Water Mass Transformations Driven by Ekman Upwelling and Surface Warming in Subpolar Gyres

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ABSTRACT

The Sverdrup relationship when applied to the Southern Ocean suggests that some isopycnals that are deep in the eastern Pacific will shoal in the Atlantic. Cold waters surfacing in the South Atlantic at midlatitudes would be warmed by the atmosphere. The potential for water mass transformations in this region is studied by applying a three-layer planetary geostrophic model to a wide ocean basin driven by the Ekman upwelling typical of the Southern Ocean surface winds. The model uses a simple physically based parameterization of the entrainment of mass into the surface layer with zonally symmetric atmospheric surface fields to find steady-state subpolar gyre solutions. The solutions are found numerically by specifying suitable boundary conditions and integrating along characteristics. With reasonable parameter settings, transformations of more than 10 Sverdrups (1 Sv = 10^6 m^3 s^-1) of water between layers are obtained. The water mass transformations are sensitive to the strength of the wind stress curl and the width of the basin and relatively insensitive to the parameterization of the surface heat fluxes. On the western side of the basin where the cold waters are near the surface, there is a large region where there is a local balance between the Ekman pumping and the exchange of mass between layers. Simple formulas are derived for the water mass transformation rates in terms of the difference between the maximum and minimum northward Ekman transports integrated across the basin and the depths of the isopycnal layers on the eastern boundary. The relevance of the model to the Southern Ocean and the Atlantic meridional overturning circulation is briefly discussed.

1. Introduction

The meridional overturning circulation (MOC) in the Atlantic Ocean is part of a complex global circulation in which dense water formed by surface cooling in the North Atlantic flows in deep western boundary currents into the Southern Hemisphere before resurfacing south of the Antarctic Circumpolar Current (ACC) in a region of surface warming (Marshall and Speer 2012). Despite intensive study, there are still differing conceptual views of the dynamics and energetics of this “conveyor belt” circulation (Huang 2010, his section 5.4). One reason for this is perhaps the challenging variety of dynamics encountered along the path of the current including hydraulically controlled flow over the sills of the Greenland–Iceland–Scotland Ridge (Pratt and Whitehead 2008); turbulent entrainment downstream of the sills (Legg 2012); intense mesoscale activity and recirculation gyres in the region of the Gulf Stream current (Hurlburt et al. 2008); equatorial dynamics that appear to pose a barrier to cross-equatorial flow (Kawase 1987); and the “open” channel-like nature of the Southern Ocean where the form drag of the time-varying flow on the time-mean flow plays a key role in the zonal momentum balance (Marshall et al. 1993; Gille 1997; Rintoul et al. 2001).

As explained in Wunsch and Ferrari (2004) and Vallis (2006, his section 15.2), consideration of the energy budget shows that it is very unlikely that a consistent dynamical model of the MOC could be developed that is driven solely by surface buoyancy fluxes. Following Cox (1989), various modeling studies have found that the wind stresses in the South Atlantic have an important influence on the Atlantic MOC. Toggweiler and Samuels (1995) emphasized the role of the zonal-mean Ekman transport at the northern edge of the Drake Passage, and Webb and Suginoara (2001, p. 213) concluded that “a large fraction of the water mass conversion, associated with the upwelling branch of the thermohaline circulation, occurs in the surface layers of the Southern Ocean.” Sijp and England (2009) and many other authors (e.g.,
Spooner et al. (2013) have also investigated the dependence of the MOC on the latitude of the maximum wind stress and on the southern annular mode (SAM). Roquet et al. (2011) show that a large fraction of the work done by the surface winds on the ocean occurs in the Southern Ocean at latitudes just to the north of the Drake Passage, and Figs. 1 and 2 in Radko and Marshall (2006) show that the maximum of the zonal wind stress is well to the north of the Drake Passage at most longitudes.

Fućkar and Vallis (2007) have also shown that the strength of the ACC can depend on the production of deep water in the Northern Hemisphere (NH) because of its impact on the meridional density gradient across the ACC and the associated thermal wind shear. Gnanadesikan and Hallberg (2000), Allison et al. (2010), and Munday et al. (2011) explore these interdependencies of the ACC and MOC. Good overviews of the extensive recent work on these interdependencies are given by Drijfhout et al. (2013), Munday et al. (2013), and Radko and Kamenkovich (2011).

Conceptual models for the role of the Southern Ocean in the MOC have been proposed by Samelson (2009), Radko and Kamenkovich (2011), and Nikurashin and Vallis (2011). These papers and much of the literature on the ACC focus on the fact that it circulates “freely” around the entire globe and interpret the circulation using the transformed Eulerian mean (TEM) formalism (Marshall and Radko 2003; Treguier et al. 2007). There are, however, very large zonal variations in the ACC. It is deflected more than 10° north along the Atlantic coast after passing through the Drake Passage and returns southward rather slowly so that the current is north of the northern edge of the Drake Passage at more latitudes than it is south of it.

Consistent with this there are large zonal asymmetries in estimates of the net surface heat flux between about 45° and 60°S, with net surface heating strongest in the Atlantic and Indian Ocean sectors. As noted by Wacongne et al. (2003) and evident in Fig. 5.2 of Josey et al. (2013), this east–west asymmetry appears to be a robust feature of net surface heat flux products. If the subsurface circulation is supposed to be nearly adiabatic, the surface heat fluxes are of great interest (Walin 1982) as they represent the major transformations between water masses, and the subsurface circulations must connect regions of similar surface density and equal and opposite area-integrated heat fluxes. On this basis one might expect meridional circulations to connect the region of heat loss in the high-latitude North Atlantic with this region of heat uptake in the South Atlantic and south Indian Oceans. More generally it is reasonable to suppose that an understanding of the geographical distribution of the net surface heat fluxes is an essential foundation for an understanding of meridional circulations and their associated water mass transformations.

A simple explanation of the zonal variations in the ACC can be given for a three-layer fluid whose lowest layer is at rest (i.e., a reduced-gravity model with two active layers) using the Sverdrup relation (Döös 1994). Consider the change in depth of the isopycnals driven by Ekman upwelling along the east–west section illustrated in Fig. 1. This section is located just to the north of the Drake Passage and spans the entire breadth of the Southern Ocean from the western boundary at the east coast of South America to the basin’s eastern boundary at the west coast of South America. Denoting the densities of the layers by \( \rho_i \) and the heights of the layer interfaces by \( \eta_i \), as in Fig. 1, and defining the reduced gravities of the layer interfaces by

\[
\rho_0 g_i = g (\rho_{i+1} - \rho_i),
\]

in which \( g \) is the gravitational acceleration and \( \rho_0 \) is a constant density, the well-known Sverdrup relation can be written in the form

\[
g_z (\eta_{1E}^2 - \eta_{1W}^2) + g_x (\eta_{2E}^2 - \eta_{2W}^2) = -\frac{2f}{\beta \rho_0} \frac{\partial \tau_x}{\partial y} (x_E - x_W).
\]

In (2), \( \tau_x \) is the surface wind stress in the zonal direction; \( f \) is the Coriolis parameter; \( \beta \) is its north–south gradient; \( x \) and \( y \) are respectively eastward and northward coordinates; and the \( E \) and \( W \) subscripts denote values at

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**Fig. 1.** Vertical section through a model with two active layers. Coordinates \( x \) and \( z \) increase eastward and with height, respectively. Height \( z = 0 \) at the ocean surface, \( h_i \) is the depth of the \( i \)th layer, and \( \eta_i \) is the height of the interface at the base of the \( i \)th layer measured from \( z = 0 \).
the eastern and western sides respectively of the region considered. As explained in more detail in section 2e, the contributions to the time mean of (2) from the eddy fluxes, which are typically parameterized as a diffusive flux of the depths of the isopycnals following Gent et al. (1995), should exactly cancel and so do not appear in (2). Just north of the Drake Passage, \( \partial r_i / \partial y > 0 \), so the rhs of (2) is positive, which implies that the depth at least one of the interfaces on the western side of the basin is less than that on the eastern side. The depths of the layers on the eastern side of the basin that will outcrop on the western side can be found by setting \( \eta_{1w} = \eta_{2w} = 0 \) in (2).

Denoting the latitude of the section by \( \phi \) and Earth's radius by \( R \) and, for simplicity, taking the section to be at 60°S, \( f \beta^{-1} = R \tan \phi \approx -10^7 \text{m} \) and \( x_E - x_W = 2 \pi R \cos \phi \approx 2 \times 10^7 \text{m} \). The maximum time-mean zonal-mean zonal wind stress in the Southern Hemisphere is about 0.15 N m\(^{-2}\), and the half-width of the jet is about \( 1.5 \times 10^6 \text{m} \) (15° of latitude), so \( \partial r_i / \partial y \approx 10^{-7} \text{N m}^{-3} \). Reasonably representative values of the reduced gravities can be obtained by taking all the layers to have a (i.e., the same) salinity of 35 ppt and their temperatures to be \( 10^\circ \), 4°, and \(-2^\circ \). The equation of state then gives \( g_1 \approx 2g_2 \approx 0.008 \text{m s}^{-2} \). Inserting these numbers into (2) one finds that

\[
\eta_E^2 + \frac{1}{2} \eta_E^2 > \frac{2 \times 10^7 \times 10^{-7} \times 2 \times 10^7}{8 \times 10^{-3} \times 10^4} = 5 \times 10^6 \text{m}^2.
\]

(3)

This shows that both an upper layer of 1000-m depth and a second layer whose interface is 2000 m deep will either outcrop or shoal well within the basin.

A proper description of the dynamics of the Southern Ocean would include a representation of the zonal asymmetry just described. It would also allow the wind stress within the Drake Passage to be nonzero and represent the transmission of the surface stress through the water column to the form stress on the bathymetric ridges (Rintoul et al. 2001; Olbers et al. 2007). At present most theories of the ACC do not include both ingredients, either treating the ACC as channel-like (as described above) or basinlike (Stommel 1957; Webb 1993; Hughes 2002). Although Tansley and Marshall (2001) and Nadeau and Straub (2009) present idealized simulations and conceptual models containing both ingredients, in the interests of clarity and to avoid unnecessary complexity, this paper limits itself to the consideration of a simpler problem, namely, that of the circulation driven by strong Ekman upwelling within a very wide basin. The resulting circulations might be termed supergyres following Speich et al. (2007) and Ridgway and Dunn (2007), although their focus was on subtropical gyres rather than the subpolar ones discussed here. This problem can be solved using the planetary geostrophic equations and techniques developed to study the global ocean circulation (Veronis 1978) and the ventilated thermocline and warm pool (Luyten et al. 1983; Luyten and Stommel 1986; De Szoeke 1995; Dewar et al. 2005). The main focus here is on the entrainment of cold water into the surface layer from the layer below, which occurs on the western side of the basin and is driven by a combination of Ekman upwelling and a net heat flux from the atmosphere into the ocean.

Section 2 summarizes the mathematical formulation and the solution techniques employed. Section 3 presents numerical solutions for a three-layer model of a very wide basin driven by the Southern Ocean wind stresses already discussed. Section 4 uses a conceptual idealization of the numerical results to derive simple formulas for the rate of water mass transformations. It also discusses the relevance of the calculations to the flow in a Southern Ocean that contains a circumpolar channel and the Atlantic MOC. Section 5 summarizes and discusses the main simplifications used in the paper and opportunities for further work.

2. Governing equations and methods of solution

a. Governing equations

The planetary geostrophic equations will be taken to govern the motions in the ocean interior (i.e., outside “sidewall” boundary layers). These equations are appropriate for large-scale motions in the ocean with small Rossby number (see, e.g., Vallis 2006, his sections 3.3 and 5.2). For simplicity, as in section 1, they are written in Cartesian coordinates with \( x \) increasing eastward and \( y \) increasing northward (the derivations are easily extended to spherical polar coordinates; see Stommel and Arons 1960; Straub 1996). Figure 1 illustrates the notation that will be used for the case of an ocean with three layers. The quantity \( h_i \) denotes the thickness of the \( i \)th layer and \( \eta_i \) denotes the height of the internal interface at the bottom of the \( i \)th layer measured from the constant geopotential surface \( z = 0 \). The depth of the ocean bathymetry \( H(x, y) \) may vary in regions where the lowest layer is at rest (reduced gravity) but for simplicity is taken to be flat in regions where the lowest layer is directly forced (either by buoyancy or wind forcing). The density, geopotential, and horizontal velocities of the \( i \)th layer are denoted by \( \rho_i, \phi_i, \) and \( u_i = (u_i, v_i) \), respectively. Using the rigid-lid approximation

\[
\eta_0 = 0,
\]

(4)

then the layer thicknesses and heights of the interfaces are related by
\[ h_i = \eta_{i-1} - \eta_i, \quad 1 \leq i \leq N. \]  

(5)

Hydrostatic balance then gives

\[ \phi_{i+1} = \phi_i + g_i \eta_i, \quad 1 \leq i < N, \]  

and the planetary geostrophic approximations to the horizontal momentum and continuity equations are given by

\[ f \mathbf{k} \times \mathbf{u}_i = -\nabla \phi_i + \frac{\tau_i}{\rho_i h_i}, \quad \text{and} \]  

(7)

\[ \frac{\partial h_i}{\partial t} + \nabla \cdot (h_i \mathbf{u}_i) = Q_{i,i+1} - Q_{i-1,i}. \]  

(8)

In (7) and (8), which apply for \( 1 \leq i \leq N \), \( \mathbf{k} \) is the unit vector parallel to the \( z \) axis; \( \tau_i = (\tau_{ix}, \tau_{iy}) \) is the turbulent momentum flux absorbed in the \( i \)th layer (the wind stress on the layer if it is at the surface and sufficiently deep); and \( Q_{i,i+1} \) is the volume flux per unit area from layer \( i + 1 \) to layer \( i \) resulting from the surface buoyancy fluxes across the ocean surface. Transfer of water into or out of additional layers at the top or bottom is not allowed, so

\[ Q_{0,1} = Q_{N,N+1} = 0. \]  

(9)

The wind stresses driving the numerical solutions presented in section 3 have no zonal variation, but the derivations presented in this section are more general, so \( \tau \) should be considered in this section to be a function of \( x \) and \( y \).

b. Parameterization of buoyancy fluxes

Following Marsh (2000) and using a simplified form of the ideas presented in appendix A of Bleck (2002), \( Q_{i,i+1} \) will be specified by

\[ (\rho_{i+1} - \rho_i)Q_{i,i+1} = \frac{\alpha_T Q_{i,i+1}^H}{c_p} - \alpha_S S_i (P_i + R_i - E_i), \]  

(10)

in which \( Q_{i,i+1}^H \) is the net heat flux and \( (P_i + R_i - E_i) \) is the net freshwater flux [the precipitation and river inflow minus the evaporation (meters per second)] into the \( i \)th layer when it is at the surface; \( c_p \) is the specific heat capacity of the water; \( \alpha_T \) and \( \alpha_S \) are the thermal and saline expansion coefficients (\( \alpha_T = -\partial \rho/\partial T, \alpha_S = -\partial \rho/\partial S \)); and \( S_i \) is the salinity content (kilograms per cubic meter) of the \( i \)th layer. As illustrated in Fig. 2, (10) is derived by applying the surface buoyancy fluxes in a way that conserves heat and results in no changes to the density of the layers. For negative buoyancy fluxes (surface cooling or evaporation) the fluxes are applied to the right amount of water in the surface layer to form water with the density of the layer below it, and this water is detrained into that layer. For positive surface buoyancy fluxes the fluxes are applied to the whole of the surface layer, and the right amount of water is detrained into the surface layer for it to retain its original density. In both cases the amount of water that is exchanged (entrained into or detrained from the upper layer) is that required to compensate the density change in the surface layer due to the buoyancy flux. This representation of the buoyancy forcing is consistent with the analysis of Walin (1982). [A factor of \((1 - S/\rho_i)\) in the denominator of the last term on the rhs of (10) has been neglected.]

There is a fundamental asymmetry in the response of a stably stratified fluid to intense surface warming and cooling. Intense surface cooling makes the surface layers convectively unstable and can support water mass exchange between two layers when the interface between them is many hundreds of meters deep. Intense surface warming will result in a stable layer at the surface. Energy input by the surface winds (wind mixing energy) will mix the layer to a few tens of meters (in the roughest conditions), but it is physically unrealistic to expect the winds to drive mass transfers into the upper layer unless it is shallow (e.g., it is shoaling because of Ekman pumping).

Thus, where the heat flux into the ocean is positive, the entrainment flux in (8) and (10) should be taken to decrease markedly with the depth of the upper layer. For simplicity and definiteness it will be taken to decay exponentially with the depth of the upper layer over a length scale \( \lambda_Q \). The surface heat flux will be represented following Haney (1971) as proportional to the difference between the atmosphere and ocean surface temperatures, \( T_A \) and \( T_i \), respectively. Freshwater fluxes will not be considered in this paper and \( Q_{i,i+1}^H \) in (10) will be specified by

\[ Q_{i,i+1}^H = r_Q (T_A - T_i) e^{\eta/\lambda_Q}, \quad T_A > T_i. \]  

(11)

Any surplus surface heat input will result in a very shallow warm layer that will be ignored. For the case of surface cooling, detrainment of water from the mixed layer should not take place until the atmospheric surface temperature is cold enough to form water that is denser than that in the layer below the surface. For the case in which the surface and subsurface layers have the same salinity, the detrainment will be parameterized by

\[ Q_{i,i+1}^H = r_Q (T_A - T_i), \quad T_A < T_i. \]  

(12)

For intermediate atmospheric temperatures there will be no entrainment or detrainment:

\[ Q_{i,i+1}^H = 0, \quad T_i < T_A < T_i. \]  

(13)

A model with only a small number of layers that uses (11)–(13) is likely to have zero values of \( Q_{i,i+1}^H \) over quite
wide regions of the domain; the fluxes that can be obtained using (11)–(13) in a two-layer model are quite limited. This is the main reason why section 3 presents a three-layer model driven by winds rather than a two-layer model. Together (12) and (13) result in a surface flux field that has discontinuities where the $i$th layer is at the surface along lines where $T_A = T_{i+1}$. For some purposes it certainly would be preferable to specify (12) in a different manner to avoid this, but the solutions discussed in this paper do not encounter such discontinuities.

Most authors developing analytical models of flows driven by interfacial fluxes have chosen to represent $Q_{i,i+1}$ as a prescribed constant or as a relaxation toward prescribed layer depths. For example, Luyten and Stommel (1986) and Radko and Kamenkovich (2011) prescribe $Q_{i,i+1}$ as a function of latitude, and Samelson (2011) specifies $Q_{i,i+1}$ as a relaxation of the layer depth toward a specified field, while Pedlosky and Spall (2005) represent it as a similar relaxation and a diffusive flux proportional to $\nabla^2 h_i$. Schloesser et al. (2012) allow the temperature of their upper layer to vary in response to the local atmospheric temperature but also specify entrainment and detrainment fluxes as relaxations toward prescribed layer depths. Although (10)–(13) provide a somewhat crude specification of the surface fluxes and it is clearly desirable that the model results are not sensitive to the choice of the parameters $r_Q$ and $l_Q$, their derivation has the sort of physical basis required to enable explanations of the geographical distribution of net surface heat fluxes to be explored.

c. Wind stresses and the eastern boundary conditions

The driving by the wind stress should not be decayed with depth using a formula similar to (11) because it acts on the fluid through Ekman pumping and rotating fluids are stiff. But when the upper layer is sufficiently shallow, some of the wind stress will penetrate into the layer below it. Again for simplicity and definiteness, if the $i$th layer is at the surface the derivations in this section will take

$$\tau_i = \tau(1 - e^{h_i/\lambda_i}), \quad \tau = \tau_i + \tau_{i+1},$$

where $\tau$ is the surface wind stress. The numerical solutions presented in the next section follow Luyten et al. (1983) in taking all the wind stress to be absorbed in the top layer that has nonzero depth.
It will be assumed that the eastern boundary layers are unable to absorb any normal flow so that no normal flow boundary conditions apply to all the layers. This assumption is commonly made (e.g., Luyten et al. 1983; Johnson and Marshall 2002) and can be justified for multilayer flows with dissipative boundary layers in some cases, but a full justification is beyond the scope of this paper. Denoting the eastern boundary by $x_E(y)$ and the vector parallel to it by $p_E$, and using (7), it follows that

$$\left(\tau_i - \rho_0 h_i \nabla \phi_i\right) \cdot p_E = 0, \quad x = x_E(y). \quad (15)$$

For simplicity it will also be assumed that the wind stress parallel to the boundary is zero (as is the case when the winds are zonal and the boundaries lie due north–south). Using (6) and (15), this implies that the geopotentials and the heights of the interfaces are constant along eastern boundaries

$$\phi_i[x_E(y), y] = \phi^0_i = \phi_{iE}, \quad \eta_i[x_E(y), y] = \eta^0_i = \eta_{iE}. \quad (16)$$

Samelson (2009) emphasizes the critical role of the layer depth along the eastern boundary in his experiments in setting the thermocline structure. Greatbatch and Lu (2003) also obtained almost no variation in the interface depth along the eastern boundary in their experiments (see their Fig. 1). In both of the above papers consistency with the eastern boundary condition (16) was not imposed but arose from the model dynamics that closed the circulations with western rather than eastern boundary currents. As there are no salinity variations in the model considered here, (16) implies that the surface temperature along the eastern boundary is the same at all latitudes. Taking this temperature to be that characteristic of midlatitudes, this implies that the ocean will be cold relative to the atmosphere near the eastern boundary in equatorial regions. Hence, there will be a very large net surface heat flux warming the oceans in this region in this model. Clearly in the “real” ocean contributions to the boundary condition from the time-varying flow (Cessi et al. 2010) as well as salinity variations need to be taken into account, but it is reasonable to suggest that this argument gives a first-order explanation of this important aspect of the geographical distribution of the net surface heat fluxes in the Pacific and Atlantic Oceans.

The western boundary layers for all the layers will be assumed to be able to absorb and recirculate whatever flow is determined from the eastern boundary conditions and the interior equations.

d. Potential vorticity conservation equation

Multiplying (7) by $h_i$ and substituting the resulting expression for $h_i u_i$ into (8), one obtains

$$\frac{\partial h_i}{\partial t} + \int \frac{\partial \phi_i}{\partial y} \frac{\partial h_i}{\partial x} = Q_{i,i+1} - Q_{i-1,i} - C_i, \quad 1 \leq i \leq N. \quad (17)$$

In (17), $J$ denotes the Jacobian derivative, and $C_i$ is an expression for the Ekman pumping in the $i$th layer:

$$J(\phi, h) = \frac{\partial \phi}{\partial x} \frac{\partial h}{\partial y} - \frac{\partial \phi}{\partial y} \frac{\partial h}{\partial x}, \quad \text{and} \quad (18a)$$

$$\rho_0 C_i = \mathbf{V} \cdot \left(\frac{\tau_i}{f}\right). \quad (18b)$$

The total Ekman pumping (summed over the two layers nearest the surface) $C$ will be calculated from $\tau$ as in (18b). Because $C$ depends linearly on $\tau$, $C = C_1 + C_{i+1}$. The quantities $C$ and $C_i$ are singular at the equator, and it transpires that for many purposes their product with $f^2 \beta^{-1}$ is more illuminating, so it is useful to introduce

$$G_i = \frac{\beta^2 C_i}{\beta}, \quad \text{and} \quad (19a)$$

$$\rho_0 G_i = \tau_{xi} + \frac{f}{\beta} \left(\frac{\partial \tau_{yi}}{\partial x} - \frac{\partial \tau_{xi}}{\partial y}\right). \quad (19b)$$

At the equator $\rho_0 G_i$ is equal to the zonal wind stress, while at midlatitudes it is dominated by the second term on the rhs of (19b) that is proportional to the wind stress curl.

In a layer where $\tau_i = 0$, (7) implies that $J(\phi, \cdot) = f(u_i \cdot V \cdot \), so in unforced layers (17) expresses conservation of the planetary geostrophic form of the potential vorticity (the Coriolis parameter divided by the layer depth).

e. Standard solutions for two active layers

Consider a region in which new surface layers are not forming and more specifically the region $R_1$ illustrated in Fig. 3, which is defined to be the region next to the eastern boundary where none of the internal layers have outcropped. Its western boundary $B_1$ lies along the line where one of the two upper layers outcrops: $\eta_{iB_1} = 0$ or $\eta_{iB_1} = \eta_{iB_1}$. As demonstrated by Luyten and Stommel (1986), steady-state solutions of (6) and (17) in this region can be obtained using the method of characteristics. As only the two upper layers are directly forced, the lower layers can be taken to be at rest, and hence

$$\phi_3 = \phi_{3E} = 0, \quad (20)$$

the constant value of $\phi_{3E}$ being taken to be zero for convenience. The steady-state solutions of (6) and (17) for the two active layers must then satisfy
The Sverdrup relation contains the Sverdrup relation integrating w.r.t. $x$ along $B$. The three-layer model. One of the internal layer interfaces outcrops along $B$. Region $R_1$ is bounded to the west by $B_1$ and to the east by $x = x_E$. Region $R_2$ is bounded to the east by $B_1$.

$$\begin{align*}
\phi_2 &= -g_2 \eta_2, \\
\phi_1 &= -g_1 \eta_1 - g_2 \eta_2, \quad (21a) \\
J(\phi_1, -\frac{\eta_1}{f}) &= Q_{1,2} - C_1, \quad \text{and} \\
J(\phi_2, \frac{\eta_1 - \eta_2}{f}) &= -Q_{1,2} - (C - C_1). \quad (22b)
\end{align*}$$

Summing (22a) and (22b), using (19a) and (21), and integrating w.r.t. $x$ from the eastern boundary, one obtains the Sverdrup relation

$$g_1 \eta_1^2 + g_2 \eta_2^2 = 2\Gamma(x, x_E; y) + g_1 \eta_{1E}^2 + g_2 \eta_{2E}^2. \quad (23)$$

in which

$$\Gamma(x_a, x_b; y) = \int_{x_b}^{x_a} G(x', y) \, dx'. \quad (24)$$

Clearly the functional relationship (23) between $\eta_1$ and $\eta_2$ allows $\eta_2$ to be determined once $\eta_1$ is known. Using (16) it also allows the nonlinear advection term $J(\eta_2, \eta_1)$ to be reduced to

$$g_2 \eta_2 J(\eta_2, \eta_1) = J[\Gamma(x, x_E; y), \eta_1]. \quad (25)$$

Multiplying (22a) by $\eta_2$, using (21b) to eliminate $\phi_1$ and then using (23) and (25), one obtains

$$-\beta g_1 \eta_1 (\eta_2 - \eta_1) \frac{\partial \eta_1}{\partial x} + fJ[\Gamma(x, x_E; y), \eta_1]$$

$$= \eta_2 (f^2 Q_{1,2} - \beta G_1) + \beta G \eta_1. \quad (26)$$

Equation (26) is of the form

$$a \frac{\partial \eta_1}{\partial x} + b \frac{\partial \eta_1}{\partial y} = Y_1. \quad (27)$$

The method of characteristics can be used to solve it (Luyten and Stommel 1986) because $a, b,$ and $Y_1$ depend only on $\eta_1$ (i.e., they do not depend on the spatial derivatives of $\eta_1$). When the wind stress is confined to the surface layer (so that $G_1 = G$), the wind stress is independent of $x$, and $x_E$ is independent of $y$:

$$a = -\beta g_1 \eta_1 (\eta_2 - \eta_1) + f(x_E - x) \frac{dG}{dy}, \quad (28)$$

$$b = fG, \quad \text{and}$$

$$Y_1 = f^2 \eta_2 Q_{1,2} - \beta G (\eta_2 - \eta_1). \quad (30)$$

If the wind stress is allowed to penetrate into the second layer and assumed to be given by (14), the method of characteristics could still be applied, but the expressions for the characteristics, particularly the form for $b$, are more complex. This more complex case is not considered further in this paper.

Solutions to (27)–(30) are obtained by integrating along the characteristics’ curves $[x(s), y(s)]$ given by

$$\frac{dx}{ds} = a, \quad \text{and} \quad (31a)$$

$$\frac{dy}{ds} = b. \quad (31b)$$

Along these characteristics, using first (31) then (27), one sees that

$$\frac{d\eta_1}{ds} = \frac{\partial \eta_1}{\partial x} \frac{dx}{ds} + \frac{\partial \eta_1}{\partial y} \frac{dy}{ds} = a \frac{\partial \eta_1}{\partial x} + b \frac{\partial \eta_1}{\partial y} = Y_1. \quad (32)$$

Solutions can be obtained for part of region $R_1$ by integrating (27)–(30) together with (23) westward from the eastern boundary conditions (16).
f. Western boundary conditions

As is well known and confirmed in section 3, the characteristics originating from the eastern boundary do not cover the whole domain. Ideally one would find a solution including the boundary layers as in Huang and Flierl (1987), but following Dewar et al. (2005) underdetermined “boundary” values for the layer depths at the western boundary of the domain will be specified. Two sets of boundary values are used in section 3. In the first set, referred to as shadow zone (SZ) solutions, the depth of the second layer is held constant and the depth of the upper layer is reduced until the Sverdrup relation is satisfied. If the upper layer outcrops before the Sverdrup relation is satisfied, the depth of the next layer is reduced until it is or it too outcrops. In the second set, referred to as western boundary conditions (WBC) solutions are found by integrating (39) along characteristics to find \( \eta_2 \) using (38). 

\[
\phi_{3B_1} = \phi_{3E}, \quad g_2 \eta_{2B_1}^2 = 2\Gamma(x_{B_1}; x_E; y) + g_1 \eta_{1E}^2 + g_2 \eta_{2E}^2. \tag{34}
\]

Within region \( R_3 \) the steady-state solutions of (6) and (17) for the two remaining layers must satisfy

\[
\phi_2 = \phi_3 - g_2 \eta_2, \tag{35}
\]

\[
J\left( \phi_2, \frac{-\eta_2}{f} \right) = Q_{2,3} - C_2, \quad \text{and} \quad (36a)
\]

\[
J\left( \phi_3, \frac{H + \eta_3}{f} \right) = -Q_{2,3} - (C - C_2). \tag{36b}
\]

Summing (36a) and (36b) and assuming that the bathymetry is flat, one obtains the Sverdrup relation in the form:

\[
g_2 \eta_2^2 + 2H \phi_3 = 2\Gamma(x, x_{B_1}; y) + g_2 \eta_{2B_1}^2 + 2H \phi_{3B_1}. \tag{37}
\]

Using (34) within (37) to eliminate \( \eta_{2B_1} \) and \( \phi_{3B_1} \), one obtains

\[
g_2 \eta_2^2 + 2H \phi_3 = 2\Gamma(x, x_E; y) + g_1 \eta_{1E}^2 + g_2 \eta_{2E}^2 + 2H \phi_{3E}. \tag{38}
\]

This is the standard Sverdrup relation; the lhs of (38) omits \( g_1 \eta_1^2 \) only because \( \eta_1 = 0 \). So the Sverdrup relation again provides a functional relationship between \( \eta_2 \) and \( \phi_3 \). Using (35) and (38) in (36a), following the same derivations as for (25) and (26), one obtains

\[
-\beta g_2 \eta_2 (H + \eta_2) \frac{\partial \eta_2}{\partial x} + fJ\left[ \Gamma(x, x_E; y), \eta_2 \right]
= H(f^2 Q_{2,3} - \beta G_2) + \beta G \eta_2. \tag{39}
\]

Note that (39) is isomorphic to (26); formally replacing \( g_1, \eta_1, \) and \( \eta_2 \) in (26) by \( g_2, \eta_2, \) and \( -H \), respectively, one obtains (39). Hence, in region \( R_2 \) solutions are found by integrating (39) along characteristics to find \( \eta_2 \) and diagnosing \( \phi_3 \) using (38).

If layer two rather than layer one outcrops along \( B_1 \), so that \( \eta_1 = \eta_2 \) in region \( R_2 \), then repeating the steps leading to (38) one obtains

\[
(g_1 + g_2) \eta_1^2 + 2H \phi_3 = 2\Gamma(x, x_{E}; y) + g_1 \eta_{1E}^2 + g_2 \eta_{2E}^2 + 2H \phi_{3E}. \tag{40}
\]

and repeating the steps leading to (39) one obtains

\[
-\beta (g_1 + g_2) \eta_1 (H + \eta_1) \frac{\partial \eta_1}{\partial x} + fJ\left[ \Gamma(x, x_E; y), \eta_1 \right]
= H(f^2 Q_{1,3} - \beta G_1) + \beta G \eta_1. \tag{41}
\]

g. Three-layer solutions in outcropping regions

Consider now the solution in region \( R_2 \) in Fig. 3, and consider first the case in which the upper layer that outcrops is the upper layer at \( B_1 \) so that \( \eta_{1B_1} = 0 \). Then, using (20) and (23), one finds that at the boundary between regions \( R_1 \) and \( R_2 \),

\[
\phi_{3B_1} = \phi_{3E}, \quad g_2 \eta_{2B_1}^2 = 2\Gamma(x_{B_1}; x_E; y) + g_1 \eta_{1E}^2 + g_2 \eta_{2E}^2. \tag{34}
\]
Solutions for $\eta_i$ are found by integrating (41) along characteristics and $\phi_s$ is diagnosed using (40).

h. Specification of standard configuration solved numerically

The salinity in all the layers in all the configurations is set to 35 ppt. In the standard configuration, the temperature of the layers is set to $10^\circ$, $4^\circ$, and $-2^\circ$C, and the depths of the layer interfaces at the eastern boundary are set to be 1000, 2000, and 4000 m. The densities of the layers are calculated using a spline fit to the equation of state using the values tabulated in Gill (1982, his appendix 3).

The ocean basin is specified to have due north–south boundaries, and its width is set to be 360° longitude. The geometry is Cartesian with the conversion from longitude $\lambda$ and latitude $\varphi$ (both in radians) to Cartesian coordinates $x$ and $y$, being calculated at 60°S, so that

$$x = \frac{1}{2} R \lambda, \quad y = R \varphi,$$  \hspace{1cm} (42)

and the basin is just over 20 000 km wide. The Coriolis and beta parameters are given by

$$f = 2 \Omega \sin \varphi, \quad \beta = 2 \Omega R^{-1} \cos \varphi.$$  \hspace{1cm} (43)

The atmosphere is specified as having no zonal variation, so $T_A(\varphi)$ and the wind field is taken to be zonal so that $\tau_y = 0$ and $\tau_x(\varphi)$. For simplicity the wind field is given in all configurations, other than configuration J, by

$$\tau_x(\varphi) = \tau_{\text{offset}} + \frac{1}{2} \tau_{\text{range}} \sin^2 \left( \frac{\pi(\varphi - \varphi_{\text{min}})}{\varphi_{\text{max}} - \varphi_{\text{min}}} \right).$$  \hspace{1cm} (44)

and $T_A(\varphi)$ is taken to vary linearly with $\varphi$:

$$T_A(\varphi) = T_{\text{min}}(\varphi_{\text{min}}) + \frac{T_{\text{max}} - T_{\text{min}}}{\varphi_{\text{max}} - \varphi_{\text{min}}} (\varphi - \varphi_{\text{min}}).$$  \hspace{1cm} (45)

To ensure that positive buoyancy fluxes at large depths are not driving the circulations, where $T_A > T_i$ in place of (11), a truncated exponential function is used:

$$Q_i^{H} = r_Q(T_A - T_i) e^{-\eta_i \lambda_Q} - e^{-3} - \eta_i < 3 \lambda_Q,$$

$$= 0 - \eta_i \geq 3 \lambda_Q.$$  \hspace{1cm} (46)

The truncated exponential above is equal to 1 at the surface and decays to zero at $\eta_i = -3 \lambda_Q$. As there are no freshwater fluxes or salinity variations within the model

$$\rho_{i+1} - \rho_i = -\rho_0 c_p (T_{i+1} - T_i)$$  \hspace{1cm} (47)

and (10) simplifies to

$$Q_{i,i+1} = \frac{-Q_i^{H}}{\rho_0 c_p (T_{i+1} - T_i)},$$  \hspace{1cm} (48)

In the standard (std) configuration, $\varphi_{\text{min}} = 65^\circ$, $\varphi_{\text{max}} = 50^\circ$, $\tau_{\text{range}} = 0.15 \text{ N m}^{-2}$, $\tau_{\text{offset}} = 0$, $T_{\text{min}} = 4^\circ$, $T_{\text{max}} = 10^\circ$, $r_Q = 30 \text{ W m}^{-2} \text{ K}^{-1}$, and $\lambda_Q = 30$ m.

Two sets of quantities are used to summarize the results. The first set consists of area integrals of the flux from layer $j$ to layer $i$ defined by

$$I_Q(i,j) = \iint Q_{ij} \, dx \, dy,$$  \hspace{1cm} (49)

where the integral is taken over the area covered by the recirculating gyre. This integral has been calculated by interpolating the values from the characteristics onto a regular grid using the python numpy griddata package.\(^1\)

Before interpolation the coordinates of the points are rescaled by dividing the $x$ coordinate by the width of the domain and the $y$ coordinate by the north–south extent of the domain. This normalization onto the unit square greatly reduces spurious grid-scale noise otherwise introduced by interpolation. The second set consists of north–south integrals of the zonal transport in each layer across a north–south section very close to $0.5^\circ$ longitude from) the western boundary:

$$I_Z(i) = \int h_i u_i \, dy = - \int h_i \frac{\partial \eta_i}{\partial y} \, dy.$$  \hspace{1cm} (50)

The integrals in (50) have been calculated by taking values from the characteristics where they cross the section, sorting them and calculating the second expression using simple centered differencing.

Appendix A describes technical details of the solution methods used and the tests that were made to ensure that the resolution of the numerical calculations presented in the next section is sufficient to give reliable values for $I_Q$ and $I_Z$.

3. Solutions

a. Ekman upwelling of cold water in the standard configuration

Figure 4 shows the paths of the characteristics calculated numerically for the “standard” configuration described in section 2h using (Fig. 4a) the SZ and (Fig. 4b) the UPV conditions on the western boundary. The starting locations of the characteristics are identical for

\(^1\) http://docs.scipy.org/doc/scipy-0.14.0/reference/generated/scipy.interpolate.griddata.html.
the two cases, and varied line types and colors have been used to enable the characteristics to be easily compared. There are some small differences between the two sets of characteristics; for example, near 58°S the characteristics are more broadly spaced near 70°W using SZ conditions and near 130°W using UPV conditions, but it is clear that the characteristics for the two solutions are quite similar. Appendix B provides an explanation of the broad shape of the paths of these characteristics.

Figures 5a and 5b present the depths of the layer interfaces $-\eta_1$ and $-\eta_2$, respectively, obtained using the UPV BCS. The layer interfaces shoal first where $G$ is a maximum at around 58°S between 120° and 150°W. The shallowing of these layer depths in a westward-rising ridge is consistent with the discussion of Fig. 1 in the introduction. Nonzero values of $Q_{1,2}$ can only occur in regions where the layer interfaces are less than 100 m. The contours in Figs. 5a and 5b have been chosen to include depths of 10, 20, and 50 m. Figure 5a shows that the upper layer outcrops when it shallows, while Fig. 5b shows that the second layer has a depth between 20 and 50 m over most of the region where it is shallow. Corresponding maps of the layer interfaces for the SZ BCS (not shown) are qualitatively similar to those in Fig. 5, the main difference being that the region of shoaling for the upper layer is slightly larger than that of Fig. 5a. Figure 5 also displays the fields of (Fig. 5c) $\phi_1$ and (Fig. 5d) $\phi_3$ for the same configuration as Figs. 5a and 5b. The geopotential in the bottom layer is within the numerical truncation error uniform in the region where $-\eta_1 > 90$ m (the contours enclosing the deep red color in Fig. 5b have been set to ±0.001 m² s⁻² to show this). In the region where the upper layers have shoaled, the geopotential field in the lower layer has to be nonzero for the solution to satisfy the

Fig. 4. Paths of characteristics as a function of longitude and latitude in the standard configuration with (a) SZ and (b) UPV BCS.
Sverdrup constraint. From (7) the zonal flow in layer \(i\), \(u_i\), satisfies 
\[ f u_i = -\frac{\partial \phi_i}{\partial y} \]
so the flow at the surface is eastward in the northern half of the region and westward in the southern half. As the variation in \(f\) across the region is small, the flow is almost symmetric about the middle of the region. The zonal wind stress is more westward in the northern half of the region than the southern half so the area integral of the scalar product of the surface wind stress and the surface currents is positive. This implies that the work done by the winds on the circulation is positive. Maps of \(f_i\) for the SZ solutions (not shown) are qualitatively similar to those of Fig. 5.

In both the SZ and UPV standard configurations at all the points where the upper layer shoals, \(T_A < T_1\), so there is no transformation of water from layer two to layer one (i.e., \(Q_{1,2} = 0\)). Configurations in which this is not the case are considered later in this section. Figure 6a presents the flux \(Q_{2,3}\), the rate of transformation of water from layer three to layer two, obtained using the UPV BCS. Again this field for the SZ BCS (not shown) is qualitatively similar. Over most of the region where \(Q_{2,3}\) is nonzero it lies between \(0.8 \times 10^{-8}\) and \(1.2 \times 10^{-8}\) m s\(^{-1}\). The values are much larger than this in a small region in the northern half of the region near the western boundary, indicating that there is a rapid adjustment from the boundary conditions used there. Figure 6b presents \(C - Q_{2,3}\) evaluated only in the region where \(Q_{2,3} > 0\). It is clear that in most of this region the difference between \(C\) and \(Q_{2,3}\) is smaller in magnitude than \(Q_{2,3}\) itself. This point is discussed in detail in the following subsection.

Finally, Fig. 7 presents sections near the western boundary (evaluated at 359.5°). These sections are important because they determine the net flows out of the region in each layer. Figure 7 illustrates the depths of the upper interface \(-\eta_1\) (solid) and lower interface \(-\eta_2\) (dotted) in the case of (Fig. 7a) SZ and (Fig. 7b) UPV BCS. The values between about 50° and 52°S are determined by the boundary conditions and clearly differ appreciably. Where \(-\eta_1\) is decreasing, in the UPV case the layer depth \(h_2\) is approximately constant, while in the SZ case \(\eta_2\) is constant. The transports in Sverdrup for each layer out of the boundary integrated northward from the south are also presented for (Fig. 7c) SZ BCS and (Fig. 7d) UPV BCS. The net transport into each layer through the western boundary \(I_Z\), defined in (50), is given by the transport at the northern edge of the region (at about 49°S). At this latitude layer one is being directly driven by the winds so the integrated transport is sensitive to the latitude at which it is calculated. The integrated transport for layers two and three using the SZ BCS and for layer three using the UPV BCS are almost independent of latitude at the northern edge of the region, but that for layer two using the UPV BCS is sensitive to latitude. Most of the net transports are a small fraction of the full gyre transports at 58°S, and the
fraction is particularly small for layer three using the UPV BCS.

The net transformation rates $I_{Q}(2, 3)$ and zonal transports $I_{Z}(2)$ and $I_{Z}(3)$ obtained for the standard configuration using SZ and UPV BCS are listed in the first row of Table 2. The net transformation rates obtained using the two BCS agree rather closely, both being about 13.5 Sverdrups ($1\text{ Sv} = 10^{6} \text{ m}^3 \text{s}^{-1}$). The net zonal transports into layer three also agree closely, both being about 13.3 Sv. Clearly in both cases, $I_{Q}(2, 3)$ and $I_{Z}(3)$ also agree quite closely. The net zonal transports into layer two do not agree, however, $I_{Z}(2)$ being about 9 Sv in the case of the SZ BCS and 1.2 Sv in the case of the UPV BCS.

b. Sensitivity of Ekman upwelling solutions to parameter choices

Table 1 describes the configurations investigated in the first of two sets of experiments exploring the dependence of the volume transports between layers [(49)] and the integrated zonal transports [(50)] on the surface forcing and other key parameters. Table 2 summarizes the results.

In configurations A and B, the range of the wind stress $\tau_{\text{range}}$, defined in (44), is respectively half and double that in the standard configuration. Figure 8 shows the $Q_{2,3}$ field for these two configurations. Both the area and the magnitude of the field are greater (smaller) in configuration B (A) than in the standard configuration. Consequently, as shown in Table 1, $I_{Q}(2, 3)$ increases more rapidly with the range of the wind stress than it would if they were linearly related.

In configurations C and D the Haney coefficient is respectively half and double that in the standard configuration, while in configurations E and F the depth scale over which the entrainment is confined is respectively half and double the standard value. The transformation rate $I_{Q}(2, 3)$ is increased (decreased) by less than 10% by doubling (halving) the Haney coefficient or the entrainment depth.
A qualitative explanation of the results for configurations A–E and D1–F1 can be obtained by considering the evolution of the depth of the upper layer in the region where the cold waters in the lowest layer are shoaling (i.e., close to the surface) using a combination of additional configurations in which the Haney coefficient was quadrupled (D1) and the entrainment depth scale quartered (E1) and quadrupled (F1) confirm that $I_Q(2, 3)$ is relatively insensitive to the parameterization of the heat flux over quite a wide range of parameter values. An integration with $r_Q = 15 \text{ W m}^{-2}$ was also attempted, but layer two was found to outcrop, and the calculation of $I_Q(2, 3)$ was not completed.

### Table 1. Description of changes made to the standard configuration in each member of the first set of configurations investigated.

<table>
<thead>
<tr>
<th>Letter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Halved wind strength ($r_{\text{range}} = 0.075 \text{ N m}^{-2}$)</td>
</tr>
<tr>
<td>B</td>
<td>Doubled wind strength ($r_{\text{range}} = 0.3 \text{ N m}^{-2}$)</td>
</tr>
<tr>
<td>C</td>
<td>Halved Haney coefficient ($r_Q = 15 \text{ W m}^{-2} \text{ K}^{-1}$)</td>
</tr>
<tr>
<td>D</td>
<td>Doubled Haney coefficient ($r_Q = 60 \text{ W m}^{-2} \text{ K}^{-1}$)</td>
</tr>
<tr>
<td>D1</td>
<td>Quadrupled Haney coefficient ($r_Q = 120 \text{ W m}^{-2} \text{ K}^{-1}$)</td>
</tr>
<tr>
<td>E</td>
<td>Quartered half-depth for entrainment ($\lambda_Q = 7.5 \text{ m}$)</td>
</tr>
<tr>
<td>E1</td>
<td>Halved half-depth for entrainment ($\lambda_Q = 15 \text{ m}$)</td>
</tr>
<tr>
<td>F</td>
<td>Doubled half-depth for entrainment ($\lambda_Q = 60 \text{ m}$)</td>
</tr>
<tr>
<td>F1</td>
<td>Quadrupled half-depth for entrainment ($\lambda_Q = 120 \text{ m}$)</td>
</tr>
<tr>
<td>G</td>
<td>Changed $\eta_{\text{E}}$ to $-1500 \text{ m}$ and $\eta_{2\text{E}}$ to $-3000 \text{ m}$</td>
</tr>
<tr>
<td>H</td>
<td>Changed $\eta_{\text{E}}$ to $-500 \text{ m}$ and $\eta_{2\text{E}}$ to $-1000 \text{ m}$</td>
</tr>
<tr>
<td>I</td>
<td>Used $\sin\pi\phi'$ rather than $\sin^2\pi\phi'$ wind profile</td>
</tr>
<tr>
<td>J</td>
<td>Set $\tau_{\text{offset}} = -0.5 r_{\text{range}}$</td>
</tr>
<tr>
<td>K</td>
<td>Halved width of wind profile ($\phi_{\text{max}} = 50^\circ \text{S}, \phi_{\text{min}} = 57.5^\circ \text{S}$)</td>
</tr>
</tbody>
</table>

### Table 2. Volume transports (Sv) obtained in the set of configurations defined in Table 1. The quantities $I_Q(n, m)$ and $I_Z(n)$ are defined by (49) and (50), respectively.

<table>
<thead>
<tr>
<th></th>
<th>SZ</th>
<th>UPV</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$I_Q(2, 3)$</td>
<td>$I_Q(2)$</td>
</tr>
<tr>
<td>std</td>
<td>13.5</td>
<td>13.3</td>
</tr>
<tr>
<td>A</td>
<td>1.55</td>
<td>1.44</td>
</tr>
<tr>
<td>B</td>
<td>39.9</td>
<td>39.5</td>
</tr>
<tr>
<td>C</td>
<td>12.8</td>
<td>12.6</td>
</tr>
<tr>
<td>D</td>
<td>14.3</td>
<td>13.9</td>
</tr>
<tr>
<td>D1</td>
<td>15.0</td>
<td>14.3</td>
</tr>
<tr>
<td>E</td>
<td>12.9</td>
<td>12.6</td>
</tr>
<tr>
<td>E1</td>
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<td>12.9</td>
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<tr>
<td>F</td>
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<td>14.2</td>
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<tr>
<td>F1</td>
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<td>16.2</td>
</tr>
<tr>
<td>G</td>
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<td>1.03</td>
</tr>
<tr>
<td>H</td>
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<td>24.9</td>
</tr>
<tr>
<td>I</td>
<td>13.5</td>
<td>13.3</td>
</tr>
<tr>
<td>J</td>
<td>12.0</td>
<td>11.7</td>
</tr>
<tr>
<td>K</td>
<td>21.0</td>
<td>20.7</td>
</tr>
</tbody>
</table>
of (30) and (32) reexpressed for layers two and three rather than layers one and two:

\[ \frac{d\eta_2}{ds} = Y_2 = f^2 \eta_3 Q_{2,3} - \beta G(\eta_3 - \eta_2). \]  

(51)

In this region both \( Q_{2,3} \) and \( G \) are positive and \( h_3 \gg h_2 \). Using (19a), (51) then reduces to

\[ \frac{dh_2}{ds} \approx f^2 h_3(Q_{2,3} - C). \]  

(52)

When the first term on the rhs of (52) is larger (smaller) than the second, the layer depth will increase (decrease) and the first term will decrease (increase) as \( s \) increases. Consequently integrating forward along the characteristic, the first and second terms will become equal in value. The depth of the upper layer and the volume flux from the lower layer to the one above it will in this sense adjust and the volume transfer between layers will be controlled by \( C \), the curl of the wind stress. Were the wind stress allowed to penetrate into the second layer the two terms would still come into balance, but the control of the mass transfer by the wind stress curl would be less direct; (26) shows that it would depend on \( G_2 \) as well as \( G \). The remaining weak dependence of the transport \( I_Q(2, 3) \) on the Haney coefficient and entrainment depth arises from areas where \( Y_2 \) is nonzero and the upper-layer depth is adjusting toward its “equilibrium” value.

Configurations \( G \) and \( H \) illustrate respectively the impact of increasing and decreasing the depths of the layer interfaces on the eastern boundaries. The results are qualitatively somewhat similar to those of configurations \( A \) and \( B \). In configuration \( G \), the wind stress field is barely able to raise the cold water to the surface and hence the mass flux between layers is very limited. Note that the increase in \( I_Q(2, 3) \) in configuration \( B \) is much greater than that in \( H \). Consistent with the last
paragraph, this can be qualitatively explained by the fact that both the area and the strength of the mass flux is larger in configuration B, while it is only the area that is larger in configuration H.

Configurations I and J indicate the dependence of the results on the detailed structure of the wind field. In configuration I, the \( \sin^2 \pi \Psi \) wind profile in (44) is replaced by a \( \sin \pi \Psi \) profile \( \Psi = (\phi - \phi_{\text{min}})/\left(\phi_{\text{max}} - \phi_{\text{min}}\right) \). The results are barely distinguishable from the standard ones. In configuration J, \( \tau_{\text{offset}} = -0.5 \tau_{\text{range}} \) instead of its standard value of zero so that the wind stress varies between \( \pm \tau_{\text{range}} \) rather than from 0 to \( \tau_{\text{range}} \). There is a small impact on the results, \( I_Q(2, 3) \) decreasing by just over 10%. The decrease arises because \( G \) depends on \( \tau_x \) as well as its north–south derivative [see (19)] and the decrease in \( \tau_x \) reduces \( G \) by about 10%.

In configuration K, the width of the wind stress profile wind is halved; \( \phi_{\text{max}} - \phi_{\text{min}} \) is halved while \( \phi_{\text{max}} \) is kept unchanged. The north–south extent of the gyre is halved, while \( G \) at corresponding points within the gyre is doubled, and one would expect these two impacts to have small net impact. At the same time the distance between the eastern edge of the shoaling region and the eastern boundary is halved so one would expect \( I_Q(2, 3) \) to increase significantly as indeed it does.

A second set of configurations that has been investigated is summarized in Tables 3 and 4. In these configurations the Ekman upwelling is moved northward, or the line of latitude at which \( T_A = T_1 \) is moved southward, and nonzero transformations of water from layers two and three into layer one can be obtained. Configurations L, M, and N have been designed so that the latitude at which \( T_A = T_1 \) is moved 5°N relative to the wind field in all three cases. To see this, recall that, as described in section 2h, the north–south gradient of \( T_A \) is 0.4°C per degree of latitude. So in configuration M the 2°C increase in \( T_A \) displaces the \( T_A \) profile 5° southward, in configuration L the 2°C decrease in \( T_1 \) also displaces it 5° southward, and in configuration N the wind stress is displaced 5° northward. Configurations O, P, and Q are similar in concept to L, M, and N, respectively, except that the relative displacement is 10° rather than 5°.

The volume transformations between layers in configuration P using shadow zone boundary conditions are illustrated in Fig. 9. In this configuration on some characteristics the second layer shallows so much in the region where \( T_A > T_1 \) that it outcrops internally, and along part of the characteristic there is a direct volume transfer from layer one to layer three. The area integral of this volume transfer for each configuration is listed in column c of Table 4, and the area integrals of the volume transfers between the two other pairs of layers are listed in columns a and b. The zonal transports into layers two and three at the western boundary are listed in columns d and e. Column f presents the total heat flux [petawatts (PW), \( 10^{15} \text{W} \)] calculated using (48):

\[
Q^H_2 + Q^H_3 = \rho_0 c_p \left[(T_1 - T_2)Q_{1,2} + (T_2 - T_3)Q_{2,3}
+ (T_1 - T_3)Q_{1,3}\right].
\]  

The total heat fluxes across the ocean surface in these configurations range from 0.32 to 1.0 PW.

Comparing the results for the std, L, and O configurations, one sees that (as one might have anticipated) there is a progressive increase in the volume transfers from both layers two and three into layer one. This is more marked for the SZ BCS but also true for the UPV

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Table 3. Description of changes made to the standard configuration in each member of the second set of configurations investigated.

<table>
<thead>
<tr>
<th></th>
<th>Reduced temperature of upper layer from 10° to 8°C.</th>
<th>Uniformly increased ( T_A ) by 2°C.</th>
<th>Moved wind stress field 5° northward.</th>
<th>Reduced temperature of upper layer from 10° to 6°C and of midlayer from 4° to 2°C.</th>
<th>Uniformly increased ( T_A ) by 4°C.</th>
<th>Moved wind stress field 10° northward.</th>
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<td>P</td>
<td>Q</td>
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Table 4. Volume transports (Sv) (a) \( I_Q(1, 2) \), (b) \( I_Q(2, 3) \), (c) \( I_Q(1, 3) \), (d) \( I_Z(2) \), (e) \( I_Z(3) \), and (f) the total heat flux (PW) obtained in the set of configurations defined in Table 3.

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BCS. All these points hold also for the std, M, and P configurations and the std, N, and Q configurations. The total heat flux also increases progressively within each of these three sets of configurations for both SZ and UPV BCS, this point being again more marked for the SZ BCS. These points are all consistent with a progressive increase in the fraction of the shoaling area where $T_A$ is greater than the surface water temperature. It is also

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**FIG. 9.** The fluxes ($10^{-6}$ m s$^{-1}$) between layers in configuration P using SZ BCS: (a) $Q_{1,2}$, (b) $Q_{1,3}$, and (c) $Q_{2,3}$.
notable that in each configuration the transfers from layer three and the zonal transports into layer three for SZ BCS are closely similar to those for UPV BCS.

The transports from layer three to one and the total heat flux are also stronger in M than N and in P than Q whether the BCS are of type SZ or UPV. This is due to the reduction in $g$ in configurations N and Q resulting from the reduction in $\frac{f^2}{\beta}$ when the wind stress is moved northward [see (19b)]. The differences between the results for L and M probably arise partly from the reduction in $T_2$ in configuration L that by (48) results in an increase in $Q_{1,2}$ and partly from the reduction in $r_2$ that reduces $g_1$ and makes it easier for the Ekman pumping to raise the denser water to the surface [see (2)]. The final configuration R combines all the changes introduced in the L, M, and N configurations. There is an appreciable increase in the total heat flux and transformation between layers that is particularly pronounced in the SZ solution.

4. Discussion

a. A conceptual model of the numerical results

The results presented in the previous section can be summarized by simple pictures and formulas for the rates of water mass transformations.

Consider first a two-layer model and Fig. 10a, which is based on the figures for the interfacial fluxes obtained for the standard configuration (Fig. 6a) and configurations A and B (Fig. 8). From the Sverdrup relation for the two-layer case, the value of $x_1$ at which the upper layer first shoals because of the Ekman upwelling, $x_1$ is given by

$$g_1 \eta_1 = 2G(x_E - x_1).$$

A simple expression for the rate at which volume is obtained by layer one from layer two in the SZ is then given by

$$I_0 = \int \int Q_{1,2} \, dx \, dy \approx C(x_1 - x_W) \Delta y,$$

where $\Delta y$ is the north–south extent of the outcropping region.

In the Southern Ocean to a good approximation $f \rho_0 C = -\partial \tau / \partial y$, so one can write

$$f \rho_0 C \Delta y = -\gamma \tau_{\text{range}}$$

and expect $\frac{1}{2} \leq \gamma \leq 1$. The value of $\gamma$ obtained in each of the numerical experiments described in the previous subsection has been calculated using the output from the experiments together with

$$\gamma = \frac{-f \rho_0 I_0}{\tau_{\text{range}}(x_1 - x_W)}.$$
boundary, $\gamma = 0.53$. Clearly in general $\gamma$ should be specified as a function of $x_1 - x_w$, but for many purposes it can be taken to be a "constant" between 0.5 and 0.9.

Using (54)–(56) and the relationship between $C$ and $G$ [(19a)] one finds that

$$I_O = C(x_1 - x_E + x_E - x_w)\Delta y$$

$$= -\frac{\beta g_1 \eta_{LE}^2}{2r^2} \Delta y - \frac{\gamma r_{range}(x_E - x_w)}{f\rho_0}.$$  (58)

Equation (58) describes how the rate of transformation of water from layer two into layer one (a volume flux in cubic meters per second) depends on two terms. The first term is proportional to the square of the depth of the interface between the layers on the eastern boundary and the north–south distance between the latitudes at which the winds are a maximum and a minimum. Recalling that $r/(f\rho_0)$ is the northward Ekman flux, one sees that the second term is equal to the difference between the maximum and minimum northward Ekman fluxes integrated across the basin.

A similar picture and formula can be derived for a three-layer model. Figure 10b, which is based on Fig. 9, depicts the outcropping regions and the areas where there are fluxes between layers. Let the water that is obtained by (i.e., transformed into that of) layer one, from either layer two or layer three, be denoted by $I_{O1}$. Then,

$$I_{O1} = \int (Q_{1,2} + Q_{1,3}) dx dy \approx C(x_1 - x_w)\Delta y,$$  (59)

where $x_1$ is the longitude at which the upper layer first shoals and $\delta_1$ depends on the fraction of the region where the upper layer shoals that is equatorward of the latitude at which $T_A = T_1$. If the latitudes at which the wind stress curl is a maximum and $T_A = T_1$ coincide, then an appropriate value for $\delta_1$ would be somewhat less than $1/2$ because the upper layer can be expected to shoal equatorward of the line where $T_A = T_1$. The Sverdrup relation for a shadow zone will be used to provide an expression for $x_1$:

$$2r^2C(x_1 - x_E) = -\beta g_1 \eta_{LE}^2.$$  (60)

This expression should give results that agree well with those obtained using SZ BCS. One could alternatively determine $x_1$ by setting $\eta_1 = 0$ and $\eta_2 = \eta_{LE} - \eta_{LE}$ in (23) to give results more consistent with those obtained using UPV BCS. Clearly (59) and (60) can be combined with (56) to give a relationship corresponding to (58).

Similarly, let the water extracted from layer three in the SZ into either layer one or layer two be denoted by $I_{X3}$. Then,

$$I_{X3} = \int (Q_{1,3} + Q_{2,3}) dx dy \approx C(x_2 - x_w)\Delta y.$$  (61)

Here, it has been assumed that $T_A = T_2$ is poleward of the region in which the second layer shoals. The Sverdrup relation can be used again to obtain an expression for $x_2$:

$$2r^2C(x_2 - x_E) = -\beta g_2 \eta_{LE}^2,$$  (62)

and an expression corresponding to (58) can be derived.

**b. Relevance to the Southern Ocean**

Up to this point the calculations have avoided consideration of the implications of the break in the eastern and western boundaries resulting from the Antarctic Circumpolar Channel. A proper consideration of the interaction between this channel and the gyre circulation discussed in section 3 is well outside the scope of this paper, but a number of relatively simple points are worth making here.

Consider first the case in which there is a channel and the winds are zero across its entirety. Then $G = 0$ at the northern edge of the channel and the gyre circulation previously discussed will hold to the north of the channel (assuming the time-mean effects of the mesoscale motions are either neglected or ignored). In the channel, there are steady-state solutions to the governing equations that have no pressure variations in the bottom layer (i.e., reduced-gravity solutions) and no zonal variation, providing the differences between the temperatures of the near-surface layers and the atmosphere do not result in net surface heat fluxes. A simple solution is obtained by taking the lowest layer to fill the full depth of the fluid south of the channel. The channel then effectively isolates the waters to its south from those to its north. The transport through the channel is determined by the depths of the interfaces at the eastern boundary (Fućar and Vallis 2007) because those are the depths of the interfaces along the entire northern boundary of the channel, and, neglecting the variations in the Coriolis parameter across the channel, using (5)–(6) one sees that

$$-f \sum_{i=1}^{N} h_i u_i = \sum_{i=1}^{N} (-\eta_i + \eta_{i-1}) \frac{\partial \phi_i}{\partial y}$$

$$= \sum_{i=1}^{N} -\eta \frac{\partial \phi_i}{\partial y} + \eta_{i-1} \frac{\partial}{\partial y} (\phi_{i-1} + g_{i-1} \eta_{i-1})$$

$$= -\eta_N \frac{\partial \phi_N}{\partial y} + \sum_{i=1}^{N-1} g_i \eta_i \frac{\partial \eta_i}{\partial y}.$$  (63)
As the bottom layer is at rest, the first term on the rhs of (63) is equal to zero, and (63) implies that

\[ \sum_{i=1}^{N} \int_{y_N}^{y_y} h \frac{\partial h}{\partial y} \, dy = - \sum_{i=1}^{N-1} \int_{y_N}^{y_y} g \frac{\eta_{i+1}}{f} \frac{\partial \eta_i}{\partial y} \, dy \]

\[ \approx - \sum_{i=1}^{N-1} \frac{g \eta_{i+1}}{2f_{\text{mean}}}, \]

where \( f_{\text{mean}} \) is the mean value of the Coriolis parameter within the channel. Using the same values for \( g, g_2, \eta_{1E}, \) and \( \eta_{2E} \) as in the standard configuration and evaluating \( f_{\text{mean}} \) at 60°S gives

\[ \sum_{i=1}^{N} \int_{y_N}^{y_y} h \mu_i \, dy \approx \frac{0.008(1000)^2(1 + 2)}{1.410^{-4}\sqrt{3}} = 8 \times 10^7 \text{ m}^3 \text{ s}^{-1} = 80 \text{ Sv}. \]

Consider next applying the solution techniques described in sections 2 and 3 to a configuration that is identical to that considered so far northward of a certain latitude \( \phi_C \) but has a periodic channel southward of that latitude. The characteristics northward of \( \phi_C \) can be calculated exactly as before. Clearly the western boundary conditions used so far may not be entirely appropriate for this case and the time-mean effects of mesoscale motions can be expected to affect the solution in both the channel and the basin, but one might suppose that this solution north of \( \phi_C \) would be at least qualitatively realistic despite the presence of the channel. Exploration of this possibility is outside the scope of this paper.

Third, Enderton and Marshall (2009) and Ferreira et al. (2010) have presented results for a number of configurations of aquaplanet coupled climate models with meridional ridges extending from the North Pole either to the South Pole or to 35° or 40°S. The surface temperature fields in the full ridge experiments have east–west variations shown in Fig. 9 of Enderton and Marshall (2009) that are consistent with the solutions presented in section 3. The numerical experiments of Tsumino and Sugino-hara (1999) and Klinger et al. (2003, 2004) are also directly relevant to the solutions presented here as they simulate an ocean basin straddling the equator that is cooled at the surface in one hemisphere and subject to Ekman upwelling in the other. It would be interesting to do a more detailed assessment of any of these simulations using the ideas and solutions presented in this paper.

It is also worth reemphasizing the point made in the paragraph following (24) that the Sverdrup relation holds instantaneously for time-varying fields and as a time mean. So within the latitudes of the circumpolar channel it will hold between two points that are not separated by “boundary layer” currents. This means that it is quite plausible that the shoaling discussed in this paper will occur at some latitudes within the channel as well as at latitudes north of it. More detailed study of this hypothesis would be worthwhile but is outside the scope of this paper.

Solutions with a subpolar gyre to the north of Drake Passage and an entirely separate zonal flow within the channel do not resemble the flow in the Southern Ocean. It is reasonable, however, to hypothesize that if the imposed wind stress field were gradually displaced southward into the latitudes of the channel, the westward flow in the southern half of the gyre would also move south and “merge” with the eastward flow in the channel to leave a single circumpolar flow that deflects sharply northward immediately to the east of Drake Passage (cf. Stommel 1957). It is quite likely that mesoscale motions would significantly modify this “transition.”

Finally, it is worth noting that the solutions of section 3 and Fig. 10 capture the east–west asymmetry in the net surface heat fluxes just north of the Drake Passage mentioned in the introduction and provide an explanation of the net surface warming of the ocean in the southern Atlantic and southern Indian Oceans that appears in a number of net surface heat flux products. They also suggest that these regions could be of global importance for transforming large quantities of very dense water into less dense water.

c. Relevance to the Atlantic MOC

This subsection describes briefly, without going into technical details, how the calculations of this paper can be extended to provide a simple, but quantitative, model of the cross-equatorial Atlantic MOC.

Suppose again that the ocean is represented by only two or three layers of constant temperature and that the depths of these layers do not vary along the eastern boundary for the reasons discussed in section 3c. At high northern latitudes the atmosphere is colder than the surface layer and surface heat loss transforms warm water from the upper layers into cold water in the lower layers. A simple parameterization of the deep convection involved in this process is given by (12). In the Southern Hemisphere, north of Drake Passage, this cold water is transformed back into the warmer surface layers as described in the schematic presented at the start of this section. It is assumed that near-surface and deep western boundary currents are able to close the circulation generated by these diabatic fluxes. For steady-state solutions to be obtained, the global net surface heat flux into each layer must be zero, so the water mass...
transformations driven by surface cooling and surface warming must total zero for each layer. For prescribed surface wind stresses and surface temperatures [i.e., $T_A$ in (11)–(13)], it transpires that this determines the depths of the isopycral layers on the eastern boundary. As noted while discussing Fig. 5, the wind stresses in the Southern Ocean do work on the surface circulation so the overall circulation can be consistent with the energetic constraints mentioned in the second paragraph of the introduction.

5. Summary

The initial motivation for this study was to understand the geographical distribution of the principal features of the net surface heat fluxes and through that to develop an understanding of the cross-equatorial MOC. As shown in the introduction, the wind stress curl in the Southern Ocean north of the Drake Passage is strong enough and the ocean basin is wide enough for cold waters lying at depths of 1000–2000 m on the eastern side of the Pacific to be driven to the surface in the Atlantic where the atmosphere is known to be cooled by the ocean. The resulting transformation of cold waters into warmer waters would need to “match” the generation of cold dense waters in the North Atlantic in a steady-state MOC. The presence of the circumpolar channel and the ACC in the region of Ekman upwelling however complicates analysis of the situation. To obtain simple and well-defined solutions illustrating how the Ekman pumping can give rise to water mass transformations, the problem has been simplified in this paper by focusing on the circulation in a wide basin driven by Ekman upwelling.

Starting from this motivation, section 2 writes down the familiar planetary geostrophic equations for $N$ active layers driven by wind and buoyancy forcing [(4)–(10)], with particular attention to the formulation of the buoyancy forcing in (11)–(13). Although this parameterization of the buoyancy formulation is straightforward, it does not appear to have been widely used. It would be useful to apply it to the ventilated thermocline problem as well as the problem discussed here as it could provide a rationale for the choice of location of outcropping isopycnals that is otherwise somewhat arbitrary.

As usual, the planetary geostrophic equations reduce to a single equation [(17)] for the evolution of the potential vorticity in each layer that is to be solved using no normal flow boundary conditions at the eastern boundary [(15)]. When the wind stress is perpendicular to the boundary, this condition implies that the depths of the layer interfaces do not change with latitude along the eastern boundary in steady-state solutions [(16)]. If, as in the model studied here, there are no salinity variations, this implies that the surface temperature along the eastern boundary is the same at all latitudes. Taking this temperature to be that characteristic of mid-latitudes implies that the ocean surface near the eastern boundary will be cold relative to the atmosphere in equatorial regions and warm in polar regions. It would be valuable to investigate further the validity of this simple explanation for the geographical distribution of the net surface heat fluxes and to derive solutions for flows driven by heat loss to the atmosphere in polar regions and by wind stresses and surface heat fluxes near the equator.

Unlike the ventilated thermocline problem for subtropical gyres, in subpolar gyres where the upper layers are losing mass because of the Ekman upwelling (or surface heat loss) the lower layers can remain stagnant and steady-state solutions for three active layer models can be found following characteristics using formulas similar to those found by Veronis (1978) and Luyten and Stommel (1986) for models with two active layers. The derivations for the two-layer case follow Luyten and Stommel (1986) closely, except that they allow the longitude of the eastern boundary $x_E(y)$ to vary with latitude and the wind stress to penetrate into the second layer. They also make more explicit use of an expression for the interaction between the heights of the interfaces in (25) that helps to explain why the method of characteristics can be applied to find solutions and simplifies the derivations for three layers in outcropping regions. The standard configuration and diagnostics used in section 3 are described in section 2h, and appendix A presents details of the numerical methods and of tests that confirm the adequacy of the techniques and resolution used for the standard configuration.

Numerical solutions for this standard configuration are presented in section 3a. Sensitivity tests presented in section 3b show that the fluxes between layers are sensitive to the strength of the wind stress curl but relatively insensitive to the formulation of the buoyancy flux (the Haney coefficient and mixing depth) and other aspects of the wind forcing. These results are interpreted as arising from a “balance” between the Ekman pumping and the flux between the near-surface layers, which holds in much of the region where the upper layers have shoaled. Solutions that have interfacial fluxes between all three layers are also presented. These solutions are obtained by moving the maximum in the Ekman pumping northward or the profile of atmospheric surface temperature $T_A$ southward.

Section 4 discusses some interpretations and applications of these numerical results. Section 4a suggests a
simple representation of the numerical results and derives simple formulas that express the rate of water mass transformation in terms of the difference between the maximum and minimum northward Ekman transports and the depths of the interfaces between the isopycnals on the eastern boundary. As described in the penultimate paragraph of the introduction, a proper description of the dynamics of the Southern Ocean would include a number of complex factors and is outside the scope of this paper. However, section 4b discusses a number of ideas for how the calculations could be made relevant to the Southern Ocean. The simple case when the wind stress in Drake Passage is zero is solved, and it is suggested that the basin solutions may provide “northern boundary conditions” for channel solutions when the wind stress is nonzero within Drake Passage. It is also suggested that the calculations provide an interpretation of the origin and role of the region of ocean surface warming in the southern Atlantic and Indian Oceans. Section 4c takes this point further by outlining how the ideas presented in this paper can be used to provide a simple, but quantitative, model of the Atlantic MOC.

Finally, it may be helpful to summarize and discuss the main assumptions that have been made in this paper and to outline some related opportunities for future work. First, it is somewhat difficult to reconcile the restrictions imposed by using a model with a small number of layers of constant density with the desire to represent surface heat fluxes. The parameterization proposed in (11)–(13) is consequently rather crude and it would be valuable to investigate whether solutions obtained using multilevel ocean circulation models with simple configurations, such as the closed basin discussed in this paper, resemble those found here. Second, only steady-state solutions have been considered. More specifically seasonal variations have been ignored despite the fact that in the NH they play an important role in modifying the ventilation of the thermocline (Williams et al. 1995) and are likely to be important also in the Southern Ocean. The time-mean impact of the ocean mesoscale on the circulations has also been ignored despite the fact that it plays a key role in transferring momentum between layers particularly (but not only) in the Southern Ocean. This has been partially recognized by the use of the uniform potential vorticity western boundary condition for the second layer in some of the numerical solutions, but a natural extension of this paper would be to consider the time-mean solutions of (4)–(10) in which the continuity equation includes a parameterization of the time-mean transports by the time-varying flow as in Radko and Kamenkovich (2011). Finally, it has also been assumed that where the lowest layer of the ocean is directly forced by buoyancy fluxes either these fluxes can be neglected or the ocean bathymetry is flat (so that a simple Sverdrup functional relationship holds). Hautala and Riser (1989), Salmon (1992), and Spall (2001) determine steady solutions with variable bathymetry for cases in which the bathymetry does not give rise to closed, resonantly forced characteristics.

Many of the other assumptions have been made for the sake of transparency rather than to obtain a soluble problem. Solutions could in principle be calculated for an ocean basin with a curving coastline and surface wind stresses and atmospheric surface temperatures that vary with longitude as well as latitude.

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APPENDIX A

Details of Numerical Methods and Sensitivity Tests

For each configuration a number of “initial” locations at the eastern and western boundaries are chosen from which characteristics are calculated. These locations must be adjusted until a good coverage of the whole gyre is obtained. At the western boundary the initial interface depths are determined using the shadow zone or the uniform potential vorticity boundary conditions. If the depth of the upper layer determined from these boundary conditions is zero, it is set to a minimum value $h_{\text{min}} = 0.1$ m so that it is able to deepen appropriately in response to surface heating. The temperatures of the two upper layers and the reduced gravity of the upper interface are then set and the characteristic is followed by integrating equations of the form (31) and (32) using the standard Heun second-order predictor–corrector method. The length of the predictor step is limited so that the fractional change in the upper-layer depth is less than $h_{\text{frac}}$, and the distance integrated is less than $s_{\text{max}}$ m. In all configurations, $h_{\text{frac}} = 0.01$. After each step the lower-layer depth or the geopotential in the bottom layer is diagnosed from the Sverdrup relation using (23), (38), or (40). The integration is stopped when the characteristic starts to leave the model domain or when the upper layer outcrops, that is, its depth becomes smaller than the minimum value $h_{\text{min}}$ or when the second layer...
On the basis of the above tests it was decided to use the numerical choices made in test two in the experiments reported in section 3. From Table A1 it is clear that the calculation of \( I_Q(2, 3) \) is not very sensitive to the choice between the two interpolation methods investigated. The nearest neighbor interpolation is used in section 3 as it gives smaller differences between the standard test and test two.

Qualitatively similar results to those presented in Table A1 were obtained using SZ BCS in place of UPV BCS. The main difference between the results is related to the fact that the values of \( I_Q(2) \) obtained using SZ BCS are much larger than those obtained using UPV BCS and hence are relatively robust to the changes in resolution investigated.

**APPENDIX B**

**An Explanation of the Shape of the Paths of the Characteristics**

The broad shape of the paths of the characteristics in Fig. 4 can be understood by consideration of (27)–(30). The wind stress field is such that \( C \) and \( G \), as defined by (18b) and (19), are both positive within the region covered by the characteristics, which means that there is Ekman upwelling there, and \( G = 0 \) at the northern and southern edges of this region. From (29) and (31b) it follows that along the characteristics \( y \) decreases as \( s \) increases within this region, so in this sense the direction of the characteristics is southward, and at the northern and southern edges of the region \( b = 0 \), so the characteristics are directed east–west (except at points like that on the northern boundary of the region near the eastern boundary where \( a = 0 \)). In the region covered by the characteristics, the atmospheric surface temperature is warmer than the model's second layer, so nonzero heat fluxes \( Q_{1,2} \) are only possible where the upper layer shoals. Along the northern and southern edges of the region, it then transpires from (30) that

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**Table A1. Description of numerical sensitivity tests conducted using the standard configuration.**

<table>
<thead>
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<th>Standard choices</th>
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<td>Halved the number of characteristics (keeping the first and last the same).</td>
</tr>
<tr>
<td>2</td>
<td>Doubled the number of characteristics (keeping the first and last the same).</td>
</tr>
<tr>
<td>3</td>
<td>Doubled the maximum separation of points along each characteristic.</td>
</tr>
<tr>
<td>4</td>
<td>Halved the maximum separation of points along each characteristic.</td>
</tr>
<tr>
<td>5</td>
<td>Halved the number of points on the north–south grid.</td>
</tr>
<tr>
<td>6</td>
<td>Doubled the number of points on the north–south grid.</td>
</tr>
</tbody>
</table>

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**Table A2. Volume transports obtained in the set of numerical sensitivity tests defined in Table A1. The quantities \( I_Q(i, j) \) and \( I_Z(i) \) are defined by (49) and (50), respectively.**

<table>
<thead>
<tr>
<th>UPV (Sv)</th>
<th>( I_Q(2, 3) ) nearest</th>
<th>( I_Q(2, 3) ) linear</th>
<th>( I_Z(2) )</th>
<th>( I_Z(3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>std</td>
<td>12.775</td>
<td>12.744</td>
<td>2.042</td>
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</table>

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outcrops, that is, its depth becomes less than that of the upper layer. If layer one has outcropped the calculation is continued using layers two and three. If the second layer has outcropped, the calculation is continued using layers one and three.

A series of tests was undertaken doubling and halving the resolution of the characteristics and the grids used by the numerical code in order to assess the resolutions required to produce reliable values of \( I_Q \) and \( I_Z \). The values of \( I_Q(2, 3) \), \( I_Z(2) \), and \( I_Z(3) \) obtained in the tests defined in Table A1 using UPV boundary conditions are presented in Table A2. In all the tests the value of \( I_Q(2, 3) \) was calculated using a nearest neighbor interpolation method (column 2) and a linear interpolation method (column 3). In the standard case the number of characteristics uniformly covering the western boundary from 49.4° to 57.6°S was chosen to be 42; the number uniformly covering the eastern boundary from 49.5° to 65°S was chosen to be 32; the maximum distance between points at which values were output was chosen to be 20 km; and the number of points on the north–south grid between 40° and 70°S at which the winds are specified and the \( I_Q(2, 3) \) integral is calculated was set to 200. In tests one and two the number of characteristics on which the solutions were calculated was halved and doubled, respectively. Halving the number of characteristics has an appreciable impact on the integrals, particularly \( I_Z(2) \). The impact of doubling the number of characteristics is less pronounced but still appreciable for \( I_Z(2) \). In tests three and four the maximum distance between points along characteristics \( s_{\text{max}} \) is respectively doubled and halved. The standard value of \( s_{\text{max}} = 10^3 \) m is evidently adequate. The integrals are somewhat sensitive to the halving and doubling of the north–south resolution of the grid explored by tests five and six, but the differences between the standard test and test five are generally smaller than those between the standard test and test two.
\[ Y_1 = 0 \] because both \( Q_{12} = 0 \) and \( G = 0 \). Consequently, \( \eta_1 \) and \( \eta_2 \) have constant values along these northern and southern edges equal to \( \eta_{1E} \) and \( \eta_{2E} \), respectively. The shape and sign of \( a \) can also be determined from consideration of (28). The first term on the rhs of (28), which is proportional to a Rossby wave speed, is always negative. The second term is linearly proportional to the distance from the eastern boundary. In the northern (southern) half of the region \( dG \) is positive (negative), so the second term’s contribution to \( a \) is negative (positive) in the northern (southern) half of the region. The shape of the paths of the characteristics can be understood by combining the above information. For example, the direction of characteristics starting at the western boundary in the upper half of the region is initially southeastward. Moving in this direction along the characteristic toward the center of the region, \( |b| \) increases and \( |a| \) decreases and the characteristic must cross into the region where \( a \) changes sign, while \( b \) is always negative.

REFERENCES


