</td>
<td>$7 \times 10^{-3}$ m$^{-2}$</td>
</tr>
<tr>
<td>$\phi^{1,1}$, $\alpha^{1,1}$</td>
<td>$L^2$</td>
<td>$2.5 \times 10^9$ m$^{-2}$</td>
</tr>
<tr>
<td>$\phi^{2,0}$, $\alpha^{2,0}$</td>
<td>$U^2H^2$</td>
<td>$5 \times 10^{-4}$ m$^{-4}$</td>
</tr>
<tr>
<td>$\phi^{2,2}$, $\alpha^{2,2}$</td>
<td>$L^4$</td>
<td>$6.3 \times 10^9$ m$^{-4}$</td>
</tr>
<tr>
<td>$E^{1,0}$</td>
<td>$UH$</td>
<td>$1.7 \times 10^9$</td>
</tr>
<tr>
<td>$E^{2,0}$</td>
<td>$UH^2$</td>
<td>$3 \times 10^{-3}$ m$^{-1}$</td>
</tr>
<tr>
<td>$E^{2,2}$</td>
<td>$L^2$</td>
<td>$1 \times 10^{-5}$ m$^{-2}$</td>
</tr>
<tr>
<td>$E^{1,1}$</td>
<td>$L^2H$</td>
<td>800 m</td>
</tr>
</tbody>
</table>

The conclusions derived from this work would change if $I = 2048$, $J = 60$, and a time step of 10 s had been used.

5. Results

Comparisons between the numerical model results and weakly nonlinear theory are now made using density (1.1). We begin in section 5a with a discussion of the $z$ dependence of the $B$ profiles extracted from the model results and of the choice of the $\alpha$ values. In section 5b comparisons of the model and theoretical wave evolutions are made. Sensitivity to the $\alpha$ values is considered in section 5c.

Dimensional values for all quantities will be used in this section as well as in section 6. Here $B$ is nondimensionalized via $B = HB$ (quantities with a tilde represent dimensionless quantities). The nondimensionalization (3.3) then leads to $(\phi, c) = (\tilde{\phi}, U\tilde{c})$. With these scalings one can systematically determine the scalings for the other quantities. These are shown in Table 1 along with units for the various terms.

a. The $B$ profiles and choice of $\alpha$ values

Figures 4a,b show the linear vertical structure function $\phi$ and its first derivative. Plots of the $\phi^{1,0}$ and $E^{1,0}$ for various $\alpha$ values are shown in Figs. 5a–l. The large values of $\alpha^{1,1}$ and $\alpha^{3,2}$ used are a consequence of the magnitude of the corresponding vertical structure functions. Dividing by the scaling values given in Table 1 gives small values for their dimensionless counterparts.

The use of (4.1)–(4.3) to initialize the numerical model can only give an approximation of a fully nonlinear, rightward propagating, mode-one wave. As the initial wave is very wide, the $O(\mu)$ terms are negligible. Thus the initialization depends on $\alpha^{1,0}$ and $\alpha^{2,0}$ and not on any other of the $\alpha$ values. The values of $\alpha^{1,0}$ and $\alpha^{2,0}$ were chosen so as to minimize the adjustment of the initial wave after three hours for an initial amplitude of $a = 20$ m. The size of the adjustment was determined by measuring the variation of the wave profiles calculated at seven different heights ($z = 40, 80, \cdots, 280$ m). Based on these results model runs are initialized using $\alpha^{1,0} = -0.008$ s m$^{-2}$ and $\alpha^{2,0} = 0.0$ s$^{-2}$ m$^{-4}$. In

![Fig. 4. The linear vertical structure function $\phi(z)$ and its derivative $\phi'(z)$](image-url)
Fig. 5. Higher-order vertical structure functions for different $\alpha^j$ values. (a,b) $\phi^{0,0}$ and $E^{0,0}$ for $\alpha^{1,0} = 0$ and $-0.008 \text{ s m}^{-2}$, (c,d) $\phi^{0,1}$ and $E^{0,1}$ for $\alpha^{1,1} = 0$, 3000 and 6000 m$^3$, (e,f) $\phi^{2,0}$ and $E^{2,0}$ for $(\alpha^{1,2}, \alpha^{2,0}) = (0, 0), (-0.008 \text{ s m}^{-2}, 0)$, and $(-0.008 \text{ s m}^{-2}, 3 \times 10^{-5} \text{ s}^2 \text{ m}^{-4})$.

Fig. 6 the adjustment after three hours for such a case is shown. The spreading of the $B$ profiles after three hours is barely discernible at this scale and is small enough to ensure that the solutions of the weakly nonlinear evolution equations are virtually identical at $t = 24$ h when initialized with $B$ profiles extracted at any of these $z$ values for times ranging between 0 and 3 hours. As long as $\alpha^{1,0}$ and $\alpha^{2,0}$ are chosen in the ranges of $(-0.01, -0.008) \text{ s m}^{-2}$ and $(0, 3 \times 10^{-5}) \text{ s}^2 \text{ m}^{-4}$ respectively, the model results are not sensitive to their values provided that the initial depression amplitude $a$ is adjusted so that the initial isopycnal displacements are the same. For values outside these ranges the $z$ dependence of the $B$ profiles increases significantly, making the solutions of the evolution equations dependent on the $z$ value used to extract the initial wave profile (particularly for the large amplitude case).

Values of $\alpha^{0,1}$ and $\alpha^{0,2}$ are constrained by the iteration procedure used to solve (3.16) (appendix B).

Choosing $\alpha^{0,1}$ in order to minimize $|E^{0,1}|$ subject to the solvability constraint that $E^{0,1} \leq 0$ gives $\alpha^{0,1} \approx 3000 \text{ m}^2$. Given this value we choose $\alpha^{0,2} = 2.4 \times 10^{-7} \text{ m}^3$ so as to minimize the maximum value of $E^{0,2}$ while satisfying the constraint that $E^{0,2} \geq 0$. Larger values of $\alpha^{0,1}$ result in $E^{0,1}$ becoming positive for some $z$ values, while decreasing $\alpha^{0,2}$ results in some negative values for $E^{0,2}$. Solutions of (3.16) can still be obtained using the iteration procedure as long as these positive and negative values are not too large. Results are not very sensitive to the values of $\alpha^{1,1}$ and $\alpha^{1,2}$ as discussed below.

Figure 7 shows the $B$ profiles after 24 hours extracted in four different ways using $\alpha^{1,0} = -0.008 \text{ s m}^{-2}$, $\alpha^{1,1} = 3000 \text{ m}^2$, $\alpha^{0,2} = 2.4 \times 10^{-7} \text{ m}^3$. The other $\alpha$ values are set to zero. In the first (Fig. 7a), first-order nonlinear theory without dispersion is used. Significant variation of the $B$ profiles is found, most evident in the wave troughs where the spread is nearly one-third of...
their mean value. Inclusion of the second-order nonlinear term (proportional to $E_{2.0}^2$) makes the agreement worse (Fig. 7b). No solution is obtained in the troughs for $z = 280$ m. The addition of the first-order dispersive term (proportional to $E_{0.1}^1$) results in significant improvements (Fig. 7c). The $B$ profiles are now identical except in the troughs where $B$ and $B_x$ have their largest magnitude. This significant improvement is expected. First-order weakly nonlinear theory (i.e., the KdV equation) predicts that the individual waves will ultimately separate into a series of rank ordered solitons for which a balance between nonlinear and dispersive terms is vital. Adding the second-order nonlinear term again makes the agreement worse (not shown). Use of all the second-order terms (Fig. 8d) gives the best agreement, with the spread in the leading trough being about 5% of the median value of $-31.4$. This median value is almost the same as the median value obtained using first-order theory.

Tests with $\alpha_{b,1}^1$ ranging between $-160$ and $160$ m showed a barely discernible change. At the bottom of the troughs of each wave $B_z = 0$, so the term in (3.16) proportional to $E_{1.1}^1$ is identically zero here. Values of $\alpha_{a,1}^1$ ranging between $-60$ and $0$ m showed no change in the size of the spread of the $B$ profiles at the minima; however, the mean value changed slightly, ranging from about $-32$ m for $\alpha_{a,1}^1 = -60$ m to $-31$ m for $\alpha_{a,1}^1 = 0$ m. The iteration procedure broke down for values of $-80$ and $20$ m. We use $\alpha_{a,1}^1 = \alpha_{b,1}^1 = 0$ in the following.

Similar results are obtained using different values of $\alpha_{b,1}^1$ and $\alpha_{a,2}^1$ and are summarized in Table 2. The spread of the $B$ profiles amplitudes in the first wave is minimized for $\alpha_{0,1}^1 = 0$. Thus, this value is preferred to
which includes all the second-order terms. Modifications to these equations are also considered. In these equations $\epsilon$ and $\mu$ are equal to one (the equations are solved using dimensional values), they are included only to indicate ordering. The evolution equations were always initialized with the model results at $t = 1$ h in order to allow the model wave to adjust, although several tests showed that identical results were obtained if the equations were initialized at $t = 0$ or $t = 2$ h.

For the background density (1.1) the linear long-wave propagation speed is $c = 0.842$ m s$^{-1}$ and the values of the four coefficients of the evolution equations, which are independent of the $\alpha$ values, are $r_{10} = 8.31 \times 10^{-3}$ m$^{-1}$, $r_{01} = -2.91 \times 10^{3}$ m s$^{-1}$, $r_{02} = -2.08 \times 10^{3}$ m$^{2}$ s$^{-1}$, and $r_{11u} = 64.2$ m. Table 3 gives representative values of the other two coefficients.

The case with $a = 10$ m is considered first. For this case the background flow speed is $-c$. In Fig. 8a the median model wave profile is compared with the solutions of the KdV, eKdV, and feKdV equations at $t = 24$ h. The most significant difference between the predicted and model waves is in the wave amplitudes. The wave amplitudes of the KdV and eKdV solutions are about 10% too large at the front of the undular bore. Fourteen kilometers behind the bore front they are approximately three times larger than the model waves. The wave amplitudes of the feKdV solutions are in much better agreement with the model results, particularly near the tail of the bore. The improved agreement of the feKdV solution with the model result indicates that the model waves being much smaller than the KdV and eKdV waves is not a consequence of numerical dissipation in the model results, corroborating the resolution tests.

The wave front in the KdV solution lags the model wave front by about 150 m, while wave fronts in the solutions of the other two equations lead by about 50 m. These represent very small differences in wave propagation speeds relative to the fluid, through which the wave has propagated about 78 km. Convergence tests suggest that at higher resolutions the model wave front would move ahead slightly, by $10-20$ m. The wavelengths are in excellent agreement for the first seven to eight waves. Near the tail of the undular bore the wavelengths in the KdV and eKdV solutions are longer than those in the model results.

Figure 8b compares the surface velocities predicted by the solutions of the three evolution equations with the model surface velocities. For the KdV solutions the first three terms on the right of (3.6a) are used to calculate $u$. The fourth term is added for the eKdV solution. All terms are included for the feKdV solution. The general features are the same as in Fig. 8a. In particular, the peak surface velocity in the leading wave of the KdV solution is about 0.40 m s$^{-1}$, approximately 10%
larger than the value of 0.36 m s\(^{-1}\) in the model results. The eKdV solution is slightly better (peak velocity of 0.39 m s\(^{-1}\)). The agreement is poor in the tail of the bore. The feKdV solution is again much better, with the peak surface velocity in the leading wave being 0.37 m s\(^{-1}\). Small amplitude short-wavelength waves are apparent in the feKdV solution ahead of the undular bore.
TABLE 2. Range of $B$ minimums in first wave at 24 h, second-order theory: $a_{1}^{0.1}$ and $a_{1}^{0.1}$ are 0.

<table>
<thead>
<tr>
<th>$a_{1}^{0.1}$ (m$^2$)</th>
<th>$a_{1}^{0.2}$ (m$^2$)</th>
<th>$a_{1}^{0.3}$ ($^2$ m$^{-2}$)</th>
<th>Min (m) ($a = 10$)</th>
<th>Max (m) ($a = 10$)</th>
<th>Min (m) ($a = 20$)</th>
<th>Max (m) ($a = 20$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-3000.0$</td>
<td>$5.0 \times 10^4$</td>
<td>0.0</td>
<td>16.27</td>
<td>16.46</td>
<td>29.3</td>
<td>30.9</td>
</tr>
<tr>
<td>$0.0$</td>
<td>$1.2 \times 10^4$</td>
<td>0.0</td>
<td>16.67</td>
<td>16.85</td>
<td>30.2</td>
<td>31.6</td>
</tr>
<tr>
<td>$0.0$</td>
<td>$2.4 \times 10^7$</td>
<td>0.0</td>
<td>16.60</td>
<td>16.81</td>
<td>29.9</td>
<td>31.6</td>
</tr>
<tr>
<td>$0.0$</td>
<td>$1.2 \times 10^2$</td>
<td>$3 \times 10^{-3}$</td>
<td>16.59</td>
<td>16.79</td>
<td>29.8</td>
<td>31.3</td>
</tr>
<tr>
<td>$3000.0$</td>
<td>$2.4 \times 10^7$</td>
<td>0.0</td>
<td>17.09</td>
<td>17.30</td>
<td>31.1</td>
<td>32.8</td>
</tr>
<tr>
<td>$3000.0$</td>
<td>$2.4 \times 10^7$</td>
<td>$3 \times 10^{-3}$</td>
<td>16.99</td>
<td>17.23</td>
<td>30.6</td>
<td>32.4</td>
</tr>
<tr>
<td>$3000.0$</td>
<td>$3.85 \times 10^7$</td>
<td>0.0</td>
<td>16.97</td>
<td>17.23</td>
<td>30.7</td>
<td>32.6</td>
</tr>
<tr>
<td>$6000.0$</td>
<td>$3.85 \times 10^7$</td>
<td>0.0</td>
<td>17.54</td>
<td>17.81</td>
<td>32.1</td>
<td>34.1</td>
</tr>
<tr>
<td>$9000.0$</td>
<td>$6.7 \times 10^7$</td>
<td>0.0</td>
<td>17.91</td>
<td>18.32</td>
<td>32.6</td>
<td>35.5</td>
</tr>
</tbody>
</table>

$\alpha_{10} = -8 \times 10^{-3}$ S m$^{-2}$

$\alpha_{10} = -5 \times 10^{-3}$ S m$^{-2}$

(not shown). This is a consequence of the dispersive relation

$$\sigma = ck + r_{01} k^3 - r_{02} k^5$$

(5.4)

of the linearized feKdV equation. Here $\sigma$ and $k$ are the frequency and wavenumber of a sinusoidal wave. The linear feKdV phase speed $\sigma/k$ is plotted as a function of $k$ in Fig. 9, along with the KdV phase speed [given by (5.4) with $r_{02} = 0$]. Included in the figure is the true linear propagation speed $c(k)$ obtained by solving the eigenvalue problem

$$\phi''(z) + \left(\frac{N^2(z)}{c^2} - k^2\right)\phi(z) = 0$$

$$\phi(0) = \phi(H) = 0,$$

(5.5)

for $c(k)$. In the short-wave limit the propagation speed and group velocity go to $-\infty$ for the linearized KdV equation and to $+\infty$ for the linearized feKdV equation, whereas the true linear phase speed and group velocity go to zero. Good agreement with the true linear propagation speeds is obtained for $k$ less than about 0.006 m$^{-1}$ (wavelengths $> 1$ km). The disagreement grows rapidly for $k > 0.01$ m$^{-1}$. That the dispersion relation for the linearized KdV equation is inappropriate for short waves is well known [see Benjamin et al. (1972) for a detailed discussion]. These authors considered in detail the regularized long-wave (RLW) equation (also known as the BBM or PBBM equation)

$$B_t = -cB_x + 2r_{10}BB_x - \frac{r_{01}}{c} B_{xx},$$

(5.6)

which is derived from the KdV equation by noting that $B_x = -B/c$ to $O(\epsilon, \mu)$. This equation was first considered by Peregrine (1966) and has been used by a number of authors in place of the KdV equation because of its improved stability properties. Its linear dispersion relation is

$$\sigma(k) = \frac{k c}{1 - r_{01}k^2/c},$$

(5.7)

for which the propagation speed goes to zero as $k \to \infty$. Solutions of the RLW equation are very similar to solutions of the KdV equation. For the case with an initial amplitude of 20 m considered below the waves in the RLW solutions were about 1.5% smaller at the bore front and half the amplitude in the bore tail. Peak-to-peak wavelengths were slightly longer. Thus, the RLW solution is in slightly better agreement with the model results but not significantly so except in the bore tail.

In Fig. 10 comparisons of the model $B$ profile with the solutions of the KdV and eKdV equations are made for the case with an initial depression amplitude of 20 m (the same case depicted in Fig. 1). A solution of the feKdV equation could not be obtained (see below). For this amplitude the $B$ profiles extracted from the model results depend on $z$ (Fig. 7d), with the amplitude of the leading wave varying between 30.6 and 32.2 m as $z$ varies between 40 and 280 m (in 10-m intervals). The profile shown in the figure is that obtained for $z = 160$ m, close to the median profile.

TABLE 3. Dependency of $r_{01}$ and $r_{11}$ on $a_{1}^{0.1}$ and $a_{1}^{0.1}$.

<table>
<thead>
<tr>
<th>$a_{1}^{0.1}$ (m$^2$)</th>
<th>$a_{1}^{0.1}$ (m$^2$)</th>
<th>$r_{01}$ (m$^{-3}$ s)</th>
<th>$r_{11}$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.0$</td>
<td>$0.0$</td>
<td>$2.35 \times 10^{-3}$</td>
<td>$4.40 \times 10^2$</td>
</tr>
<tr>
<td>$0.0$</td>
<td>$3000.0$</td>
<td>$2.35 \times 10^{-3}$</td>
<td>$3.40 \times 10^2$</td>
</tr>
<tr>
<td>$-8 \times 10^{-3}$</td>
<td>$0.0$</td>
<td>$2.09 \times 10^{-3}$</td>
<td>$5.80 \times 10^2$</td>
</tr>
<tr>
<td>$-8 \times 10^{-3}$</td>
<td>$3000.0$</td>
<td>$2.09 \times 10^{-3}$</td>
<td>$4.80 \times 10^2$</td>
</tr>
</tbody>
</table>
The differences between the theoretical results and the model results are qualitatively the same as the \( a = 10 \) m case but are now larger. The waves at the front of the undular bore in both theoretical solutions are close to 25% larger than the model waves. The error in the theoretical wave amplitude grows toward the tail of the bore. The theoretical wavelengths (peak to peak) are about 10% too short at the front of the undular bore but become too long toward the tail of the bore. The KdV wave front lags the model wave front by almost 600 m while the eKdV wave front leads by about 350 m. These represent differences in propagation speeds of about -0.7% and 0.4%, respectively. The results of the resolution tests suggest that the model wave front is within 100 m of a converged solution. Because the model wave front moves ahead with increased resolution, this would improve the agreement of the eKdV solution in this regard.

Many difficulties were encountered in solving the feKdV equation. Very small time steps were required (0.00625 s vs 1.0 s for the KdV and eKdV equations). In addition, problems with instabilities were encountered that were time step independent. Short-wave instabilities were circumvented by occasionally applying a biharmonic damping term. This was accomplished by setting

\[ B \rightarrow B + K_s B_{xxx} \]  

(5.8)

every 50 time steps. A value of \( K_s = 50 \) was used to remove the small gridscale waves. Increasing \( K_s \) to 400 gave identical results. For initial depression amplitudes of 15 and 20 m solutions of the feKdV equation were unstable on length scales comparable to those of the waves in the developing undular bore. As a consequence the feKdV solutions broke down by \( t = 10 \) and 20 h respectively, with the solutions becoming very different from the model results at much earlier times. Setting \( \tau = 0 \) or \( \tau = 0 \) resulted in an earlier breakdown of the solution.

An alternative evolution equation, with a different linear dispersion relation, was tried. The linearized feKdV equation is
Fig. 9. Linear propagation speeds $c$ as a function of wavenumber $k$: true linear propagation speed (solid), linear KdV equation (dotted), linear feKdV equation (dashed), analytic approximation to linear propagation speed (dash-dot).

$$B_i = -cB_x + r_{01}B_{xxx} + r_{02}B_{xxxx}.$$  

The right side of this equation can be written as

$$\int_{-\infty}^{\infty} K(x - \xi)B(\xi, t) d\xi,$$  

where

$$K(x) = \frac{-i}{2\pi} \int_{-\infty}^{\infty} \sigma(k) e^{ikx} dk.$$  

(Whitham 1967; Fornberg and Whitham 1978) with $\sigma(k)$ given by (5.4). It is straightforward (numerically) to use alternate expressions for $\sigma(k)$. An obvious choice is the true linear dispersion relation $\sigma(k) = kc(k)$ obtained by solving the eigenvalue problem

(5.5) (Whitham 1967). An approximate analytic dispersion relation of the form

$$c(k) = \sigma(k)/k = \left(\frac{a}{k^2 + b}\right)^{1/2}$$  

(5.12)

can also be used. For mode-one internal waves in a fluid with constant buoyancy frequency $N$ and depth $H$, (5.12) is the exact linear dispersion relation when $a = N^2$ and $b = (\pi/H)^2$. For the analytic approximation the constants $a$ and $b$ are chosen so that the first two terms of the Taylor series expansion, in powers of $k^2$, agree with the KdV propagation speed $c + r_{01}k^2$. This gives

$$c(k) = \sigma(k)/k = \left(\frac{c^3}{c - 2r_{01}k^2}\right)^{1/2}.$$  

(5.13)

This analytic approximation is included in Fig. 9. Replacing the linear terms of the feKdV equation with (5.10) and using one of the alternative dispersion relations results in the modified feKdV equation. This change is easily implemented using the pseudospectral method of Fornberg and Whitham (1978) to numerically solve the evolution equations.

In Fig. 11 solutions of the feKdV equation (5.3) and the modified feKdV equation, using the true linear dispersion relation, are compared with the model results for the case with an initial amplitude of $a = 10$ m. The modified equation is in slightly better agreement with the model results at the front of the undular bore but is in slightly worse agreement in the bore tail.

In Fig. 12 solutions of the modified feKdV equation, using both the analytic dispersion relation (5.13) and the true linear dispersion relation obtained by solving (5.5), are compared with the model results for the case with an initial amplitude of $a = 20$ m. There is a dramatic improvement (compared to the KdV and eKdV

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Fig. 10. Profiles of $B$: Solutions of KdV and eKdV equations at $t = 24$ h compared with model results for initial amplitude of $a = 20$ m. Model results using $z = 160$ m (gray), KdV solution (dotted), and eKdV solution (dashed). $a$ values as in Fig. 7.
solutions) in the wave amplitude at the front of the bore where the difference between the predicted and model wave amplitudes is now less than 2%. The theoretical amplitudes are still far to large 15 km behind the front but are two-thirds of the amplitude of the corresponding waves in the KdV and eKdV solutions, a significant improvement. Farther back the amplitudes of the KdV, eKdV, and modified feKdV waves are similar. The bore front lags the model front by about 300 m. The solutions of the modified feKdV equation using the true linear propagation speed $c(k)$ and the approximation (5.13) are similar near the front of the undular bore. The theoretical peak–peak wavelengths are in good agreement near the front of the bore but are too large in the tail of the bore. Toward the rear the phases of the two theoretical solutions drift apart due to differing peak–peak wavelengths.

When the KdV and eKdV equations were modified in the same way as the feKdV equation, the result was very different. Consider the large amplitude ($a = 20$ m) case. As expected, waves in the bore tail were substantially reduced in size, resulting in much better agreement with the model results. However, for the modified KdV equation the waves in the front of the bore were about 25% larger than the corresponding waves in the KdV equation, substantially increasing the difference between the theoretical and model wave amplitudes. This increase in amplitude can be attributed to the decrease in the horizontal spread of the wave train (most prominent at the wave tail) caused by the large decrease in the propagation speed gradient $c_k$ in wavenumber space. The larger waves are also propagating more rapidly so that the wave front is farther ahead. The waves in the modi-
fied eKdV equation became so large that the solution broke down.

c. Sensitivity to $\alpha$ values

A number of runs were done to determine the sensitivity of the comparisons of the theoretical and model waves to the $\alpha$ values. A summary of the qualitative sensitivity is given in Table 4.

The KdV equation is independent of the $\alpha$ values, so solutions of this equation are sensitive to them only through the initial wave profile. Increasing $\alpha^{1.0}$ from $-0.008$ to $-0.006 \text{ s m}^{-2}$ increases the initial wave amplitude and also increases the $z$ dependence of the wave profiles. For the small amplitude case ($a = 10 \text{ m}$) the variation of $B$ is not very large. The increased wave amplitude results in an increase in the propagation speed, and as a consequence the wave front moves forward by 100 m and now lags the model wave front by only 50 m. The leading wave amplitude (measured down) increases from about 19.2 to 19.5 m. This increase in amplitude is compensated by an increase in the amplitude of the extracted wave profile, which increases from about 17.2 to 17.6 m (see Table 2). Thus, not only is there difference in wave front positions reduced by two-thirds, the amplitude difference decreases slightly as well, from 12% to 11%. For the larger amplitude case the extracted profiles become sensitive to $z$, so that the KdV solutions depend on the value of $\alpha$ used for calculating the initial wave profile $B$. This results in a considerable variation in the solutions at $t = 24 \text{ h}$. As mentioned in section 5a, values of $\alpha^{1.0}$ between about $-0.01$ and $-0.008 \text{ s m}^{-2}$ worked equally well in minimizing the $z$ dependence of the wave profiles at early times. The value $-0.008 \text{ s m}^{-2}$ was used rather than the median value of $-0.009 \text{ s m}^{-2}$ because of the improvement in the position of the wave front and wave amplitudes with the larger value of $\alpha^{1.0}$; however, this choice does not significantly affect the results.

The position of the wave front in the solutions of the eKdV and feKdV equations does not exhibit the same sensitivity to the value of $\alpha^{1.0}$ because of the dependence of $r_{20}$ on $\alpha^{1.0}$, as discussed in section 3.

For example, as $\alpha^{1.0}$ increases from $-0.008$ to $-0.006 \text{ s m}^{-2}$, the wave front of the eKdV solution moves back only 6 m. The amplitude again increases from 19.2 to 19.5 m.

All of the evolution equations are independent of $\alpha^{2.0}$. Changing $\alpha^{2.0}$ from 0 to $3 \times 10^{-4} \text{ s}^2 \text{ m}^{-4}$ slightly decreases the initial wave profile amplitude and hence changes the solutions of the evolution equations somewhat. This change is not very significant. For example, the wave front of the KdV solution is moved back by 20 m with negligible amplitude change.

The KdV and eKdV solutions are independent of $\alpha^{0.1}$ and $\alpha^{0.2}$, as both the initial wave profile and the equations themselves are independent of them. The extracted profile to which they are being compared, however, does depend on their values. Increasing $\alpha^{0.1}$ from 3000 to 6000 $\text{ m}^2$, and $\alpha^{0.2}$ from $2.4 \times 10^7$ to $3.85 \times 10^7 \text{ m}^4$ increases the leading wave amplitude from 17.2 to 17.7 m. Hence, by increasing $\alpha^{0.1}$ sufficiently one could perhaps make the extracted wave amplitudes agree with the KdV wave amplitudes at the wave front. The iteration procedure used to solve for $B$, however, begins to breakdown for $\alpha^{0.1}$ much larger than 6000 m. Also, for $\alpha^{0.1} = 6000 \text{ m}^2$, the wave profile has a larger dependence on $z$. The calculation of the surface velocities: from the KdV and eKdV solutions depends on $\alpha^{0.1}$ but not on $\alpha^{0.2}$. Increasing $\alpha^{0.1}$ from 3000 to 6000 $\text{ m}^2$ decreases the amplitude of the leading wave in the KdV solution to 0.38 m s$^{-1}$, halving the error.

The feKdV equation depends on the value of $\alpha^{0.1}$ through the parameter $r_{110}$, which decreases as $\alpha^{0.1}$ increases. For the small amplitude $a = 10 \text{ m}$ case increasing $\alpha^{0.1}$ from 3000 to 6000 $\text{ m}^2$ results in a decrease in the leading wave amplitude of about 10%, while the extracted wave profile increases by about 3%. The wave front moves back by approximately 150 m. In addition, the first three waves are closer together, but the separation distances between the other waves is not significantly changed. As a result the third and following waves are moved ahead by about 100 m. Increasing $\alpha^{0.1}$ thus increases the difference between the model and theoretical wave profiles. When $\alpha^{0.1}$ is decreased to 1000 $\text{ m}^2$, the theoretical solution breaks down.

---

Table 4. Dependency of various quantities on $\alpha$ values and the effect of increasing $\alpha^{1.0}$, $\alpha^{2.0}$, and $\alpha^{0.1}$.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>$\alpha$ Dependency</th>
<th>$\alpha^{1.0}$</th>
<th>$\alpha^{2.0}$</th>
<th>$\alpha^{0.1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial wave profile</td>
<td>$\alpha^{1.0}$, $\alpha^{2.0}$</td>
<td>increase</td>
<td>decrease</td>
<td>—</td>
</tr>
<tr>
<td>Final wave profile</td>
<td>all $\alpha^{1/2}$</td>
<td>none</td>
<td>decrease</td>
<td>increase</td>
</tr>
<tr>
<td>KdV equation</td>
<td>none</td>
<td>increase amplitude, forward shift</td>
<td>slight backward shift</td>
<td>—</td>
</tr>
<tr>
<td>eKdV equation</td>
<td>$\alpha^{1.0}$</td>
<td>increase amplitude, no shift</td>
<td>slight backward shift</td>
<td>—</td>
</tr>
<tr>
<td>feKdV equation</td>
<td>$\alpha^{1.0}$, $\alpha^{0.1}$</td>
<td>increase amplitude, no shift</td>
<td>slight backward shift</td>
<td>decrease amplitude, backward shift</td>
</tr>
<tr>
<td>Modified feKdV equation</td>
<td>$\alpha^{1.0}$, $\alpha^{0.1}$</td>
<td>increase amplitude, no shift</td>
<td>slight backward shift</td>
<td>increase amplitude, forward shift</td>
</tr>
</tbody>
</table>
For the larger wave case the sensitivity to the \( \alpha \) values is similar. For the modified feKdV equations increasing \( \alpha^{0.1} \) from 3000 to 6000 \( \text{m}^2 \) results in a 5\% increase in the wave amplitude and a forward shift of all the waves by about 150 \( \text{m} \), halving the difference in wave front position. This is illustrated in Fig. 13 where the surface currents in the model results are compared with those in the theoretical solutions for two cases, one with \( \alpha^{0.1} = 3000 \text{ m}^2 \) and \( \alpha^{0.2} = 2.4 \times 10^7 \text{ m}^4 \) and one with \( \alpha^{0.1} = 6000 \text{ m}^2 \) and \( \alpha^{0.2} = 3.85 \times 10^7 \text{ m}^4 \). Significant improvement in amplitudes at the front of the bore is obtained with the larger \( \alpha^{0.1} \) value, with the KdV solutions overestimating the surface currents by about 15\% versus 25\% for the \( \alpha^{0.1} = 3000 \text{ m}^2 \) case. The modified feKdV solution (using the true linear dispersion relation) underestimates the surface currents by about 9\% when \( \alpha^{0.1} = 3000 \text{ m}^2 \) and by about 7\% when \( \alpha^{0.1} = 6000 \text{ m}^2 \). While the larger value of \( \alpha^{0.1} \) results in better agreement, it is at the cost of an 18\% increase in the dependence of the extracted wave profiles on \( z \) (see Table 2).

6. Summary and discussion

In this paper the validity of first- and second-order weakly nonlinear theories for the description of the formation of an internal wave undular bore from an initial wide depression has been explored. This was accomplished by making comparisons with the results of a fully nonlinear numerical model. Because the weakly nonlinear theory is based on the governing equations of the numerical model, differences between theoretical predictions and the numerical model results can be attributed to inadequacies of the theory. This allows the range of validity of the theory to be ascertained and illustrates the manner in which the theory is inadequate. This knowledge can be used to determine whether discrepancies between observations and theoretical predictions are due to weaknesses of the theory or due to inadequately modeled or missing physical phenomena (rotation, dissipation, interaction with other waves, etc.). It may also aid in the development of better theories. The modified feKdV equation is an example of this.

Shallow-water, weakly nonlinear theory assumes the existence of a separable wave profile \( B(x, t) \). It is based on an asymptotic expansion in powers of two small parameters: the nonlinear parameter \( \epsilon \), a measure of the amplitude to depth ratio; and the dispersive parameter \( \mu \), a measure of the square of the depth to horizontal wavelength ratio. The theory yields evolution equations for \( B \): the Korteweg–de Vries (KdV) equation at first-order in \( \epsilon \) and \( \mu \), the extended KdV (eKdV) equation when second-order nonlinearity is included, and the full extended KdV (feKdV) equation when all the second-order terms are included. The vertical structure of the waves are described by a number of vertical structure functions. These functions are determined only up to an arbitrary multiple of the linear vertical structure function \( \phi \). This results in a number of undetermined constants (the \( \alpha \)).

The numerical model was initialized with a single depression, the front and rear of which were given by hyperbolic tangent profiles with a long flat region in between. Attention was focused on the leading front of the depression, which evolves into an undular bore. The initialization was done using the results of the weakly nonlinear theory. Use of second-order nonlinear terms was essential for the amplitudes considered here. The dispersive terms were unimportant for the initialization because of the broad front and rear.

The existence of a separable wave profile was explored by extracting \( B(x, t) \) at various values of \( z \) from the model density field \( \rho(x, z, t) \). Because only a finite number of terms are used in the expression relating \( B \) to \( \rho \), the extracted \( B \) profiles depend on \( z \). In principle, as the number of terms retained in the asymptotic expansion for \( \rho \) increases this dependence should disappear. It was shown that second-order theory resulted in a significant reduction in the \( z \) dependence of the extracted wave profiles. For the case of an undular bore in which individual waves are about 75\% of the breaking amplitude, the variation of the \( B \) profiles obtained using first and second-order theory was about 10\% and 5\% of the median value respectively (for \( z \) in 10 m intervals ranging between 40 and 290 m above the bottom in a fluid of depth 300 m). For waves of one-half amplitude the variation of the \( B \) profiles using second-order theory was about 1.5\%. This suggests that the assumption of the existence of a separable wave profile \( B(x, t) \) is valid.

The results depend on the choice of values for the arbitrary constants. The constants associated with the vertical structure functions for the nonlinear terms (\( \alpha^{1.0} \) and \( \alpha^{2.0} \)) could be restricted to a range of values \([ (-0.01, -0.008) \text{ s} \text{ m}^{-2} \text{ and } (0, 0.00003) \text{ s} \text{ m}^{-4} \text{, respectively}] \) on the basis of minimizing the dependence of the \( B \) profiles on \( z \) for early times (before the curvature of the profiles became large enough for the dispersive terms to be important). Values of \( \alpha^{0.1} \) and \( \alpha^{0.2} \) were chosen to minimize the absolute values of the \( O(\mu) \) and \( O(\mu^2) \) vertical structure functions \( E^{0.1} \) and \( E^{0.2} \) associated with the density perturbation, subject to satisfying the constraints imposed by the iteration method (namely that \( E^{0.1} \leq 0 \) and \( E^{0.2} \geq 0 \)) used to solve (3.16). This constraint is perhaps unphysical and could possibly be removed by an alternative solution method. A range of values of \( \alpha^{0.1} \) and \( \alpha^{0.2} \) were considered and the results suggest that these choices are reasonable, at least for the density profile considered. The two remaining constants were taken as zero. They do not affect the evolution equations and the extracted wave profiles were not very sensitive to their values.

With the existence of a separable wave profile confirmed, the wave profile evolution predicted by the three evolutions equations was compared with the evo-
ution in the numerical model. Two cases were considered in detail for which density profile (1.1) was used. Comparisons were made after 24 hours. For the large amplitude case the waves in the undular bore after 24 hours were about 75% of the breaking amplitude. These waves had isopycnal displacements of about 30 m, 10% of the fluid depth. In the smaller amplitude case the waves were about half the size.

Positions of the undular bore front were in excellent agreement with the model results even for the large amplitude case for which the predicted wave propagation speeds were within 1% of that in the model results (averaged over 24 hours). The greatest difference between the theoretical results and the model results was in the wave amplitudes. The waves in the KdV and eKdV solutions overpredicted the model wave amplitudes at the front of the undular bore by about 10% and 25% for the small and large amplitude cases respectively. Much larger errors were found in the tail of the undular bore. For the small amplitude case the wave amplitudes in the solution of the second-order feKdV equation were in much better agreement with the model waves, being within 2% of the model wave amplitudes. For the large amplitude case the feKdV equation became unstable and broke down. This instability is associated with the dispersion relation of the linearized feKdV equation for which the propagation speed goes to infinity as the wavelength goes to zero. The linear terms (5.9) can be written in the form of an integral (5.10) with its kernel $K$ given in terms of the dispersion relation $\sigma(k)$. Replacing $\sigma(k)$ with alternate expressions (Whitham 1967; Fornberg and Whitham 1978) results in a modified feKdV equation. Both the true linear dispersion relation (given by (5.5)) and an analytic approximation (5.13) were considered. Solutions obtained using either of these were very similar near the front of the bore but drifted apart in the bore tails due to differing peak-peak wavelengths. They were in much better agreement with the model results than were solutions of the KdV or eKeV equations, with the theoretical wave amplitude being within 2% of the model wave amplitude at the front of the undular bore.
Substantial errors persisted in the tail of the undular bore. Theoretical peak–peak wavelengths were in good agreement with the model results near the front of the bore but were too large in the bore tail.

The large errors in the undular bore tail have not been studied in detail; however, this result is possibly a consequence of energy in the shorter wavelengths propagating too slowly, which results in the undular bore being too long. After much longer times all of the waves that can be seen in the figures separate into a train of solitary waves followed by a dispersive wave train (not shown). The waves in the dispersive wave train are much smaller than the waves seen in any of the figures. At $t = 24$ h they cannot be seen.

Modifying the KdV and eKdV equations in the same way resulted in much larger wave amplitudes at the front of the bore, significantly increasing the error. The RLW equation was also considered. To obtain this equation the KdV equation is modified by replacing the term $r_0 B_{xx}$ by $-(r_0/c)B_{xx}$. The KdV and RLW equations are formally equivalent to first order. The linearized RLW equation has much improved dispersive properties in the short-wave limit. The solution of the RLW equation was not significantly different from that of the KdV equation at the front of the bore where the wave amplitudes were only 1.5% smaller (a slight improvement). There was significant improvement in the solutions in the bore tail, with the wave amplitudes cut in half. Time steps for the RLW equation could be about 10 times larger than those used for the KdV equation so the RLW equations is not only more accurate but is also much more efficient numerically.

The above results were obtained using density (1.1). Some model runs were done using a second density (1.2), which had the same density change and maximum gradient but differed in that it has a pycnocline 100 m thick centered at about 200 m with an upper mixed layer about 50 m thick (see Fig. 2). The first-order nonlinear effects for this profile are substantially weaker than for the first ($r_{10} = 3.39 \times 10^{-3}$ m$^{-1}$ vs $8.31 \times 10^{-3}$ m$^{-1}$). In contrast, the first-order dispersive effects are stronger, with $r_{01} = -5.1 \times 10^3$ m$^3$ s$^{-1}$ versus $-2.9 \times 10^3$ m$^3$ s$^{-1}$ for density (1.1). As a consequence the formation time for the undular bore was substantially increased and the wavelength of the individual waves is substantially larger. For the case with an initial amplitude of 20 m, the bore consisted of only six waves after 36 hours.

Interestingly the optimal values for $\alpha^{1.0}$ and $\alpha^{2.0}$ were the same for both density profiles. Choosing $\alpha^{0.1}$ on the basis of minimizing the magnitude of $E^{0.1}$ while requiring $E^{0.1} \leq 0$ gives $\alpha^{0.1} \approx 4000$ m$^{-1}$. With this value for $\alpha^{0.1}$ choosing $\alpha^{0.2}$ so that $E^{0.2} \geq 0$ while minimizing $|E^{0.2}|$ results in $\alpha^{0.2} \approx 2.8 \times 10^7$ m$^4$. The similarity of these four values to the values determined in a similar fashion for the first density profile is presumably related to the fact that both have the same density change between the surface and the bottom as well as approximately the same peak density gradient.

For an initial amplitude of $a = 10$ m the extracted profiles at $z = 40, 80, \cdots, 280$ m were virtually identical (using second-order theory). In Fig. 14 solutions of the KdV, eKdV, and the modified feKdV equations are compared with the model results at 36 hours. The KdV wave front is about 1 km in front of the model wave front and the leading waves are slightly more than 10% larger than the corresponding model waves. The eKdV and modified feKdV solutions are in much better agreement, particularly the latter.

Problems were encountered for the larger amplitude case, which had an initial amplitude of $a = 20$ m. The isopycnal displacements of the resulting waves in the undular bore were similar in amplitude to the corresponding case using density (1.1). The iteration pro-
procedure for calculating $B$ from the density perturbation did not converge for $z$ between about 200 and 280 m. This happened for both the first- and second-order theories and is related to the fact that $E^{1.0}$ is positive in this range. The quadratic equation

$$\frac{b}{N^2} = B\phi + B^2 E^{1.0}$$

has no solution for large negative $b$. With an alternative initial $B^{(0)}$ (e.g., $b/N^2\phi$), the iteration procedure continually increases the amplitude of $B$ so that $B_x$ grows. As a consequence $B^2 E^{1.0}$, which is positive, becomes larger while $B_x E^{0.1}$ becomes more negative. The nonlinear term dominates and $B \rightarrow -\infty$ during the iteration. Decreasing $\alpha^{1.0}$ from $-0.008$ to $-0.026$ s m$^{-2}$ made $E^{1.0}$ negative everywhere and the iteration procedure worked with $\alpha^{2.0} = 0$ s$^2$ m$^{-4}$. Increasing $\alpha^{2.0}$ increases $E^{2.0}$. With a value of $\alpha^{2.0} = 0.0001$ s$^2$ m$^{-4}$ solutions could be obtained with $\alpha^{1.0} = -0.0015$ s m$^{-2}$. Both these cases are unacceptable as they result in a significant dependence of $B$ on $z$ (>50% of their median values). This illustrates a breakdown in the theory with the truncated asymptotic series no longer being valid.

In Fig. 15 the surface velocities predicted by the weakly nonlinear evolutions equations, relative to the fluid in front of the undular bore, are shown. The KdV solution is over 4 km in front of the wave front. In fact, the second wave in the solution is also ahead of the leading wave in the model results. The linear propagation speed is about 1.276 m s$^{-1}$. The upstream flow speed used for the model run was $c_u + 0.02 = 1.297$ m s$^{-1}$. Thus, the wave has propagated about 173 km through the fluid. The leading wave in the KdV solutions has propagated an additional 4 km, which represents a 2.3% error in the propagation speed (in contrast to a $-0.7\%$ error in the corresponding case using the first stratification). The maximum surface current predicted by the KdV solution is about 25% too large. Both the eKdV and modified eKdV solutions are in much better agreement with the model results in terms of both the position of the wave front and the wave amplitudes. Apart from the amplitude of the leading wave the eKdV solutions is clearly the best of the two.

One problem left unanswered by this study is the question of how to choose the $\alpha$ values. The choice of $\alpha^{1.0}$ and $\alpha^{2.0}$, based on minimizing the $z$ dependence of the $B$ profiles at early times when the flow curvature is small, is relatively well defined. The tedious procedure by which this is done is not very desirable. A theoretical basis for computing these values given a density profile is desirable. Even more problematic is the choice of the other $\alpha$ values, although picking $\alpha^{0.1}$ and $\alpha^{0.2}$ on the basis of minimizing their maximum absolute values subject to $E^{0.1}$ and $E^{0.2}$ being less than and greater than 0 respectively appears to work well.

The results discussed here indicate that significant improvements are gained by using second-order theory. Use of second-order theory results in a greatly reduced $z$ dependence of the extracted $B$ profiles. The modified eKdV equation, using either the true linear dispersion relation or the analytic approximation (5.13), is significantly better than either the KdV or the eKdV equations, which consistently significantly overestimate wave amplitudes in the undular bore. This suggests that observed waves that are smaller than those predicted by first-order weakly nonlinear theories may not only be a consequence of dissipation but may also be due to missing higher-order nonlinear and dispersive effects. Because the KdV and eKdV wave amplitudes are so similar, it is the higher-order nonlinear-dispersive and dispersive terms that are crucial for obtaining better predictions of wave amplitudes. The significant improvements to the eKdV equation that resulted from

**Fig. 15.** Surface currents at $t = 36$ h using the second density profile $\delta_2$. Initial amplitude $a = 20$ m. Model result (gray), KdV solution (short dashes), eKdV solution (long dashes), and modified eKdV solution using analytic dispersion (solid). The KdV solution is passing out of the front of the domain and entering in the back due to periodic boundary conditions. $\alpha$ values as in Fig. 14.
modifications of the linear-dispersive terms suggests that similar modifications of the nonlinear-dispersive
terms is worth investigating.

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Council of Canada and the Canadian Department of Fisheries and Oceans.

**APPENDIX A**

**Expressions for S^1(z) and T^i(3) Functions**

and the Integrals \( I_s^i \)

\[
S^{1,0} = \frac{(N^2)}{c^3} \phi^2 \quad T^{1,0} = -\frac{1}{2} \frac{(N^2)}{c^2} \phi^2
\]

\[
S^{0,1} = -\phi \quad T^{0,1} = 0
\]

\[
S^{2,0} = 2 \frac{(N^2)}{c^3} \phi \phi^{1,0} - \frac{1}{2} \frac{(N^2)}{c^4} \phi^3 + 3 r_{10} \frac{(N^2)}{c^4} \phi^2
- r_{10} \frac{N^2}{c^4} \phi \phi' - \frac{8}{3} r_{10} \frac{N^2}{c^5} \phi^{1,0} - 4 r_{10} \frac{N^2}{c^6} \phi
\]

\[
T^{2,0} = -\frac{(N^2)}{c^2} \phi \phi^{1,0} - r_{10} \frac{(N^2)}{c^3} \phi^2 + \frac{1}{2} \frac{(N^2)}{c^4} \phi^3
+ \frac{r_{10}}{3} \frac{N^2}{c^5} \phi \phi' + \frac{4}{3} r_{10} \frac{N^2}{c^6} \phi^{1,0} + \frac{4}{3} r_{10} \frac{N^2}{c^7} \phi
+ \frac{3}{2} r_{10} \phi \phi' + 2 \frac{r_{10}}{c} \phi^{1,0}
\]

\[
S^{1,1} = -6 \phi^{1,0} + \frac{2}{3} \frac{(N^2)}{c^3} \phi \phi^{0,1} + 3 r_{10} \frac{(N^2)}{c^4} \phi^2
- 3 r_{10} \frac{N^2}{c^4} \phi \phi' - 4 r_{10} \frac{N^2}{c^5} \phi^{1,0} - 4 r_{10} \frac{N^2}{c^6} \phi^{0,1}
- 12 r_{10} r_{10} \frac{N^2}{c^7} \phi
\]

\[
T^{1,1} = -\frac{(N^2)}{c^2} \phi \phi^{0,1} - r_{10} \frac{(N^2)}{c^3} \phi^2 + \frac{r_{10}}{c} \frac{N^2}{c^4} \phi
+ 2 \frac{r_{10}}{c} \frac{N^2}{c^5} \phi^{1,0} + 2 \frac{r_{10}}{c^2} \phi^{0,1} + 4 r_{10} r_{10} \frac{N^2}{c^6} \phi
+ \frac{3}{2} r_{10} \phi \phi' + 6 r_{10} \frac{N^2}{c^5} \phi^{1,0}
\]

\[
S^{1,1} = -6 \phi^{1,0} + \frac{2}{3} \frac{(N^2)}{c^3} \phi \phi^{0,1} + 3 r_{10} \frac{(N^2)}{c^4} \phi^2
+ 3 r_{10} \frac{N^2}{c^4} \phi \phi' - 12 r_{10} \frac{N^2}{c^5} \phi^{0,1} - 18 r_{10} r_{10} \frac{N^2}{c^6} \phi
\]

\[
T^{1,1} = -\frac{(N^2)}{c^2} \phi \phi^{0,1} - r_{10} \frac{(N^2)}{c^3} \phi^2 - \frac{r_{10}}{c} \frac{N^2}{c^4} \phi
+ 6 r_{10} \frac{N^2}{c^5} \phi^{0,1} + 6 r_{10} r_{10} \frac{N^2}{c^6} \phi
\]

\[
S^{0,2} = -\phi^{0,1} - 2 r_{01} \frac{N^2}{c^3} \phi^{0,1} - 3 r_{01} \frac{N^2}{c^4} \phi
\]

\[
T^{0,2} = r_{01} \frac{N^2}{c^2} \phi^{0,1} + r_{01} \frac{N^2}{c^3} \phi
\]

The integrals \( I_s^i \):

\[
I = 2 \int_0^1 \phi^{1,0} (z) dz,
\]

\[
I^{1,0} = -\frac{3}{2} \int_0^1 \phi^{0,1} dz, \quad I^{0,1} = -c \int_0^1 \phi^2 dz
\]

\[
I^{2,0} = -\int_0^1 \left\{ 2 \phi^{1,0} \phi^{0,1} + 2 \frac{N^2}{c^2} \phi^2 \frac{\phi^{1,0}}{c} + \frac{2}{c} \phi^{4,0} \right\} dz
\]

\[
+ \frac{8}{3} r_{10} r_{10} \phi^{0,1} + 4 \frac{r_{10}}{c} \phi^{1,0} \right\} dz
\]

\[
I_{a,1}^{1,1} = -2 \int_0^1 \left\{ c \phi^{1,0} + \phi^{2,0} \phi^{0,1} + \frac{N^2}{c^2} \phi^2 \phi^{0,1}
+ 3 r_{10} \phi^{3,0} + 2 r_{01} \phi^{1,0} \phi^{0,1}
+ 2 r_{10} \phi^{0,1} + 6 \frac{r_{10} r_{10}}{c} \phi^{1,0} \right\} dz
\]

\[
I_{b,1}^{1,1} = -2 \int_0^1 \left\{ 3 \phi^{1,0} + \phi^{2,0} \phi^{0,1} + \frac{N^2}{c^2} \phi^2 \phi^{0,1}
+ \frac{3}{2} r_{10} \phi^{3,0} + 9 \frac{r_{10} r_{10}}{c} \phi^{2,0} \right\} dz
\]

\[
I^{0,2} = -\int_0^1 \left\{ c \phi^{0,1} + 2 r_{01} \phi^{1,0} \phi^{0,1} + 3 \frac{r_{01}}{c} \phi^{2,0} \right\} dz
\]

**APPENDIX B**

**Calculating Wave Profiles from the Density Field**

The following iteration procedure is used to determine \( B(x, t) \) from the numerical model results at time
\( t \). First the density perturbation \( b/g \) at some fixed \( z \) is computed as a function of \( x \) at the chosen time. An initial estimate \( B^{(0)} \) for \( B(x, t) \) is obtained by solving the quadratic equation

\[
\epsilon E^{1,0} B^{(0)}^2 + \phi B^{(0)} = \frac{b}{N^2}, \quad (B.1)
\]

for which the appropriate solution, dictated by the requirement that \( B^{(0)} \rightarrow b/N^2 \phi \) as \( \epsilon \rightarrow 0 \), is

\[
B^{(0)} = -\phi + (\phi^2 + 4 \epsilon b E^{1,0}/N^2)^{1/2} \quad (B.2)
\]

We then iterate via

\[
\phi B^{(j+1)} + \mu E^{0,1} B^{(j+1)} + \mu^2 E^{2,0} B^{(j+1)} = S^{(j)} \quad (B.3a)
\]

\[
B^{(j+1)} = \gamma B^{(j)} + (1 - \gamma) B^{(j+1)} \quad (B.3b)
\]

where the relaxation parameter \( \gamma \in (0, 1) \) is typically around 0.3 and
\[ S^{(j)} = \frac{b}{N^2} - \epsilon E^{1.0} B^{(j)^2} - \epsilon^2 E^{2.0} B^{(j)^3} \]

\[ - \epsilon \mu E^{1.1}_a \left( B^{(j)} B^{(j)}_{xx} - \frac{1}{2} B^{(j)^2}_{xx} \right) - \frac{\epsilon \mu}{2} E^{1.1}_b B^{(j)^2} \].

(B.4)

Since \( z \) is fixed, \( \phi, N^2 \), and the \( E^{k,j}_t \) are constants in these equations. Equation (B.3a) is solved using discrete Fourier transforms, which gives

\[ \hat{B}^{(j+1)}_n = \frac{\hat{S}^{(j)}_n}{1 - \mu(2\pi n/L)^2 E^{0.1} + \mu^2 (2\pi n/L)^4 E^{0.2}}, \]

where

\[ \hat{f}_n = \sum_{m=0}^{N-1} f_m e^{i2\pi mn/N} \]

is the discrete Fourier transform with inverse

\[ f_m = f(mL/N) = \frac{1}{N} \sum_{n=-N/2}^{N/2} \hat{f}_n e^{-i2\pi mn/N} \]

and \( L \) is the domain length.

The iteration procedure usually converges rapidly if the denominator on the right side of (B.5) is positive for all \( n \). This condition is always met for sufficiently small \( n \) and is true for all \( n \) provided \( E^{0.1} \leq 0 \) and \( E^{0.2} \geq 0 \) is positive. When \( E^{0.1} \) and \( E^{0.2} \) do not meet these conditions (depending on the values of \( z, \alpha^{0.1}, \) and \( \alpha^{0.2} \)), the iteration procedure generally does not work, although it may if these conditions are close to being met. This procedure may also break down if \( E^{1.0} \) has too large a positive value (see section 6).

REFERENCES


