The Structure of Three-Dimensional Tide-Generating Currents.\textsuperscript{1}  
Part I: Oscillating Currents

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ABSTRACT

A simple method for computing first-order three-dimensional tidal currents is presented. The method involves solving separately the equations for the depth-averaged velocity and the vertical velocity gradient. The interaction between these two equations is through the bottom friction. For the depth-averaged velocity, it was found that the equation of motion could be approximated by the shallow-water equation and thus could be solved easily by a numerical method. The vertical variations of the tidal current, which are functions of the depth-averaged velocity, were computed for various forms of the vertical eddy viscosity and compared to observations.

The dynamics of the tidal current are discussed and explained through the variation in acceleration that results from the frictional and Coriolis forces, and in terms of the interaction between these two forces.

1. Introduction

Tidal current problems in shallow areas have usually been approached by solving depth-averaged equations. In some physical processes there is a need to estimate the three-dimensional tidal current. For example, the computation of sediment fluxes requires that the vertical velocity profile at each location be known because the concentration of the sediment in the water column varies with depth. To simulate these three-dimensional tidal currents in a sea or basin with irregular coastlines, a three-dimensional numerical model can be applied. However, this type of computation is expensive and inefficient, and has poor vertical resolution because of limitations in computer time and storage. In this paper, a method is developed to compute the first-order three-dimensional tidal current; it provides very good vertical resolution and does not use a three-dimensional numerical model. The second-order residual currents will be reported in Part II of this paper.

Sverdrup (1926) was the first to study the vertical distribution of tidal currents in a shallow area; he used a constant eddy viscosity and assumed a constant surface level in the direction normal to the motion of the tidal wave. Fjeldstad (1929) extended Sverdrup's model to include the exponential height function of eddy viscosity. The simplification of the tidal elevation in these studies is generally invalid for an enclosed or semi-enclosed basin because a strong surface gradient in the direction normal to the tidal wave can be set up either by a Coriolis force acting normal to a solid boundary or by waves reflected from irregular coastlines.

Generalization of the Sverdrup problem to the case of a two-dimensional horizontal pressure gradient was given by Sgibneva and Fel'zenbaum (1963), who applied a depth-independent eddy viscosity, and was extended by Kagan (1966), who used an eddy viscosity that increased linearly from the bottom to a given level and then became uniform. In these studies, the vertical distribution of the tidal current was assumed to be a function of the horizontal structure of the surface elevation, which, in Kagan's study, was considered an external parameter. The equation for surface elevation derived by Sgibneva and Fel'zenbaum was difficult to solve because of the complicated boundary conditions. In a more realistic model in which the eddy viscosity varies with depth and depends on the current amplitude, and thus on the surface elevation (Bowden et al., 1959), the equation for the surface elevation becomes very complicated.

Another method used to solve three-dimensional tidal problems is that of the application of the boundary-layer approximation, which assumes absence of friction in the upper portion of the water column (Johns and Dyke, 1971). This approach is valid in deep water, but not in shallow water with a
strong tidal current, and thus cannot be applied generally (see Section 5a).

In this paper, a simple method is developed to obtain the three-dimensional tidal currents in homogeneous (constant density) water. The method involves solving the equation for the vertical distribution of tidal currents as a function of the depth-averaged velocity, which, in turn, can be computed easily from a numerical model. Although the computations are performed using a single tidal constituent, they can be extended to include different constituents because only linear systems are considered.

During the preparation of this manuscript, it was found that a method similar to the one used in this study was employed by Nihoul (1977) to solve for three-dimensional currents generated by tides and storm surges. Nihoul (1977) applied a nonlinear expression for the vertical eddy viscosity and obtained a solution by a series expansion of variables in terms of eigenfunctions of the vertical turbulent diffusivity operator (similar to Heaps, 1972). A limitation of the study is that only some particular solutions, which may not compare well with observations, can be easily computed because the analytical solution of the eigenfunctions can only be obtained for some particular forms of the vertical eddy viscosity. In our study, we consider tidal forcing only and apply the linear forms of the vertical eddy viscosity; the computation becomes fairly straightforward and the tidal currents for various forms of the vertical eddy viscosity can be solved for easily.

The simple model described here cannot be applied in a nonlinear system because of the linearization of the equations of motion. The circulation in a nonlinear system can only be solved through a numerical model. However, it is important to have simple models because they are much more efficient in exploring the dynamics of the current system than the nonlinear three-dimensional models. There are, however, certain assumptions made in the simple model which may require input from numerical models and experimental observations. It is only through examination of both the simple and numerical models that the circulations of the tidal currents can be correctly predicted and the physics involved understood.

One of the assumptions in this study is the linear behavior (time independence) of the vertical eddy viscosity coefficient ($N$). The only observation in homogeneous and shallow water (Bowden et al., 1959) indicates that the vertical eddy viscosity tends to have maximum values about 3 h before and 3 h after high water, and that higher values occur during the flood tide than during the ebb tide. However, this nonlinear behavior of $N$ should not be emphasized because the estimated errors in the values of $N$ are very large ($\pm 50\%$ (Bowden et al., 1959)).

Note that the vertical eddy viscosity is sensitive to the stratification. The effect of the nonlinearity in $N$ on the vertical variation of the tidal current has been investigated by Johns (1976) in a numerical model of a tidal channel. The results indicate that the response of the system is effectively linear because the current profiles of the first-order components are remarkably similar to the profiles obtained from a simple linear Stokes' "shear-wave" solution. In light of the large uncertainty in the experimental values of $N$ and insensitivity of the tidal current to the nonlinear behavior of the vertical eddy viscosity, it is then quite reasonable, in order to perform efficient computations, to assume that the behavior of the vertical eddy viscosity is linear in the first approximation.

The major problem in the three-dimensional tidal computation is uncertainty about the vertical eddy viscosity coefficient. The determination of the exact form of this coefficient, which would require more experimental data, is not the objective of this paper. The main purpose here is to develop, using reasonable approximations and including various forms of the vertical eddy viscosity coefficient, an efficient method for the computation of the three-dimensional tidal current and to understand the dynamics involved. Of course, there exist in the solution some uncertainties associated with the coefficient, but these exist in all three-dimensional and two-dimensional (channel) tidal models, and can only be removed through more observations. However, the effect on tidal current computations of various forms of the vertical eddy viscosity coefficient can be predicted by the model given here. Comparison of the results with current meter observations will aid in the determination of the most appropriate form for the coefficient. A field program is now being undertaken at the head of the Bay of Fundy, and we hope it will provide some contribution toward the determination of the coefficient form and the understanding of the tidal current dynamics.

In Section 2, the method of computation is described. The analytical and numerical solutions for various forms of the vertical eddy viscosity are given in Section 3, and a numerical evaluation and a comparison to the observation are presented in Section 4. A discussion of tidal dynamics, boundary layer and shallow water approximations is found in Section 5.

2. Method of computation

For a Cartesian system with coordinates $x$, $y$ and $z$, where $x$ and $y$ are the horizontal coordinates and $z$ the vertical coordinate measured upward from the mean sea level, the equations describing the tidal motion in the homogeneous basin are
\[
\begin{align*}
\frac{\partial u}{\partial t} + u \cdot \nabla u + w \frac{\partial u}{\partial z} + f \times u &= -g \nabla \zeta + N \frac{\partial u}{\partial z} + A_h \nabla^2 u \\
\frac{\partial \zeta}{\partial t} + \nabla \cdot \int_{-D}^{z} uz \, dz &= 0
\end{align*}
\]

where \( u \) is the horizontal velocity vector, with components \( u \) and \( v \) in coordinates \( x \) and \( y \); \( w \) is the vertical velocity component; \( D \) is the depth of the bottom below the mean sea level; \( \zeta \) is the height of the water surface above the mean sea level; \( f \) is the Coriolis parameter; \( N = N(x, y, z) \) is a vertical eddy viscosity coefficient; \( A_h \) is a horizontal eddy viscosity coefficient; \( g \) is gravity; \( \nabla \) is the horizontal gradient operator; and \( \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 \) is the horizontal Laplacian operator.

In solving the depth-mean velocity equations in the two-dimensional \( x, y \) models, the diffusion effect along the coastal boundary is neglected in many studies (i.e., Ramming, 1972; Hunter, 1972; Nihoul and Ronday, 1975). It has been shown by Tee (1976, 1977) that this neglect was physically improper for an area with a strong tidal current and an irregular coastline.

To appreciate the significance of the horizontal diffusion term in three-dimensional tidal computations, we examine the equation of motion at a plane coast perpendicular to the \( x \) axis. At the coast the normal velocity \( u \) and its derivatives in the \( y \) and \( z \) directions vanish, and the \( x \) component of the momentum equation in (1) becomes

\(-f v = -g \frac{\partial \zeta}{\partial x} + A_h \frac{\partial^2 u}{\partial x^2}.
\]

If the horizontal diffusion term is neglected, we can apply the free-slip boundary condition, and the \( v \) component generally will not be equal to zero. As the surface elevation \( \zeta \) is \( z \)-independent, we can see from the above equation that \( v \) is uniform throughout the water column. Since \( v \) is equal to zero at the bottom because of the vertical diffusion terms and thus the formation of the bottom boundary layer, the only solution for \( v \) is zero at all depths. This result implies a change of our boundary condition from free-slip to non-slip. Thus, in a three-dimensional tidal problem, only the non-slip boundary condition can be applied at the solid boundary. A solution for applying this boundary condition is to include the horizontal diffusion term so that a coastal boundary layer can be introduced. The diffusion term, of course, can be neglected in the offshore area of the shallow basin where vertical friction is usually more important.

In this study, it is assumed that \( \zeta \ll d \); thus the equations for the first-order oscillating component can be obtained from (1) as

\[
\begin{align*}
\frac{\partial u}{\partial t} + f \times u &= -g \nabla \zeta + N \frac{\partial u}{\partial z} + A_h \nabla^2 u \\
\frac{\partial \zeta}{\partial t} + \nabla \cdot \int_{-D}^{z} uz \, dz &= 0
\end{align*}
\]

The following notation is used in this paper:

1) A depth-mean tidal current is denoted by a capital letter, e.g.,

\[
U = \frac{1}{D} \int_{-D}^{0} udz.
\]

2) A prime denotes a derivative with respect to \( z \), e.g., \( u' = \frac{\partial u}{\partial z} \).

3) A caret such as in \( \hat{u} \) denotes a complex function in the transform

\[ u = \text{Re}(\hat{u} e^{i\sigma t}) \]

where \( \sigma \) is a tidal frequency.

4) A zero denotes an amplitude, and \( g \) with subscript \( u, v \) or \( \zeta \) denotes a phase lag of \( u, v \) or \( \zeta \), e.g.,

\[ u = \text{Re}[u_0 e^{i(\omega t - \phi_u)}]. \]

Eq. (2) can be separated into the depth-averaged and depth-differential components

\[
\begin{align*}
\frac{\partial U}{\partial t} + f \times U &= -g \nabla \zeta + \frac{\tau_s - \tau_b}{\rho D} + A_h \nabla^2 U \\
\frac{\partial \zeta}{\partial t} + \nabla \cdot DU &= 0
\end{align*}
\]

and

\[
\frac{\partial u'}{\partial t} + f \times u' = \frac{\partial^2}{\partial z^2} Nu' + A_h \nabla^2 u',
\]

where \( \rho \) is density, \( \tau_s \) the surface stress and \( \tau_b \) the bottom stress. Since only tidal forcing is considered, the surface stress \( \tau_s \) is equal to zero. For the bottom stress, the quadratic form has usually been used in shallow-water computations:

\[
\tau_b = \rho \gamma |U| U,
\]

where \( \gamma \) is the bottom friction coefficient. This stress is approximated by the linearized form

\[
\tau_b = \rho \lambda DU,
\]

where \( \lambda \) is a linear but variable friction coefficient proportional to the current amplitude \( U_m \) and inversely proportional to the depth \( D \), i.e.,
\[
\lambda(x,y) = K \frac{\nu U_m}{D},
\]

(7)

where \( K \) is a constant. The exact linearization (by expanding (5) into a Fourier series of \( \sin(n \pi t) \) and \( \cos(n \pi t) \), where \( n = 0, 1, 2, \ldots \)) of (5) generally produces a much more complex formulation than that given in (6) and cannot be applied in simple models. In the special cases of one-dimensional and two-dimensional models with the semi-major axis equal to the semi-minor axis, Eq. (6) is equal to the exact linearized form. In any case, the bottom stress given in (5) is only an empirical formula and is derived mainly from one-dimensional cases (Dronkers, 1964) where our linearization is valid. Also, our linearization (6) accounts for the fact that the frictional effect becomes more important in areas of stronger tidal currents and/or shallower water. Thus, Eq. (6) gives a reasonable approximation to the bottom stress. For \( V_0 \neq 0, K \) is equal to \( 8/3 \pi \) (Dronkers, 1964); \( K \) is equal to 1 if the tidal ellipse becomes a circle.

In this study, for a given \( N, \tau_b \) is part of the solution of (4). In order for (3) and (4) to be consistent, the computed \( \tau_b \) should be applied in (3). By varying the vertical eddy viscosity, these computed stresses can be adjusted to approximate the empirical values [Eq. (6)]. Although there also exist some uncertainties in the empirical formulation of the bottom stress [Eqs. (5) and (6)], they are considered less serious than those regarding the vertical eddy viscosity coefficient.

The empirical bottom stress formula can also be written in terms of the bottom current, i.e., \( \tau_b = \rho g U_b U_b \), where \( U_b \) is the bottom current, defined as the current at the upper limit of the vertical frictional layer, and \( \rho g \) is another frictional coefficient which is larger than \( \gamma \) by a factor of \( (U_0/U_{b0})^2 \) \((U_{b0} \) is the amplitude of \( U_b \)). Since, as will be seen later, our computed values of \( U_0/U_{b0} \) for the realistic forms of the vertical eddy viscosity are close to the observed ratio of 0.77 (Bowden et al., 1959), the magnitudes of the bottom stress (6) are approximately the same whether they are expressed in terms of \( U \) or \( U_0 \). Only the magnitudes of the bottom stress in the empirical formula are important in the adjustment of the vertical eddy viscosity, the vertical variation of the tidal current is not affected significantly by the different forms of \( \tau_b \). Note that the magnitude of the bottom stress can vary a little bit for different forms of \( \tau_b \) if the vertical variation of the phases are taken into account. However, this slight difference is not considered important because of the uncertainty in the values of \( \gamma \) (Bowden et al., 1959).

Eq. (3) with the bottom stress \( \tau_b \) proportional to \( U \) is a shallow-water equation which governs the motion of the depth-mean current. It can be solved easily through either an analytical or a numerical method. To obtain the depth-averaged current in this study, we solve (3) through the approximation to the shallow-water equation. Section 5b will illustrate the validity of this approximation for the realistic forms of the vertical eddy viscosity.

To compute the vertical profiles of the tidal currents, we first consider the area outside the coastal boundary layer where the horizontal mixing terms can be neglected; Eqs. (3) and (4) then become

\[
\begin{align*}
\left\{ i\sigma \dot{U} - f\dot{V} &= -g \frac{\partial \zeta}{\partial x} - \hat{\tau}_{bx}(D\rho) \\
i\sigma \dot{V} + f\dot{U} &= -g \frac{\partial \zeta}{\partial y} - \hat{\tau}_{by}(D\rho) \\
i\sigma \dot{\xi} + \frac{\partial}{\partial x} D\dot{U} + \frac{\partial}{\partial y} D\dot{V} &= 0
\end{align*}
\]

(8)

and

\[
\begin{align*}
i\sigma \dot{u}' - f\dot{v}' &= \frac{\partial^2}{\partial z^2} N u' \\
i\sigma \dot{v}' + f\dot{u}' &= \frac{\partial^2}{\partial z^2} N v'
\end{align*}
\]

(9)

where \( \hat{\tau}_{bx} \) and \( \hat{\tau}_{by} \) are the components of \( \tau_b \) in the \( x \) and \( y \) directions.

Let \( \dot{L}_+ = \dot{u}' + i\dot{v}' \) and \( \dot{L}_- = \dot{u}' - i\dot{v}' \), and introduce a nondimensional variable \( \eta = z/D \); then (9) can be separated into

\[
iD^2(\sigma + f)\dot{L}_+ = \frac{\partial^2}{\partial \eta^2} N \dot{L}_+
\]

(10)

and

\[
iD^2(\sigma - f)\dot{L}_- = \frac{\partial^2}{\partial \eta^2} N \dot{L}_-
\]

(11)

The boundary conditions for \( \dot{L}_+ \) and \( \dot{L}_- \) are (i) at the surface, \( \eta = 0, \dot{u}' = \dot{v}' = 0 \), and thus \( \dot{L}_+ = \dot{L}_- = 0 \); and (ii) at the bottom, \( \eta = -1, \dot{u}' = \dot{v}' = 0 \), and thus \( \dot{L}_+ = \dot{L}_- = 0 \). From these two boundary conditions and the definition

\[
\dot{U} = \int_{-1}^{0} u d\eta,
\]

\( \dot{L}_+ \) and \( \dot{L}_- \) can be computed from (10) and (11) as a function of \( U \). A detailed description of these computations for various forms of \( N \) is given in Section 3.

After obtaining \( \dot{L}_+ \) and \( \dot{L}_- \), \( \dot{u}' \) and \( \dot{v}' \) can be computed from

\[
\begin{align*}
\dot{u}' &= \frac{1}{2} (\dot{L}_+ + \dot{L}_-) \\
\dot{v}' &= \frac{1}{2i} (\dot{L}_+ - \dot{L}_-)
\end{align*}
\]

(12)
and \( \hat{u} \) from
\[
\hat{u} = D \int_{-1}^{\eta} \hat{u}'(\eta^*) d\eta^*. \tag{13}
\]

Using the continuity equation
\[
\frac{\partial \hat{w}}{\partial z} + \nabla \cdot \hat{u} = 0 \tag{14}
\]

and \( \hat{w} = 0 \) at \( \eta = -1 \), we obtain the vertical velocity from
\[
\hat{w} = D \int_{-1}^{\eta} [\nabla \cdot \hat{u}(\eta^*)] d\eta^*. \tag{15}
\]

To obtain a final solution for the closed or semi-closed basin, the tidal current in the coastal boundary layer must be investigated. As there is no simple method of making the computation in this layer where horizontal mixing terms cannot be ignored, a three-dimensional numerical model with very fine grid spacing near the coast may be necessary for solution. In this paper, the velocity structure in the coastal boundary layer is not studied. However, in the numerical evaluation of the depth-mean velocity from (3), the inclusion of the horizontal mixing terms implies a nonslip boundary condition at the coast and thus generates the correct vorticity values which, as indicated by Tee (1976, 1977), cannot be neglected in areas of strong tidal currents and irregular coastlines. Although the tidal current in the coastal boundary layer is not solved, the three-dimensional structures of the tidal currents in the offshore area can still be predicted because the depth-average velocity \( \mathbf{U} \), on which \( \mathbf{u} \) depends, can be correctly computed from (3).

In summary, we first compute the depth-mean tidal current \( \mathbf{U} \) from (3) through the approximation to the shallow water equation. The three-dimensional tidal currents \( \mathbf{u} \) and \( \mathbf{w} \) in the offshore area can then be obtained from (13) and (15) as functions of \( \mathbf{U} \).

In the sections that follow, it is assumed that the depth-mean velocity \( \mathbf{U} \) is either estimated from observation or known from the solution of the depth-average equation (3). The vertical variation of the tidal current for different dissipative systems is then investigated. The observed data can be used either to parameterize or verify (if \( N \) is known) the models.

3. Solutions for various forms of eddy viscosity

The vertical eddy coefficient \( N \) is probably the most uncertain parameter in the tidal computation. The only investigation in homogeneous water (Bowden and Fairbairn, 1952; Bowden et al., 1959) indicated that \( N \) has a maximum \( (N_m) \) near mid-depth and decreases toward the surface and bottom. \( N_m \) was parameterized as (Bowden, 1967)
\[
N_m = 2.5 \times 10^{-3} U_m D, \tag{16}
\]

where \( U_m \) is the amplitude of the depth-mean tidal current. Stratification can reduce this \( N_m \) to a very small value. In this study, the following forms of \( N \) are used:

(i) \( N \) is independent of \( z \).
(ii) \( N \) varies parabolically with depth, e.g.,
\[
N = N_m \{ R_1 + 4(R_1 - 1)\eta + 4(R_1 - 1)\eta^2 \}, \tag{17}
\]

where \( N_m \) is the maximum vertical eddy viscosity at \( \eta = -\frac{1}{2} \), and \( R_1 \) is the ratio of \( N \) at the surface or bottom to \( N_m \).
(iii) \( N \) increases rapidly from the bottom through a thin laminar sublayer to a uniform eddy coefficient \( N_m \) in the turbulent layer, i.e.,
\[
N = \left[ \frac{\nu_0 [1 + R_2 D(\eta + 1)]^2}{N_m = \nu_0 (1 + R_2 \delta_z)^2} \right], \quad \eta \leq \eta_z \tag{18}
\]

where \( \eta_z = (\delta_z/D) - 1, \nu_0 = 1.4 \times 10^{-6} \text{ m}^2 \text{ s}^{-1} \) is the molecular eddy viscosity, and \( R_2 \) and \( \delta_z \) (or \( N_m \) and \( \delta_z \)) are parameters that will be adjusted to obtain a best fit to the experimental data.
(iv) \( N \) has the combined characteristics of (ii) and (iii); that is, for \( \eta \approx \eta_z, N \) increases rapidly according to (18), and above that it increases to \( N_m \) at \( \eta = \frac{1}{2} \eta_z \) according to (17).

A summary of these models is given in Fig. 1. Note that \( N_m \) is denoted as the maximum \( N \) in the water column. In case (i) where \( N \) is independent of \( z \), then \( N = N_m \) at all depths.

Case (i) is studied in Section 3a where a simple analytical solution is obtained. Case (ii) has been applied in a narrow channel by several authors (i.e., Johns, 1969; McGregor, 1972; Ianniello, 1977), who found that a realistic velocity profile could be obtained if \( N_m \) was two or three orders of magnitude larger than \( N \) at the bottom. In the following computation, examples will be given for \( R_1 = 0.1, 0.01 \) and 0.001. Case (iii) was developed by Johns (1970) who argued that (iii) is more reasonable than (ii) because, in case (ii), the value of \( N \) at the bottom is several orders of magnitude larger than the expected molecular viscosity (see also Johns and Dyke, 1971). An analytical solution for this case can be obtained and is given in Section 3b. Case (iv) is developed here in order to take into account the laminar sublayer as indicated by Johns and the observed maximum \( N \) near mid-depth (Bowden et al., 1959). A numerical method given in Section 3c is developed to obtain solutions for cases (ii) and (iv).

A more realistic vertical eddy viscosity may be estimated if an equation describing the balance of the turbulent energy density is included in the computation (Vager and Kagan, 1969). However, this inclusion creates nonlinearity in the equation of motion, and thus our three-dimensional tidal problem can only be solved through a complicated
numerical model, which is not the objective of this study. In any case, the tidal currents obtained from this type of computation are basically similar to those using the prescribed vertical profile of the eddy viscosity (Vager and Kagan, 1969). This also applies to other studies with nonlinear forms of eddy viscosity (Johns, 1976).

In the following discussion, cases (i), (ii), (iii), and (iv) are denoted as the homogeneous model, parabolic model, sublayer model and combined model, respectively.

\textbf{a. The homogeneous model—Analytical solution}

For \( N \) independent of \( z \), Eqs. (10) and (11) become

\[ \alpha^2 \hat{L}_+ = \frac{\partial^2}{\partial \eta^2} \hat{L}_+, \]  

(19)

\[ \beta^2 \hat{L}_- = \frac{\partial^2}{\partial \eta^2} \hat{L}_-, \]  

(20)

where

\[ \alpha = (1 + i)\left(\frac{\sigma + f}{2N}\right)^{1/2} D, \quad \beta = (1 + i)\left(\frac{\sigma - f}{2N}\right)^{1/2} D. \]

The solution of (19) is

\[ \hat{L}_+ = A e^{\alpha \eta} + A_1 e^{-\alpha \eta}, \]  

(21)

where \( A \) and \( A_1 \) are coefficients independent of \( \eta \). From the boundary condition \( \hat{L}_+ = 0 \) at \( \eta = 0 \), Eq. (21) becomes

\[ \hat{L}_+ = 2 A \sinh(\alpha \eta). \]  

(22)

Similarly, we can obtain

\[ \hat{L}_- = 2 B \sinh(\beta \eta), \]  

(23)

where \( B \) is a coefficient independent of \( \eta \). From (12),

\[ \begin{cases} \hat{u'} = A \sinh(\alpha \eta) + B \sinh(\beta \eta) \\ \hat{v'} = i^{-1}[A \sinh(\alpha \eta) - B \sinh(\beta \eta)] \end{cases} \]  

(24)

The velocity is then

\[ \begin{cases} \hat{u} = D \frac{A}{\alpha} \cosh(\alpha \eta) + D \frac{B}{\beta} \cosh(\beta \eta) + C_1 \\ \hat{v} = i^{-1}\left[D \frac{A}{\alpha} \cosh(\alpha \eta) - D \frac{B}{\beta} \cosh(\beta \eta)\right] + C_2 \end{cases}, \]  

(25)

where \( C_1 \) and \( C_2 \) are coefficients that depend only on \( x \) and \( y \). Applying the boundary condition \( \hat{u} = 0 \) at \( \eta = -1 \), and the definition

\[ \hat{U} = \int_{-1}^{0} \hat{u} d \eta, \]

we obtain

\[ \begin{cases} \frac{A}{\alpha} D = - \hat{U} + i \hat{V} \\ - \frac{B}{\beta} D = - \hat{U} - i \hat{V} \end{cases}, \]  

(26)

\[ \begin{cases} C_1 = - \left[\frac{A}{\alpha} D \cosh(\alpha) + \frac{B}{\beta} D \cosh(\beta)\right] \\ C_2 = - i^{-1}\left[\frac{A}{\alpha} D \cosh(\alpha) - \frac{B}{\beta} D \cosh(\beta)\right] \end{cases} \]

Thus, given \( \alpha \) and \( \beta \), the velocity \( u \) can be calculated as a function of \( U \).

To estimate the value of \( N \), we substitute the bottom stress \( \hat{\tau}_b = (\rho N \hat{u}_b') \big|_{\eta=-1} \) into the depth-average momentum equation (8), and obtain

\[ \begin{cases} i(\sigma + \lambda_1) \hat{U} - f \hat{V} = -g \frac{\partial \hat{z}}{\partial x} - \lambda_k \hat{U} \\ i(\sigma + \lambda_1) \hat{V} + f \hat{U} = -g \frac{\partial \hat{z}}{\partial y} - \lambda_k \hat{V} \end{cases}, \]  

(28)
effect becomes more important if the current becomes stronger or the water shallower, or both.

b. The sublayer model—Analytical solution

Let

$$\psi = [1 + R_2 D (\eta + 1)].$$

(Eq. (10)) then becomes

$$i(\sigma + f) \hat{L}_+ = R_2^2 \frac{\partial^2}{\partial \psi^2} N \hat{L}_+.$$  

(35)

The vertical eddy viscosity $N$ for a sublayer model (Eq. (18)) is

$$N = \begin{cases} 
\psi_0 \psi_2^2, & \psi \leq \psi_z \\
 N_m = \nu_p \psi_2^2, & \psi \geq \psi_z \end{cases}$$

(36)

where $\psi_z$ is a value of $\psi$ when $\eta = \eta_z$. For $\psi \leq \psi_z$, Eq. (35) becomes

$$\frac{\partial^2}{\partial \psi^2} (\psi \hat{L}_+) = \frac{i(\sigma + f)}{R_2^2 \nu_0} \hat{L}_+.$$  

(37)

For $\hat{L}_+ \propto \psi^m$, Eq. (37) gives

$$(m + 2)(m + 1) = \frac{i(\sigma + f)}{R_2^2 \nu_0}$$

(38)

and a solution for $\hat{L}_+$,

$$\hat{L}_+ = B_1 \psi^{m_1} + B_2 \psi^{m_2},$$

(39)

where $m_1$ and $m_2$ are solutions of $m$ from (38), and $B_1$ and $B_2$ are coefficients determined from the boundary conditions. Using $dz = (1/R_2) d\psi$, and the boundary condition $\hat{L}_+ = 0$ for $\psi = 1$, the integration of (39) with respect to $z$ gives

$$\hat{L}_+ = \frac{1}{R_2} \left[ \frac{B_1}{m_1 + 1} (\psi^{m_1 + 1} - 1) \right.$$  

$$\left. + \frac{B_2}{m_2 + 1} (\psi^{m_2 + 1} - 1) \right].$$  

(40)

For $\psi \geq \psi_z$, $N = N_m$ is independent of $z$. Replacing $\eta$ by $[(\psi - 1)/R_2 D] - 1$ from (34), the solution of $\hat{L}_+$ and $\hat{L}_+$ becomes (Section 3a).

$$\hat{L}_+ = B_3 \sinh \left[ \alpha \left( \frac{\psi - 1}{R_2 D} - 1 \right) \right],$$  

(41)

$$\hat{L}_+ = B_3 \frac{D}{\alpha} \cosh \left[ \alpha \left( \frac{\psi - 1}{R_2 D} - 1 \right) \right] + B_4,$$  

(42)

where $B_3$ and $B_4$ are coefficients independent of $z$.

Eqs. (40) and (42) include four unknown coefficients that can be determined as a function of $\hat{U} + i \hat{V}$ from the following conditions:

(i) At $\psi = \psi_z$, the velocities are continuous or $\hat{L}_+$ in (40) is equal to that in (42).

(ii) At $\psi = \psi_z$, the stresses are continuous, or $\hat{L}_+$ in (39) is equal to that in (41).
(iii) At \( \psi = \psi_z, (\partial / \partial \psi) N L_\psi \) is continuous. The existence of this condition can be seen from (35) with the requirement that \( L_\psi \) at \( \psi = \psi_z \) is finite or from (2) that the pressure gradient is independent of \( z \). From (36), (39) and (41), we then have

\[
\nu_0 [B_1 (m_1 + 2) \psi^{m_1+1} + B_2 (m_2 + 2) \psi^{m_2+1}] - 1 = N_m \alpha / R_2 D_ \cosh \left[ \alpha \left( \frac{\psi - 1}{R_2 D} \right) \right]. \quad (43)
\]

(iv) From the definition that the depth average of \( \hat{u} + i \hat{v} \) is equal to \( \hat{U} + i \hat{V} \), we have

\[
\int_1^{\psi} \frac{1}{R_2 D} \left[ \frac{B_1}{m_1 + 1} (\psi^{m_1+1} - 1) + \frac{B_2}{m_2 + 1} (\psi^{m_2+1} - 1) \right] d\psi + \frac{1}{R_2 D} \int_0^{\eta_m} \left[ \frac{B_3}{\alpha} \cosh \left[ \alpha \left( \frac{\psi - 1}{R_2 D} \right) \right] \right] d\psi = \hat{U} + i \hat{V}. \quad (44)
\]

Similarly, we can obtain \( \hat{L}_\psi \) as a function of \( \hat{U} - i \hat{V} \). The velocities \( \hat{u} \) and \( \hat{v} \) can then be calculated, \( \lambda_k, \lambda_i \) and \( f_i \) defined in (31) can also be calculated using the same procedure as in Section 3a. The computed \( \lambda_k \) as a function of \( d_0 \) is shown in Fig. 2.

c. Numerical solution

In rewriting (10) and (11) in a finite-different form, \( \eta \) is separated into \( n \) unequal grid increments \( \Delta_k \), where \( k \) is an integer which increases from 1 at the bottom to \( n \) at the surface. A continuous function, for example, \( L_\psi(x,y,\eta) \), is written as \( L_\psi(k) \), where \( k = 1 \) at the bottom and increases to \( (n + 1) \) at the surface.

In order to resolve the thin laminar sublayer in the combined model, it is necessary to have a very small grid spacing near the bottom. \( \Delta_k \) is designed here to increase parabolically from the bottom, i.e.,

\[
\Delta_k = \frac{k^2}{n} \sum_{k=1}^n k^2.
\]

By choosing \( n = 110, \Delta_k \) increases from \( \Delta_1 = 0.222 \times 10^{-2} \) to \( \Delta_{110} = 2.690 \times 10^{-2} \). To examine the accuracy of the solution with this value of \( n \), the numerical method is applied to the homogeneous and sublayer models, where analytical solutions can be obtained. A comparison indicates a good agreement between the numerical and analytical computations.

The finite difference form of (10) is

\[
\frac{1}{2} \sum_{k-1}^{n} \left[ \frac{N(k+1) \hat{L}_\psi(k+1) - N(k) \hat{L}_\psi(k)}{\Delta_k} - \frac{N(k \hat{L}_\psi(k) - N(k-1) \hat{L}_\psi(k-1))}{\Delta_{k-1}} \right] = i D^2 (\sigma + f) \hat{L}_\psi(k)
\]

or

\[
\hat{L}_\psi(k+1) = \frac{N(k+1) \lambda_k N(k+1) + \hat{L}_\psi(k)}{2} \times \frac{-((\gamma_k + 1) N(k) - 1/2 i D^2 (\sigma + f) \Delta_{k-1} \cdot (\Delta_k + \Delta_{k-1}))}{\Delta_k} \quad (45)
\]

From (47), (48) and (50), we obtain

\[
\hat{L}_\psi = \frac{1}{\delta_+} (C_+ F_+).
\]

where \( \Delta_+ = \Delta_{n-1} / \Delta_n \). For any specified values of \( \hat{L}_\psi \) at \( \eta = 0 \) and \( \eta = 0 \), Eq. (45) gives \( (n-1) \) linear equations with \( (n-1) \) unknowns \( \hat{L}_\psi(2), \hat{L}_\psi(3), \ldots, \hat{L}_\psi(n) \). These linear equations can be solved easily through a matrix decomposition.

To obtain \( \hat{L}_\psi \) at \( \eta = 0 \), we apply the boundary condition \( \hat{L}_\psi(n+1) = 0 \). However, at \( \eta = -1, \hat{L}_\psi(1) \) cannot be specified because the boundary condition at this depth is \( \hat{L}_\psi = 0 \). This difficulty can be overcome by using the following method of computation.

We first specify \( \hat{L}_\psi(1) = C_+(x,y) \), where \( C_+(x,y) \) is any nonzero function of \( x \) and \( y \); Eq. (45) thus gives a solution of \( \hat{L}_\psi \) that must be adjusted because \( C_+ \) is an arbitrary function. We denote this unadjusted \( \hat{L}_\psi \) as \( \hat{L}_\psi \) and the corresponding depth average currents in the \( x \) and \( y \) directions as \( \hat{U} \) and \( \hat{V} \). From (45) and the boundary condition \( \hat{L}_\psi(n+1) = 0 \), we can see that \( \hat{L}_\psi \) is a linear function of \( C_+ \) or

\[
\hat{L}_\psi = C_+ F_+,
\]

where \( F_+ \) is a function of \( x \) and \( y \), and \( \eta \) and independent of \( C_+ \). From (13) and the definition of depth-averaged velocity, we obtain

\[
\hat{U} = C_+ D \int_{-1}^{0} \int_{-1}^{\eta} F_+ d\eta d\eta^*.
\]

\( \hat{U} \) and \( \hat{V} \) will be equal to \( \hat{U} \) and \( \hat{V} \) if the boundary condition at \( \eta = 1 \) is specified correctly. Defining the adjusting parameter as

\[
\delta_+ = \frac{\hat{U} + i \hat{V}}{\hat{U} + i \hat{V}},
\]

Eq. (47) gives

\[
\hat{U} + i \hat{V} = \frac{C_+ D}{\delta_+} \int_{-1}^{0} \int_{-1}^{\eta} F_+ d\eta d\eta^*,
\]

which corresponds to \( \hat{L}_\psi \) as

\[
\hat{L}_\psi = \frac{1}{\delta_+} (C_+ F_+).
\]
\[ \dot{L}_+ = \dot{U} + i \dot{V} F_{+n}, \]  
\[ (51) \]

where

\[ F_{+n} = F_+ \left[ \int_{-1}^{0} \int_{-1}^{\eta^*} F_+ d\eta d\eta^* \right]^{-1} \]

is a normalized function describing the vertical variations of \( L_+ \). Eq. (51) indicates that the solution of \( L_+ \) is independent of \( C_+ \). This is an important conclusion because we can now obtain \( \dot{L}_+ \) from (45) without knowing the specific function of \( C_+ \). Otherwise the solution becomes difficult to obtain, as \( \dot{L}_+ \) is unknown. For example, if the tide-generating force was included, \( \dot{L}_+ \) would become a function of both \( C_+ \) and the force, and \( \dot{L}_+ \) or \( \dot{L}_+/\delta_+ \) would not be always independent of \( C_+ \).

Similar procedures can be used to obtain \( \dot{L}_- \) from (11) as

\[ \dot{L}_- = \dot{U} - i \dot{V} F_{-n}, \]  
\[ (52) \]

where

\[ F_{-n} = F_- \left[ \int_{-1}^{0} \int_{-1}^{\eta^*} F_- d\eta d\eta^* \right]^{-1} \]

and \( F_- \), corresponding to \( F_+ \) in (46), describes the vertical variation of \( L_- \). From (12),

\[ \begin{align*}
\dot{u}' &= \frac{1}{2} \dot{U} \frac{1}{D} (F_{+n} + F_{-n}) + \frac{1}{2} \frac{\dot{V}}{D} (F_{+n} - F_{-n}) \\
\dot{v}' &= -\frac{1}{2} \frac{1}{D} \frac{\dot{U}}{D} (F_{+n} - F_{-n}) + \frac{1}{2} \frac{\dot{V}}{D} (F_{+n} + F_{-n})
\end{align*} \]  
\[ (53) \]

Similar to the computation in Section 3a, we can then write

\[ \lambda_R = \operatorname{Re} \left\{ \frac{1}{2} \frac{N(1)}{D^2} \left[ F_{+n}(1) + F_{-n}(1) \right] \right\} \]

\[ \lambda_i = \operatorname{Im} \left\{ \frac{1}{2} \frac{N(1)}{D^2} \left[ F_{+n}(1) + F_{-n}(1) \right] \right\} \]

\[ f_c = f - i \frac{1}{2} \frac{N(1)}{D^2} \left[ F_{+n}(1) - F_{-n}(1) \right] \]  
\[ (54) \]

where \( F_{+n}(1) \) are the values of \( F_{+n} \) at \( \eta = -1 \). The velocity \( \dot{u}' \) can be calculated from (13) and (53).

In the homogeneous model (Section 3a), we have shown that at any location, \( \lambda_R, \lambda_i \) and \( f_c \) are functions only of \( d_0 \), and \( \dot{u}'/U \) is a function only of \( d_0 \) and \( \dot{V}/U \). We now investigate whether this conclusion is also valid for other forms of \( N \).

We first examine those models where \( N \) can be written as

\[ N = N_m P(\eta) \]  
\[ (55) \]

and where \( N \) is explicitly independent of \( D \). An example of this form of \( N \) is the parabolic model [Eq. (17)]. From (31) and (55), (10) becomes

\[ \frac{\partial^2}{\partial \eta^2} (p\dot{L}_+^*) = i2d_0^2 \left( 1 + \frac{f}{\sigma} \right) \dot{L}_+. \]  
\[ (56) \]

For a given \( f/\sigma \), this equation involves only a parameter \( d_0 \). From (51), we can then write \( \dot{L}_+^* \) as

\[ \dot{L}_+^* = \frac{\dot{U} + i \dot{V}}{D} F_{+n}(\eta, d_0). \]  
\[ (57) \]

Similar expressions can be obtained for \( \dot{L}_-^* \). As \( N(1) \) is proportional to \( N_m \), we can see from (31) that \( N(1)/\sigma D^2 \) is proportional to \( d_0 \). From (54), we can then see that \( \lambda_R/\sigma, \lambda_i/\sigma \) and \( f_c/\sigma \) are functions only of \( d_0 \). From (13) and (53), we can also see that \( \dot{u}'/U \) at each location \( (x, y, \eta) \) are functions only of \( d_0 \) and \( \dot{V}/U \).

For the sublayer and combined models, it is clear from (18) that \( P \), defined in (55), is also a function of the depth \( D \). Applying a similar procedure to that discussed above, it can be shown that \( \lambda_R/\sigma, \lambda_i/\sigma \), \( f_c/\sigma \) and \( \dot{u}'/U \) generally are also dependent on \( D \).

Fig. 2 shows plots of \( \lambda_R \) vs. \( d_0 \) for the parabolic model, and for the sublayer and combined models with \( D = 19 \) m. The results of \( \lambda_R \) vs. \( d_0 \) for the sublayer and combined models with other values of \( D \) were also computed and found to differ only slightly from those with \( D = 19 \) m.

4. Numerical evaluation and comparison to the observations

Because of the uncertainty as to the form of the vertical eddy viscosity, there exist several parameters in the model, in addition to \( N_m \) (the maximum value of the vertical eddy viscosity) which occurs in all models, there is \( R_t \) (the ratio of the vertical eddy viscosity at the surface to that at mid-depth).
in the parabolic and combined models and \( \delta_x \) (the thickness of the laminar sublayer) in the sublayer and combined models. One of these parameters is determined by approximating the computed stress to the empirical value [Eq. (6)]. The other parameters can be determined by comparing the observed and calculated tidal currents. For the homogeneous model, since there is only one parameter \( N_m \), such a comparison can be used to verify the model.

There are very few observations of the vertical variation of the tidal currents in homogeneous water. Bowden and Fairbairn’s (1952) result is probably the most complete set of analyzed data at the present time. However, due to short current records and the poor accuracy of Bowden and Fairbairn’s measurements, these data do not provide much information for comparison. For example, the data do not give any good information about the vertical profile of the phase lags and of the inclinations of the semi-major axis because of small vertical variations in the water column, or of the amplitudes of the semi-minor axis because of small ratios of \( V_0/U_0 \). Thus, comparison can only be made of the amplitude of the semi-major axis. In any case, it is not the objective of this study to choose between the parabolic, sublayer and combined models. This choice would require much more accurate data.

Two stations off Red Wharf Bay, Anglesey, North Wales, were occupied in Bowden and Fairbairn’s experiment: the West station where water depth was 19 m, and the East station where water depth was 12 m. The latitudes of these stations are about 53°20'N, which corresponds to \( f = 1.169 \times 10^{-4} \text{ s}^{-1} \). Seven records, each containing 24 h data, were taken at the West station. There are four similar records for the East station. Detailed descriptions of the data and their analyses were given in Bowden and Fairbairn (1952).

Because of its shallower water, the East station has a stronger dissipative system than the West station. The deviation of the tidal currents between these two stations could provide some verification of the models. However, the poor accuracy of the experimental data does not permit us to make such a verification. Thus, only the calculated and observed tidal currents in the West station will be presented.

The major axis of the depth-average velocity is taken in this study to be parallel to the \( x \) axis. In the West station, \( U_m = 0.58 \text{ m s}^{-1} \), \( V_0/U_0 = -0.12 \) and \( \lambda = 0.55 \times 10^{-4} \text{ s}^{-1} \). A negative value of \( V_0/U_0 \) indicates that the circulation in a tidal ellipse is clockwise; a positive value indicates an anti-clockwise circulation. The linear friction coefficient \( \lambda \) is estimated from (7) with \( K = 8/3\pi, \sigma \) equal to the semi-diurnal frequency, \( \gamma = 0.0021 \) (Bowden et al., 1959) and \( U_m \) equal to the semi-major axis. From the \( \lambda \) values and Fig. 2, values of \( d_0 \) and \( N_m \) correspond-

| Table 1. \( R_1, \delta_x, d_0 \) and \( N_m \) of various models at the West station. |
|-----------------|-----------------|-----------------|
| Forms of \( N \) | Parameters \( R_1 \) \( \delta_x \) (m) \( d_0 \) \( N_m \) (10^{-4} \text{ m}^{-2} \text{ s}^{-1}) |
| Homogeneous     | —               | —               | 1.97            | 65 |
| Parabolic       | 0.1             | 1.31            | 1.31            | 148|
| Parabolic       | 0.01            | —               | 0.99            | 259|
| Parabolic       | 0.001           | —               | 0.83            | 367|
| Sublayer        | —               | 0.1             | 1.18            | 182|
| Combined        | 0.5             | 0.07            | 1.08            | 218|
| Bowden et al.   | (1959)          | —               | 0.96            | 276|

In Table 1 are shown the various forms of \( N \) are given in Table 1. The values of \( \delta_x \) shown in the table give good agreement between the calculated and observed tidal currents in the sublayer and combined models. Also included in Table 1 is \( N_m \) as estimated from the empirical formula [Bowden, 1967, Eq. (16)].

Fig. 3 shows plots of \( u_0/U_0, g_x - g_x, v_0/V_0 \) and the inclination of the major axis. The observed values of \( u_0/U_0 \) and their standard deviations are also included. From Fig. 3a, we can see that a good fit can be obtained for the parabolic model with \( R_1 = 0.001 \), the sublayer model with \( \delta_x = 0.1 \text{ m} \), and the combined model with \( \delta_x = 0.07 \text{ m} \) and \( R_1 = 0.5 \). The \( u_0/U_0 \) near the bottom for the parabolic model with \( R_1 = 0.01 \) is smaller than the observed values.

For the homogeneous model, and the parabolic model with \( R_1 = 0.1 \), the calculated values of \( u_0/U_0 \) are too large in comparison to the observed values at the upper portion of the water column and too small at the lower portion. The \( g_x - g_x, v_0/V_0 \) and inclination of the major axis in these two models also show much stronger vertical variation than those of the others.

5. Discussion

Unless otherwise specified, the results of the combined model with \( R_1 = 0.5, \delta_x = 0.07 \text{ m} \) and \( D = 19 \text{ m} \), which were shown in Section 4 to be in agreement with the observations, are applied below. The other realistic models (the sublayer model with \( \delta_x = 0.10 \text{ m} \) and the parabolic model with \( R_1 = 0.001 \)) generally give the same results.

a. The boundary-layer approximation

In some tidal computations (i.e., Johns and Dyke, 1971), it is assumed that the vertical friction \( \partial \sigma/\partial z \) can be neglected near the surface because of the small vertical variation of the tidal current in the upper portion of the water column. Fig. 4 is plotted to determine whether or not the approximation is applicable for the West station \( (d_0 = 1.08) \). It shows
the amplitude of $\tau_z/\rho = Nu'$ for $V_0/U_0 = -0.12$. It can be seen from the figure that the boundary-layer approximation is obviously invalid for the West station.

The stresses for two less dissipative systems ($d_0 = 2.34$ and $4.57$) are also shown in Fig. 4. As $d_0$ increases, the stress decreases more rapidly from the bottom and becomes more uniform near the surface. Thus, as expected, the boundary-layer approximation is more valid in the less dissipative system.

b. The shallow water equation

In a more dissipative system (small $d_0$), it is expected that the depth-mean tidal current can be governed by the shallow water equation because the Coriolis effect on the bottom friction can be neglected and the phase of the current varies only slightly in the water column. However, as $d_0$ increases, the phase of both the current and bottom stress deviates more significantly from that of the depth mean current; thus, the validity of the shallow-water equation requires further investigation.

As indicated in Section 3 [Eq. (33)], the shallow water equation is valid only if $f_c = f$ and $\lambda_1 \ll \sigma$. Fig. 5 shows $\lambda_1/\sigma$ and $f_c/f$ as functions of $d_0$. The result of the sublayer model with $\delta_z = 0.1$ m is close to that of the combined model with $\delta_z = 0.07$ m and $R_1 = 0.5$, and is not shown in the figure. For the homogeneous model and the parabolic model with $R_1 = 0.1$, the deviations from the shallow water equation can be quite large. For example, the deviation for low $d_0$ values can be as high as 20% for the homogeneous model and 8% for the parabolic model. However, for a more realistic model (e.g., the sublayer and combined models), the deviations are less than 3% in amplitude and 2° in phase. Thus, the depth-averaged equation (3) with $\tau_0$ equal to the computed bottom stress can be approximated by the shallow water equation.

In a low dissipation system, although the deviation of the phase of the bottom stress from that of the depth mean current is found to increase with $d_0$, the deviation from the shallow water equation decreases. This is because the stress ($\tau$) is proportional to $U_0^2$ [Eqs. (6) and (7)] and the Coriolis and
other acceleration terms are proportional to $U_0$; thus as $d_0$ increases, the stress decreases more rapidly and causes lower values of $\lambda$, and $(f_s/f - 1)$.

In the shallow water equation, the second-order advective terms \((u \cdot \Delta u) + (w \partial u / \partial z)\), where the angle braces denote the depth average, have been approximated by $U \cdot \Delta U$. This approximation can be examined in this study by comparing $\langle u^2 \rangle$ with $U^2$, $\langle v^2 \rangle$ with $V^2$, and $\langle uv \rangle$ with $UV$. Fig. 6 shows these comparisons as functions of $d_0$ for $V_0/U_0 = -0.12$ and $-0.50$. The deviation of $\langle v^2 \rangle$ from $V^2$ for $V_0/U_0 = -0.12$ is not shown because the term $v(\partial u/\partial y)$ is not considered important for a small ratio of $V_0$ to $U_0$. However, the term $v(\partial u/\partial y)$ can be of the same order of magnitude as $u(\partial u/\partial x)$ because $V_0/U_0 \approx L_u/L_u$ from the scaling of the continuity equation where $L_u$ and $L_v$ are the length scales of $u$ and $v$. As shown in the figure, the amplitudes of the deviations have their maxima at intermediate values of $d_0$. As the ratio $V_0/U_0$ increases, the deviation of $\langle u^2 \rangle$ increases, and those of $\langle uv \rangle$ and $\langle v^2 \rangle$ decrease. The deviation of $\langle uv \rangle$ can be quite high for small values of $V_0/U_0$. Thus, for small ratios of $V_0$ to $U_0$, the approximation of $v(\partial u/\partial y)$ in the shallow water equation may not be appropriate.

The phases of the deviations were small (<5°). The deviations for the anticlockwise tidal ellipses ($V_0/U_0 = 0.12$ and 0.50) were generally similar to those shown in Fig. 6.

c. Dynamics of the tidal current

In this section, the tidal currents for different dissipative systems ($d_0 = 0.71, 2.34$ and $4.57$) computed from the combined model with $\delta_u = 0.07$ m, $R_1 = 0.5$, $D = 19$ and $V_0/U_0 = \pm 0.12$ are discussed. To determine the type of current system corresponding to these values of $d_0$, $U_m$ is computed from (7),

\[ U_m = \frac{1}{2} \sqrt{2gH(d_0 + \delta_u)} \]

where $\lambda$ can be obtained from Fig. 1. For $D = 19$ m, equal to the depth of the West station, the corresponding values of $U_m$ for $d_0 = 0.71, 2.37$ and $4.57$ are $1.00, 0.21$ and $0.09$ m s$^{-1}$. If $D$ increases to $60$ m, the corresponding values become $3.16, 0.66$ and $0.28$ m s$^{-1}$.

Fig. 7 shows for $V_0/U_0 = 0.12$ and various dissipative systems the vertical variation of amplitudes, phases and inclinations of the semi-major axis ($u_{ma}$). Note that in this discussion, the semi-major
To explain the dynamics of the tidal current, an equation is obtained by subtracting (2) from (3) and neglecting the horizontal diffusion terms in the offshore area:

\[
\frac{\partial}{\partial t} (u - U) = -f \times (u - U) + \frac{1}{\rho} \left( \frac{\partial \tau}{\partial z} + \frac{\tau_b}{D} \right). \tag{58}
\]

We consider first the high dissipation system so that the Coriolis term in (58) can be neglected. The stress

...
term \((1/\rho)(\partial\tau_1/\partial z) + (\tau_0/D)\) is a measure of the deviation of the curve for \(\tau(\rho)\) as a function of \(z\) from the straight line joining \((\pi(\rho) = 0 + \rho(\rho) = (\tau_0/D)\) at \(z = -D\). From Fig. 4, we can see that the \(x\) component of the stress term in (58) is negative in the lower portion and positive in the upper portion of the water column. Note that the vertical variation is less significant in the phase of the stress than in the amplitude. From (58) we can thus see that \(u_{\text{maj}}\) is smaller than \(U_0\) in the lower portion of the water column and larger in the upper portion (Fig. 7a). As \(d_0\) increases, the curve shown in Fig. 4 deviates more from a straight line. However, this does not imply that the curve of \(u_{\text{maj}}\) always deviates more from \(U_0\), because the stress term \((1/\rho)(\partial\tau_1/\partial z) + (\tau_0/D)\) also depends on the stress's amplitude, which, as shown in Section 5b, is decreasing more rapidly than the accelerating terms. Thus, we can see from (58) that only if the effect of the curve's deviation is more significant than the effect of the amplitude's decrease will the deviation of \(u_{\text{maj}}\) from \(U_0\) increase. For example, near \(n = -0.9\) (the frictional force remains dominant here, even for \(d_0 = 2.34\), and 4.57) the effect of the curve's changes is more important as \(d_0\) increases from 2.34 to 4.57 \([u_{\text{maj}}]/(U_0)\), increases, Fig. 7a); and the effect of the amplitude's changes becomes more important as \(d_0\) increases from 0.71 to 2.34 \([u_{\text{maj}}]/(U_0)\) decreases.

For the semi-minor axis, the vertical distribution of \(u_{\text{min}}\) is expected to be similar to that of \(u_{\text{maj}}\) in the high dissipation system (Fig. 8a) because, by neglecting the Coriolis term, the velocity gradients \(u'\) and \(v'\) can be separated and their equations of motion will be the same [Eq. (9)].

As \(d_0\) increases to a less dissipation system, the Coriolis term becomes important in the momentum equation because the stress decreases very rapidly (Section 5b). As \(U_0 \gg V_0\) in this discussion, the Coriolis effect, while it becomes important in the \(v\) component equation, can remain small in the \(u\) component equation. This is why, as the circulation reverses, \(u_{\text{maj}}\) changes only slightly (Fig. 8a) and \(u_{\text{min}}\) changes very significantly (Fig. 7a). The vertical variation of \(u_{\text{min}}\) can be explained from (58). Note that the Coriolis force tends to push a water particle to the right, i.e., clockwise acceleration. In the upper portion of the water column, \(u_0 > U_0\) (Fig. 7a), so the Coriolis force \(-\epsilon x(\delta - U)\) in (58) is expected to produce a clockwise acceleration. This acceleration is anticlockwise in the lower portion of the water column because \(u_0 < U_0\). With an anticlockwise circulation of the depth-mean velocity \(U\), the Coriolis term in (58) produces acceleration in the upper portion of the water column which is in the opposite direction of the circulation of \(U\) but which is in the same direction in the lower portion. Thus, we can expect that \(v_0\) or \(u_{\text{min}}\) is smaller than \(V_0\) in the upper portion and larger in the lower portion of the water column. As \(u_0\) decreases continuously from the surface, we can expect that \(u_{\text{min}}\) will increase from the surface to the deeper water (Fig. 8a). The maximum of \(u_{\text{min}}\) is found in the lower portion because, due to the frictional effect, \(u_{\text{min}}\) must decrease to zero at the bottom.

For the depth-mean velocity with clockwise circulation, a similar origin can be ascribed to the vertical variation of \(u_{\text{min}}\) shown in Fig. 8a. However, in this case the acceleration produced by the Coriolis term (Eq. 58) in the upper portion of the water column is in the same direction as the circulation of \(U\), and in the opposite direction in the lower portion. Thus, \(u_{\text{min}}\) or \(v_0\) is larger than \(V_0\) in the upper portion and decreases to less than \(V_0\) in the lower portion. Near the bottom, the frictional effect is important, and, as indicated before, it tends to reduce \(u_{\text{min}}\) toward the bottom. Thus, both the Coriolis and frictional effects tend to decelerate the clockwise mo-
tion of the tidal current. If these two effects are sufficiently strong, $u_{\text{min}}$ can be reduced to zero at some depth, and below that the circulation of the tidal current is forced to become anticlockwise or opposite to the circulation of the depth-mean velocity (Fig. 8a).

If $V_0 \approx 0$, a condition that can be assumed in a narrow channel, we can expect from the above discussion that the circulation of the tidal ellipse will be clockwise in the upper portion of the water column and anticlockwise in the lower portion.

Fig. 8b shows the ratio of $u_{\text{min}}$ to $u_{\text{maj}}$ for $V_0 = \pm 0.12 U_0$ and $d_0 = 0.71, 2.34$ and 4.57. It is interesting that although the ratio of the minor axis to the major axis of the depth-mean velocity is only 12% in this study, the ratio can be quite high near the surface or the bottom. For example, the ratio for $d_0 = 4.57$ and $V_0 = 0.12 U_0$ is as high as 40% near the bottom (Fig. 8b).

6. Summary

The three-dimensional tidal equations can be solved by first computing the depth-averaged velocity from the shallow water equation which is easily solved numerically. The calculated depth-averaged velocity is then used as an external parameter in computing the vertical variation of the tidal current using the $u'$ equation. By adjusting the value of vertical eddy viscosity, the bottom friction calculated from the solution of $u'$ can be made to approximate the value used in the depth-averaged equation. The vertical variation of the tidal current was computed for various forms of the vertical eddy viscosity and then compared to Bowden and Fairbairn’s observations (1952). A good fit was obtained for the sublayer model with $\delta_z = 0.1$ m, for the combined model with $\delta_z = 0.07$ m and $R_1 = 0.5$, and for the parabolic model with $R_1 = 0.001$. The tidal current dynamics are explained in Section 5c.

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