Continental Shelf Wave Propagation in the Mid-Atlantic Bight: A General Dispersion Relation

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ABSTRACT

The authors address the propagation of continental shelf waves in the Mid-Atlantic Bight. An analytical model of the bathymetry in the region is constructed by representing the continental shelf as a gently sloping bottom, which deepens linearly with offshore distance to the place where it meets the continental slope. Seaward of that point, the bathymetry is modeled with an exponentially decaying function of distance. The linearized, barotropic equations of hydrostatic motion, subject to the long-wave approximation, yield separate shelf and slope solutions, which are matched at the shelf break to specify the eigenfunctions. The associated eigenvalues define the dispersion relations for each of the modes. Wavenumber–frequency pairs derived from NOAA sea surface height stations along the coast are plotted on the first-mode dispersion curve, and the agreement is good. The theory also shows good agreement with the wave data of D. P. Wang.

1. Introduction and background

Low frequency, long period waves have been documented in specific continental shelf and slope regions for 50 years (Hamon 1962). Narrow, sloping shelves provide a waveguide for these subinertial oscillations so that their phases propagate with the coast on their right in the Northern Hemisphere and on the left in the Southern Hemisphere (Huthnance 1975). Suggested production mechanisms for these waves include vorticity generation by the alongshore wind component acting over a variable-depth shelf (Adams and Buchwald 1969), tidal forcing (Baines et al. 2005), and atmospheric pressure fluxes (Noble and Butman 1979), with bathymetric variation providing a restoring force (Flather 1988). These waves affect coastal ecosystems and influence ocean circulation by enhancing upwelling regimes (Middleton and Bye 2007). Measurements in South America show wave-related variances correlated with El Niño (Camayo and Campos 2006). They also contribute energy toward eddy and meander formation in energetic currents, such as the Gulf Stream (Brooks 1978), where the wave–current interaction alters the vorticity gradient of the stream and damps the waves (Brink 1990).

Because of geographic variability in shelf/slope morphology, the speed and period of coastally trapped waves can vary over wide ranges. Smith (1978) detected waves traveling at 200 km day$^{-1}$ off the coast of Peru, similar to the speeds found in the Gulf of California by Enfield and Allen (1983). At the other extreme, Theibaut and Vennell (2010) observed hurricane-generated, coastally trapped waves off Nova Scotia moving far more rapidly at 1382 ± 429 km day$^{-1}$. These waves often have frequencies near $\omega \approx 0.2f$, where $f$ is the local inertial frequency and $\omega$ is the wave frequency (Cutchin and Smith 1973). On the other hand, wavelengths much longer than the shelf width of, say, the mid-Atlantic coast, can have periods of several days to as much as a few weeks (Baines et al. 2005). For example, waves with periods near 10 days have been noted off southern Africa (Schumann and Brink 1990).

Although dispersion relations can be determined numerically for arbitrary bathymetry, it is often useful for modeling purposes to be able to discover a dispersion relation in terms of tabulated transcendental functions.
This also allows us to easily compare wave propagation characteristics on one portion of the shelf to another with the same configuration but different shelf widths, abyssal depths, and other environmental length scales. Several investigators have been successful in doing this in the case of analytical idealizations of the actual depth profiles.

Buchwald and Adams (1968) describe coastally trapped waves propagating over convex-upward exponential bathymetry meant to model the Australian continental shelf. Ball (1967) developed a dispersion relation for continental “edge waves” over an exponentially concave-upward continental slope. He derives equations relating the frequencies and wavenumbers based on the topographic parameters used to model the continental slope. Pedlosky (1987) obtained a dispersion relation for long waves, assuming a shelf width $O(20 \text{ km})$ and wavelengths of several hundred to several thousand kilometers. He assumes a linearly varying slope profile, matched to a vertical continental slope, thereby obtaining an infinite set of modes dependent on the shelf depth/bottom depth ratio. Finally, Baines et al. (2005) conducted laboratory simulations of coastal wave generation and propagation over a sloping power-law depth profile in a stratified ocean and obtained waves within their predicted speed and frequency ranges.

In this paper, we develop a topographically constrained dispersion relation for coastally trapped waves by modeling mid-Atlantic coastal depth variation with both a linearly sloping shelf and an exponential, concave-upward slope bathymetry. As such, this study combines facets of the analytical solutions of Pedlosky (1987) and Ball (1967). These two works, as well as that by Buchwald and Adams (1968), represent exact analytical solutions to the continental shelf wave problem for different shelf shapes. The solution described in the present work adds to this small suite of analytical continental shelf wave solutions and provides a dispersion relation with which to evaluate present and published wave data. Consequently, we compare our results with a decade of recent sea level data from tide gauges along the mid-Atlantic coast and against similar data collected decades earlier by Wang (1979). Section 2 describes the geography of our study area and the wave and bathymetric data used. In section 3, we summarize the derivation of the dispersion relation and compare these results with data from two geographic cases in section 4. In each case, wavenumber and frequency pairs computed from the tide gauge data and the Wang (1979) data are overlaid on our theoretical curves, and we find that our theory is consistent with these historical and present measurements. In addition, we explore the reasons for the good agreement of experiment data and provide an explanation. Section 5 summarizes our findings.

2. Data and experiment setup

a. Sea surface height records

The data is derived from the NOAA sea surface height (SSH) records [SSH records are from the NOAA Tides & Currents website: http://tidesandcurrents.noaa.gov/; verified data records are at an hourly water level interval using the mean lower low water data with units of meters in Greenwich mean time] collected from 1 January 2000 through 28 February 2010. Records were divided into 60-day segments, and the first 1024 (~42 days) were retained to facilitate taking spectra and doing coherence calculations. Time series containing data gaps or possessing individual points with egregiously high or low values were not used. Ultimately, this resulted in the synthesis of a set of 26 high-quality contemporaneous time series for all stations from Eastport, Maine, to Oregon Inlet, North Carolina. In this study, we use only those records from Sandy Hook and Atlantic City, New Jersey; Ocean City, Maryland; and Duck, North Carolina. The locations of these stations are displayed in Fig. 1.

A frequency spectrum for hourly sea level data from the tide gauge at Ocean City is displayed in Fig. 2, revealing broad energetic areas at subinertial frequencies. (The spectra at Sandy Hook and Duck were similar to the Ocean City results, and so are not shown.) Clear peaks for the diurnal, semidiurnal, and other major tidal constituents are visible, with the flatter, high amplitude, low frequency range suggesting other physically important mechanisms, which we attribute primarily to the presence of continental shelf waves.

b. Extraction of wave characteristics

To detect waves long enough to affect widely spaced stations on the U.S. East Coast, we calculate the coherence and phase between Sandy Hook and Duck, which are separated by 500 km. Sandy Hook is south of the sharp bend in the coast at the western end of Long Island, while Duck is near the Cape Hatteras coastal prominence where the Gulf Stream separates from the coast. Between these two stations, the shelf provides a waveguide, which could support long waves. The phase results in Fig. 3a show small phase differences with generally consistent sign from the extremely low frequency band up to 0.4 cycles per day (cpd). This includes the 0.1–0.15-cpd range corresponding to $0.2f$ at these latitudes, which are typical frequencies for coastally trapped waves (Pedlosky 1987; Gill 1982).

The coherence of the phase data (Emery and Thomson 2001; Mied et al. 2010) for the ensemble of 26 time series
is shown in Fig. 3b. High coherence at diurnal and semi-diurnal frequencies validates expectations. More interestingly though, coherence falls off between 0.2 and 0.9 cpd, which includes the 3–5-day periods at which atmospheric forcing also plays a role. At frequencies below 0.2 cpd, coherence increases to ~0.8. Sets of wavelength–frequency pairs were constructed from phase data having coherences greater than 0.5 by extrapolating the phase difference between stations to a full wavelength at that frequency. Wavelengths extracted in this way range from 500 to 2000 km, and are plotted in the results section.

c. Bathymetry in the Mid-Atlantic Bight

The importance of topography on the behavior of coastally trapped waves has been well documented. In our study area, shelf width, latitude, and the ratio of shelf depth to ocean depth vary. Table 1 displays the shelf width \( l \) in kilometers, the shelf/ocean depth ratio \( \gamma \), and the Coriolis parameter \( f \) at Sandy Hook, Duck and Ocean City. Ocean City represents the midpoint of the wave path from Sandy Hook to Duck. From north to south along this path, shelf width decreases by a factor of 3 and Coriolis decreases by 10%. These different environmental parameters are accounted for in the theory developed in the following section.

In their development of dispersion relations for coastally trapped waves, Pedlosky (1987) and Buchwald and Adams (1968) use either vertical continental escarpments or slopes that are “concave downward,” respectively. The linearly deepening continental shelf is typical of the Mid-Atlantic Bight, but the vertical escarpment is not. Similarly, the Brooks (1978) concave-downward bathymetry is appropriate for only a portion of the bathymetry off Cape Fear, North Carolina. Thus, although each of these morphologies describes portions of our study region, neither is valid throughout. Instead, the mid-Atlantic shelf deepens gradually to the point where it meets the continental slope, which is consistently concave upward seaward from Sandy Hook to Duck. In Fig. 4, we show an example of this configuration from 1-minute gridded elevations/bathymetry for the world (ETOPO1) data (Amante and Eakins 2009) extending perpendicular from Ocean City across shelf isobaths to the abyssal depth of 4.5 km. We model this
bathymetry with a linear near-shore shape for \( 0 \leq x < l \) and an exponential curve (the dashed line in the figure) for \( l \leq x < \infty \). Specifically, we may write

\[
H(x) = \gamma H_0 \frac{x}{l}, \quad 0 \leq x < l
\]  \hspace{1cm} (1a)

\[
H(x) = H_0 [1 + (1 - \gamma) e^{-a(x-l)}], \quad l \leq x < \infty.
\]  \hspace{1cm} (1b)

The exponential part of the bathymetry curve extends from the diamond marker in the figure, which also indicates the point used for shelf depth in the calculation of gamma. Qualitative trials of varying exponents resulted in selection of the best-fit exponent, in this case \( a = 0.0078 \text{ km}^{-1} \). Along the study area, the value of \( a \) lies in the range \( 0.006-0.008 \text{ km}^{-1} \). In contrast, the ratio of the shelf and ocean depths \( \gamma \) varies by a factor of 3, as shown in Table 1.

3. Theory

The linearized shallow-water equations governing a barotropic ocean on a rotating earth are

\[
\frac{\partial u}{\partial t} - fu = -g \frac{\partial \zeta}{\partial x},
\]  \hspace{1cm} (2a)

\[
\frac{\partial v}{\partial t} + fu = -g \frac{\partial \zeta}{\partial y},
\]  \hspace{1cm} (2b)

\[
\frac{\partial \zeta}{\partial t} + H \frac{\partial u}{\partial x} + u \frac{\partial H}{\partial x} + H \frac{\partial v}{\partial y} = 0,
\]  \hspace{1cm} (2c)

where the cross-shelf and along-shelf velocities are \((u, v)\) in the \((x, y)\) directions, respectively. The positive \( y \) direction corresponds to flow with the coast on the left. The Coriolis parameter is \( f \), \( \zeta \) is the free surface displacement, and the bathymetry \( H = H(x) \) is assumed to be a function of only the cross-shelf variable.

For continental shelf waves, the scales of the along-shelf length \((L = 2\pi/a)\), cross-shelf length \(l\), cross-shelf velocity \(u\), along-shelf velocity \(v\), and wave frequency \(\omega\) are subject to the long-wave assumptions \( l/L \ll 1 \) and \( \omega f \ll 1 \) (Hoskins 1975; Pedlosky 1987). This is equivalent to simply putting \(\partial u/\partial t = 0\) in (2a). Under this approximation, (2a) and (2b) become

\[
v = \frac{g}{f} \frac{\partial \zeta}{\partial x},
\]  \hspace{1cm} (3a)

\[
u = -\frac{g}{f} \frac{\partial \zeta}{\partial y} - \frac{g}{f^2} \frac{\partial^2 \zeta}{\partial t \partial x}.
\]  \hspace{1cm} (3b)

Using (3a) and (3b) in (2c), we obtain the equation governing continental shelf waves subject to the approximation that the along-shelf motion is geostrophic, but the cross-shelf motion is not,

\[
\frac{\partial}{\partial t} \left( H \frac{\partial^2 \zeta}{\partial x^2} + \frac{dH}{dx} \frac{\partial \zeta}{\partial x} - \frac{f^2}{g} \zeta \right) + \frac{f}{g} \frac{dH}{dx} \frac{\partial \zeta}{\partial y} = 0.
\]  \hspace{1cm} (4)

The assumption of a wave solution of the form

\[
\zeta(x,y,t) = \phi(x) \exp[i(\alpha y + \omega t)]
\]  \hspace{1cm} (5)
yields an equation for \( f(x) \),
\[
H \frac{d^2 \phi}{dx^2} + \frac{dH}{dx} \frac{df}{dx} + \left( \frac{f_0}{\omega} \frac{dH}{dx} - \frac{f^2}{g} \right) \phi = 0. \tag{6}
\]

From section 2, we see that the bathymetry may be approximated for the shelf region as a linear function of the cross-shelf variable (1a) and an exponential function seaward of the shelf break at \( x = l \) in (1b).

a. Solution on the continental shelf (0 ≤ x < l)

For the shelf profile (1a), the governing equation (6) is
\[
x \frac{d^2 \phi}{dx^2} + \frac{df}{dx} + \mu \frac{f}{T} \phi = 0, \tag{7a}
\]
where
\[
\mu = \frac{alf}{\omega} - \frac{f^2 l^2}{\gamma g H_0}. \tag{7b}
\]

The substitution
\[
\eta = \left( \frac{2x}{l} \right)^{1/2} \tag{8}
\]
transforms (7a) into
\[
\eta \frac{d^2 \phi}{d\eta^2} + \frac{df}{d\eta} + 2\mu \eta^2 \phi = 0. \tag{9}
\]
The solution to (9) is expressed as a linear sum of Bessel functions of the first and second kind so that
\[
\phi = AJ_0(2^{1/2} \mu^{1/2} \eta) + BY_0(2^{1/2} \mu^{1/2} \eta). \tag{10}
\]

With (8), the requirement that \( \phi(0) \) be finite allows us to write the solution to (9) as
\[
\phi = AJ_0(2^{1/2} \mu^{1/2} \eta). \tag{10}
\]

b. Solution on the continental slope (l ≤ x < \( \infty \))

For the exponential bathymetry (1b), (6) becomes
\[
H_0 \left[ 1 + (\gamma - 1) e^{-a(x-1)} \right] \frac{d^2 \phi}{dx^2} + H_0 a(1 - \gamma) e^{-a(x-1)} \frac{df}{dx} + \frac{f_0}{\omega} H_0 a(1 - \gamma) e^{-a(x-1)} \frac{f^2}{g} \phi = 0. \tag{11}
\]

Table 1. The Rossby radius at the shelf break \( L_S = (gH_0)^{1/2}/f \), cross-shelf length l, shelf–ocean depth ratio \( \gamma \), bathymetry exponent \( a \), Coriolis parameter \( f \), and first-mode eigenvalue \( \mu_1 \) for the sea height stations used in this study. The phase speed is computed for each station from the slope of (19a).

<table>
<thead>
<tr>
<th>Station</th>
<th>( L_S ) (km)</th>
<th>l (km)</th>
<th>( \gamma )</th>
<th>( a ) (km(^{-1}))</th>
<th>( f ) (s(^{-1}))</th>
<th>( \mu_1 )</th>
<th>Phase speed (km day(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sandy Hook</td>
<td>352</td>
<td>204</td>
<td>0.025</td>
<td>0.007</td>
<td>9.433 × 10(^{-5})</td>
<td>0.9831</td>
<td>1263.7</td>
</tr>
<tr>
<td>Cape May</td>
<td>392</td>
<td>148</td>
<td>0.029</td>
<td>0.007</td>
<td>9.1306 × 10(^{-5})</td>
<td>0.8922</td>
<td>1281.7</td>
</tr>
<tr>
<td>Ocean City</td>
<td>382</td>
<td>138</td>
<td>0.027</td>
<td>0.0078</td>
<td>9.025 × 10(^{-5})</td>
<td>0.8935</td>
<td>1051.1</td>
</tr>
<tr>
<td>Kiptopeke</td>
<td>370</td>
<td>123</td>
<td>0.024</td>
<td>0.0065</td>
<td>8.7909 × 10(^{-5})</td>
<td>0.8709</td>
<td>954.3</td>
</tr>
<tr>
<td>Duck</td>
<td>322</td>
<td>79</td>
<td>0.017</td>
<td>0.008</td>
<td>8.509 × 10(^{-5})</td>
<td>0.8231</td>
<td>661.9</td>
</tr>
<tr>
<td>Avon</td>
<td>249</td>
<td>40</td>
<td>0.010</td>
<td>0.008</td>
<td>8.412 × 10(^{-5})</td>
<td>0.7672</td>
<td>452.38</td>
</tr>
</tbody>
</table>
The change of variable

\[ \phi = \xi^p \Phi(\xi), \]  

with

\[ \xi = (1 - \gamma)e^{-a(x-l)} \]  

and

\[ p = \frac{f}{a(gH_0)^{1/2}}, \]

leads to

\[ \xi(1 - \xi) \frac{d^2 \Phi}{d\xi^2} + [(2p + 1) - 2(p + 1)\xi] \frac{d\Phi}{d\xi} \]

\[ - \left[ p(p + 1) - \frac{f\alpha}{a\omega} \right] \Phi = 0, \]  

which is the hypergeometric equation; it has regular singular points at \( \xi = 0, 1, \) and \( \infty. \) Although solutions to (13) may be written in terms of standard forms, the notation is cumbersome. We choose instead a straightforward Frobenius expansion about the point \( \xi = 0: \)

\[ \Phi(\xi) = \xi^{c} \sum_{j=0}^{\infty} \xi^j, \]  

where Abramowitz and Stegun (1972) list the appropriate values of \( c \) as 0 and \(-2p.\) The two linearly independent solutions of (13) may therefore be written as \( \Phi_1 = \xi^0 \sum_{j=0}^{\infty} A_j \xi^j \) and \( \Phi_2 = \xi^{-2p} \sum_{j=0}^{\infty} B_j \xi^j.\) With (12a,b), the solution to (11) on the continental slope is

\[ \phi(x) = A_0 \xi^p \sum_{j=0}^{\infty} A_j \xi^j + B_0 \xi^{-p} \sum_{j=0}^{\infty} B_j \xi^j. \]  

Examination of both terms in (15) reveals the first is bounded at \( \xi = 0 (x = \infty),\) but the second one is infinite there. The requirement that the solution be zero at infinity requires that we set \( B_0 = 0 \) and write the continental slope solution as

\[ \phi(x) = A_0 [(1 - \gamma)e^{-a(x-l)}]^p \sum_{j=0}^{\infty} A_j ((1 - \gamma)e^{-a(x-l)})^j. \]  

Substitution of (14) into (13) yields the recursion relation for \( A_j: \)

\[ \frac{A_{j+1}}{A_j} = \frac{(j + p + 1)(j + p) - f\alpha/a\omega}{(j + p + 1)^2 - p^2}. \]  

As \( j \) increases to infinity, the recursion ratio (16b) goes to unity, indicating that convergence of the solution can only arise from the factor within the braces in (16a), which reaches a maximum of \((1 - \gamma)\) at \( x = l.\) Therefore, convergence occurs only when \( \gamma > 0,\) which is the only physically realistic case. It is also noteworthy that the expressions for the eigenvalues \( \mu \) determined on the continental shelf (7b), the eigenvalues over the slope (13), and the recursive relation for the slope eigenfunctions (16b) each contain the frequency and wavelength only in the form of the ratio \( \omega/a.\) Thus the group velocity in the along-shelf direction (\( d\omega/dl \)) is a constant dependent only on the environmental parameters, with the consequence that the energy in waves of any length propagates at the same rate down the coast.

c. Dispersion relation and modes

We need to match the shelf and slope solutions at the shelf break \( x = l.\) In the case of no continental shelf, \( \gamma = 0 \) and (16a) is divergent at \( x = 0.\) (Ball 1967). To obtain a well-behaved solution at \( x = 0,\) he requires \( j = n \) in (16b), where \( n = 0, 1, 2, \ldots \) forces the series to truncate. This produces a sequence of polynomials, which are the continental shelf wave modes. A necessary consequence of this is that each mode has its own unique dispersion relation.

The scenario described in the present work is different, however, since the presence of the shelf means \( \gamma \neq 0.\) Convergence is not an issue, and the dispersion relation is established instead by matching the shelf and slope solutions. We require the free surface \( \zeta \) and cross-shelf velocity \( u \) to be continuous there. From (3b), (10),
and (16a), these two continuity requirements yield a system of two homogeneous linear equations for $A$ and $A_0$. The solvability condition is that the determinant vanishes:

$$
\begin{vmatrix}
J_0(2\mu^{1/2}) & -G_1(\gamma) \\
\alpha J_0(2\mu^{1/2}) - \frac{aw^{1/2}}{f} J_1(2\mu^{1/2}) & -\alpha G_1(\gamma) + \frac{aw}{f} G_2(\gamma)
\end{vmatrix} = 0,
$$

where

$$
\begin{align*}
G_1(\gamma) &= \sum_{j=0}^{\infty} \frac{A_j}{A_0} (1 - \gamma)^{p+j} \\
G_2(\gamma) &= \sum_{j=0}^{\infty} \frac{A_j}{A_0} (p + j)(1 - \gamma)^{p+j},
\end{align*}
$$

which leads to the dispersion relation

$$
J_0(2\mu^{1/2})G_2(\gamma) - \frac{\mu^{1/2}}{\alpha l} J_1(2\mu^{1/2})G_1(\gamma) = 0. \quad (17a)
$$

For fixed environmental parameters $a$, $l$, and $\gamma$, this is a transcendental equation in $\mu$. There are an infinite number of eigenvalues $\mu_n$ ($n = 1, 2, 3, \ldots$) for which (10) and (16a) can be matched. Figure 5 depicts the lhs of (18) as a function of $\alpha/\omega$. We solve (18) by iteration using the MATLAB zero-finding functions, and the first-mode results are summarized in Table 1.

4. Results

a. Continental shelf wave eigenmodes

By solving (18) as described above, we arrive at a sequence of eigenvalues, which can be used to calculate modes. Each eigenfunction is composed of two parts: a continental shelf component (10) and a continental slope solution (16a). Figure 6 displays the first four eigenmode shapes of (18) off the coast of Ocean City so that the parameters $a$, $l$, and $\gamma$ correspond to the values at this site. Several interesting facets of the modes are immediately apparent.

First, the Bessel-function behavior of the shelf solution and the hypergeometric character of the slope solution are seen in these curves. The first mode displays a maximum cross-shelf amplitude at the coast, a rapid decrease toward the shelf break point, and an asymptotic decrease toward zero over the slope. Second, with increasing mode number, additional oscillations with decreasing amplitudes are seen over the shelf. In particular, we observe that the $(n + 1)$th mode has one more zero than does the $n$th mode, consistent with general eigenfunction theory (Ince 1956). Third, the behavior of the modal functions in the neighborhood of the coast ($x = 0$) is consistent with the functional form in (10). The near-shore solution (10) behaves as $J_0(2^{1/2}x^{1/2}l^{1/2}) \sim 1 - x/l + O([x/l]^2)$ for $x/l \ll 1$ so that the amplitude decreases linearly seaward from a local maximum at the coast; this near-shore behavior is clearly seen in Figs. 6 and 7.

The relative abruptness of the transition from oscillating shelf solutions to evanescent slope solutions depends on the ratio $\gamma$. Figure 7 displays the amplitude modes for Ocean City with increasing values of theoretical $\gamma$. With a large $\gamma$ (a minimal elevation difference between the ocean floor and the shelf), continental shelf wave oscillations decrease, and equilibrium sea level is not reached until well over the deep ocean.

b. Dispersion relation

The eigenvalues $\mu_n$ include the effects of the topographic parameters as shown in (7b). Rearranging (7b) provides a more compact version of the dispersion relation

$$
\frac{\omega_n}{f} = \frac{f}{\mu_n + f^{2}l^{2}/\gamma gH_0}. \quad (19a)
$$

In the narrow shelf limit where $\mu_n \gg f^{2}l^{2}/\gamma gH_0$, (19a) becomes

$$
\frac{\omega_n}{\alpha} = \frac{f}{\mu_n}, \quad (19b)
$$

indicating that wave speed increases with increasing shelf width as in Pedlosky (1987). Our data falls within
this regime, as the squares of the ratios of the shelf widths to the Rossby radii on the continental shelf are less than the $\mu_n$, which are all $O(1)$ (as listed in Table 1). We display plots of $\omega_n$ versus $\alpha$ for the first-mode ($n = 1$) waves in Figs. 8a,b. The dispersion curves are straight lines with a slope determined by the governing topographic inputs $a$, $l$, $H_0$, and $\gamma$. The slopes of these lines also represent the theoretical average phase speed of the coastal waves propagating between the stations indicated in Fig. 1.

Figure 8a addresses first-mode wave propagation between Sandy Hook and Duck. The Ocean City station is close to midway between these two stations, so we choose it as representative of the propagation environment to specify the parameters used in (19a). We overlay the dispersion line with results from our tidal gauge data analysis (section 2) of the Sandy Hook to Duck time series. Agreement between the measured points and the theory is generally quite good, with most of the points lying nearly equidistant from the line for the entire $\alpha$ range.

Figure 8b deals with propagation between Kiptopeke and Avon. Since Duck lays approximately midway between the two stations, the straight-line dispersion relation uses Duck bathymetric parameters as representative of this region. The data points are those reported by Wang (1979) for his Kiptopeke to Avon series. The slope of the line equates to 610 km day$^{-1}$, consistent with Wang’s approximation of 600 km day$^{-1}$ over this relatively small, varying (widths from 123 to 40 km) region of the shelf.

Figures 8a,b share a common interesting feature: the length of their respective study regions is much smaller than the longest wavelengths observed. In the Sandy Hook to Duck region, the smallest $\alpha = 0.1 \times 10^{-6}$ m$^{-1}$, so the corresponding wavelength is $2\pi/\alpha = 6.3 \times 10^7$ m, which exceeds the distance between the stations ($5 \times 10^7$ m) by a factor of $10^2$. The length scale disparity for the Kiptopeke to Avon propagation and its waves is similar, though not quite as pronounced. There, the smallest wavenumber ($1.2 \times 10^{-6}$ m$^{-1}$) corresponds to waves of length $5.2 \times 10^6$ m, which is 50 times the total

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**Fig. 6.** The cross-shelf eigenmode shapes off the coast of Ocean City, as given in (18), for (a) $n = 1$, (b) $n = 2$, (c) $n = 3$, and (d) $n = 4$. The values of $a$, $l$, and $\gamma$ used in (18) are listed in Table 1. The vertical arrow indicates the edge of the continental shelf at 138 km. Each eigenfunction has been normalized so that the value at $x = 0$ is unity.
distance of 10^5 m. Moreover, the bathymetric parameters vary over the domain encompassing all of the sites. Specifically, Table 1 reveals that $g$ and $l$ change by factors of 3 and 5, respectively. Thus, neither of the propagation scenarios in Figs. 8a,b satisfies our underlying assumption (section 3) of one-dimensional propagation in a uniform coastal waveguide, which may contribute to departures from our dispersion curve. Figure 9, calculated from (19a), illustrates the significant impact of shelf width on the calculation of phase speed in the narrow shelf width environment. In narrow shelves, an increase in width causes a rapid and, fortunately, linear increase in phase speed. The values of $l$ from Sandy Hook to Avon (Table 1) fall in this linear range and are designated in the figure. Thus, a midpoint approximation between two stations in this linear regime yields a reasonable estimate of the wave speed and enables our one-dimensional theory to adequately describe propagation of these mode-1 continental shelf waves.

In the wide shelf limit where $\mu_n \ll f^2l^2/gH_0$, (19a) becomes

$$\frac{\omega_n}{\alpha} = \frac{f^2 \gamma g H_0}{\alpha^2} = \frac{\gamma f g H_0}{f^2} = \frac{\gamma f L^2}{L_D},$$

(19c)

where $L_D$ is the Rossby radius of deformation in the deep ocean and $\gamma f / l = f H_0 / H$ is an effective bathymetric $\beta$ on the shelf.

From (1) and (2), the vorticity equation is $(u_z - u_y) + f(u_x + v_y) = 0$. Substitution of a solution of the form $(u, v, \zeta) \propto e^{(\alpha y + \omega t)}$ into this vorticity equation and (2) and (3) yields the dispersion relation for topographic Rossby waves on the shelf, $\omega = (f \gamma a l) / (\alpha^2 + f^2 g H_0)$. The long-wave limit yields $\omega / \alpha = (\gamma f / l) L^2_D$, identical to (19c). We conclude that continental shelf waves on wide shelves behave as long topographic Rossby waves. This also leads to a phase speed decrease over wider shelves owing to the reduction of this effect. The difference in phase speed behavior over the narrow shelf and wide shelf regimes is clearly evident in Fig. 9.

5. Conclusions

We have developed a theory for continental shelf wave propagation in the Mid-Atlantic Bight that allows for a realistic combined linear-shelf, exponential slope bathymetry. These waves are an important constituent of the shelf dynamics because spectral analysis shows that low frequency waves in this region have energy comparable to the diurnal tide and therefore may play significant roles in the energy and momentum budgets at the continental margin.
This model to more varied examples of this type throughout the study area. Further studies may apply.

varying only slightly in degree of curvature \[\text{exponentially upward-concave bathymetric profile that}\]
greatly exceeds the length of the waveguide established by

2 \sqrt[3]{\text{shelf width} \cdot \text{shelf depth} \cdot \text{ocean depth}. This confirms the concept that the bathymetric change and accompanying vorticity gradient are important sustaining forces for these waves.

The accompanying dispersion relation shows that frequency and wavenumber are dependent upon shelf width, \(e\)-folding distance of the slope bathymetry, abyssal depth, fractional height of the shelf to abyssal depth, and latitude (via the Coriolis parameter). Observed phase speeds agree well with theoretical first-mode dispersion curves, centered in each region for subinertial waves with frequencies from 0.05\(f\) to 0.3\(f\). At frequencies below that range, the wavelength greatly exceeds the length of the waveguide established by the shelf.

In this paper, we have developed a relation for an exponentially upward-concave bathymetric profile that varies only slightly in degree of curvature \([a\text{ in (1b)}]\) throughout the study area. Further studies may apply this model to more varied examples of this type of continental margin topography. Additionally, a direction for meaningful follow-on work may be to incorporate the effects of bottom friction noted by several other studies (providing a more refined phase speed estimate) and to use this model to quantify the amplitude and energy of these continental shelf waves.

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